Packing et couverture de boules dans des graphes excluant un mineur

Journées Graphes et Algorithmes - 16 novembre 2020

Carole Muller, Université libre de Bruxelles

Travail avec Nicolas Bousquet, Wouter Cames van Batenburg, Louis Esperