Ample completion of OMs and CUOMs

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**Context**

**Conjecture** [Floyd and Warmuth, 1995]:
Every set family of VC-dimension $d$ has a sample compression scheme of size $O(d)$.

**Theorem** [Moran and Warmuth, 2016]:
Every ample set family of VC-dimension $d$ has a labeled sample compression scheme of size $d$.

**Question**:
Can any set family of VC-dimension $d$ be completed to an ample set family of VC-dimension $O(d)$?

**Our result**:
Every OM and CUOM of VC-dimension $d$ can be completed to an ample set family of VC-dimension $d$. 
Complexes of oriented matroids

\[ U = \{1, \ldots, m\} \text{ and } \mathcal{L} = \{-1, 0, +1\}^m \]
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Complexes of oriented matroids

$$U = \{1, \ldots, m\} \text{ and } \mathcal{L} = \{-1, 0, +1\}^m$$

$$(U, \mathcal{L}) \text{ OM iff }$$

(C) $$\forall X, Y \in \mathcal{L}, X \circ Y \in \mathcal{L};$$

(SE) $$\forall X, Y \in \mathcal{L} \text{ and for each } e \in U \text{ with } X_e Y_e = -1, \exists Z \in \mathcal{L} \text{ such that } Z_e = 0 \text{ and } Z_f = (X \circ Y)_f \forall f \in U \text{ with } X_f Y_f \neq -1$$

(Sym) $$\mathcal{L} = -\mathcal{L} := \{-X : X \in \mathcal{L}\}.$$
Complexes of oriented matroids

\[ U = \{1, \ldots, m\} \text{ and } L = \{-1, 0, +1\}^m \]

\((U, L)\) OM iff

\((C)\) \(\forall X, Y \in L, X \circ Y \in L;\)

\((SE)\) \(\forall X, Y \in L, \text{ and } \forall i \in U \text{ with } X_i Y_i = -1, \exists Z \in L \text{ such that } Z_i = 0 \text{ and } Z_j = (X \circ Y)_j \forall j \in U \text{ with } X_j Y_j \neq -1;\)

\((Sym)\) \(L = -L := \{-X : X \in L\}.\)

\[
\begin{pmatrix}
0 \\
+
\
-
\end{pmatrix}, \begin{pmatrix}
+
\\
-
\end{pmatrix} \Rightarrow \begin{pmatrix}
+
\\
\
0
\end{pmatrix}
\]
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\((\text{Sym})\) \(\mathcal{L} = -\mathcal{L} := \{-X : X \in \mathcal{L}\}\).

\[(X \circ Y)_i = \begin{cases} X_i & \text{if } X_i \neq 0 \\ Y_i & \text{otherwise.} \end{cases} \]
Complexes of oriented matroids

\[ U = \{1, \ldots, m\} \text{ and } \mathcal{L} = \{-1, 0, +1\}^m \]

\((U, \mathcal{L})\) **OM** iff

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(Sym) \(\mathcal{L} = -\mathcal{L} := \{-X : X \in \mathcal{L}\}.\)

\((U, \mathcal{L})\) **COM** iff

(C), (SE) and (FS) \(\forall X, Y \in \mathcal{L}, X \circ -Y \in \mathcal{L}.\)
Tope graphs

**Topes of** $\mathcal{L}$ **:** covectors without zero entries.

**Tope graph of** $\mathcal{L}$ **:** subgraph induced by its topes in the hypercube $\{+, -\}^m$. 
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Tope graph of $\mathcal{L}$: subgraph induced by its topes in the hypercube $\{+, −\}^m$.

Proposition [Bandelt et al., 2018]: COMs are uniquely determined by their topes.

Remark: tope graphs $G$ of COMs are isometric subgraphs of hypercube $Q$, i.e., $\forall u, v \in V(G), d_G(u, v) = d_Q(u, v)$. 
Gated and antipodal subgraphs

$G' \subseteq G$ gated if $\forall u \in V(G) \exists u' \in V(G')$ s.t. $\forall v' \in V(G')$ there is a shortest $(u, v')$-path through $u'$.
Gated and antipodal subgraphs

$G' \subseteq G$ **gated** if $\forall u \in V(G) \exists u' \in V(G')$ s.t. $\forall v' \in V(G')$ there is a shortest $(u, v')$-path through $u'$.
Gated and antipodal subgraphs

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$G$ antipodal if $\forall u \in V(G), \exists v \in V(G)$ s.t. $\forall w \in V(G)$, there is a shortest $(u, v)$-path through $w$. 
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**COMs**: all antipodal subgraphs are gated.

**OMs**: antipodal COMs.

**AMPs**: COMs s.t. all antipodal subgraphs are hypercubes.
pc-minors and VC-dimension

\[ \text{VC-dim}(G) = \max\{d : Q_d \text{ is a pc-minor of } G\}. \]
Classes of partial cubes

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Classes of partial cubes

**COMs** = all antipodal subgraphs are gated

**UOMs** = OMs s.t. all proper antipodal subgraphs are hypercubes

**AMPs** = COMs s.t. all antipodal subgraphs are hypercubes

**CUOMs** = COMs s.t. all proper antipodal subgraphs are UOMs

**OMs** = antipodal COMs

**UOMs** = OMs s.t. all proper antipodal subgraphs are hypercubes
Our result

**Theorem 1** [Chepoi, Knauer, and P., 2020]: Any OM of VC-dimension $d$ can be completed to an ample of the same VC-dimension.

**Theorem 2** [Chepoi, Knauer and P., 2020]: Any CUOM of VC-dimension $d$ can be completed to an ample of the same VC-dimension.
Proof of Theorem 1

Any OM of VC-dimension $d$ can be completed to an ample of the same VC-dimension.

Proposition [Björner et al., 1993]:

Any OM can be completed to a UOM of the same VC-dimension.
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\[ G \]  UOM of VC-dimension \( d \)

contraction of \( E_i \)

\[ G' \]  has an ample completion of VC-dimension \( \leq d - 1 \)
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\( G \) UOM of VC-dimension \( d \)

\( G' \) has an ample completion of VC-dimension \( \leq d - 1 \)

\( H \) is an ample completion of \( G \) of VC-dimension \( d \)

contraction of \( E_i \)

expansion on \( G_i^+ \)
Proof of Theorem 2 (1/2)

Any CUOM of VC-dimension $d$ can be completed to an ample of the same VC-dimension.

Idea: 1) Complete independently each facet of $G$ to an ample;
2) Take the union of those facet completions.
Proof of Theorem 2 (1/2)

Any CUOM of VC-dimension \( d \) can be completed to an ample of the same VC-dimension.

Idea: 1) Complete independently each facet of \( G \) to an ample; 2) Take the union of those facet completions.

Lemma 1:
\( G \) partial cube, \( H \subseteq G \) gated and \( H' \) partial cube s.t. \( H \subseteq H' \subseteq C(H) \)

(i) \( G' \) partial cube;
(ii) \( H' \subseteq G' \) gated;
(iii) \( \text{VC-dim}(G') = \max\{\text{VC-dim}(G), \text{VC-dim}(H')\} \).

\begin{center}
\begin{tikzpicture}

\draw[fill=lightgray!30] (0,0) ellipse (3 and 1);
\draw[fill=lightgray!30, thick] (-3,0) -- (-1,1) -- (1,1) -- (3,0) -- cycle;
\draw[fill=lightgray!30] (0,0) -- (-2,-2) -- (-1,-1) -- (1,-1) -- (2,-2) -- cycle;

\draw[fill=lightgray!30] (0,0) ellipse (2 and 0.5);
\draw[fill=lightgray!30, thick] (-2,0) -- (-1,1) -- (1,1) -- (2,0) -- cycle;
\draw[fill=lightgray!30] (0,0) -- (-1,-1) -- (1,-1) -- cycle;
\draw[fill=lightgray!30] (0,0) ellipse (1.5 and 0.3);
\draw[fill=lightgray!30, thick] (-1,0) -- (1,0) -- cycle;

\node at (-3,0) {$G$};
\node at (-1.5,1) {$C(G)$};
\node at (1.5,1) {$C(H)$};
\node at (0,-1) {$H$};
\end{tikzpicture}
\end{center}
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Any CUOM of VC-dimension $d$ can be completed to an ample of the same VC-dimension.

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Proof of Theorem 2 (1/2)

Any CUOM of VC-dimension \( d \) can be completed to an ample of the same VC-dimension.

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Proof of Theorem 2 (2/2)

Any CUOM of VC-dimension $d$ can be completed to an ample of the same VC-dimension.

**distance** $d(A, B) := \min\{d(a, b) : a \in A, b \in B\}$.

**mutual projection** $\text{pr}_B(A) := \{a \in A : d(a, B) = d(A, B)\}$.

**Lemma 2 :**

$A, B$ facets of a CUOM $G \Rightarrow \text{pr}_B(A) = \text{pr}_{C(B)}(C(A))$ and

$\text{pr}_A(B) = \text{pr}_{C(A)}(C(B))$. 

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![Diagram](attachment:image.png)
Conclusion

Can any set family of VC-dimension $d$ be completed to an ample set family of VC-dimension $O(d)$?

- Any partial cubes of VC-dimension 2 can be completed to an ample of VC-dimension 2;

- Any OM and CUOM can be completed to an ample of the same VC-dimension.
Thank you for your attention!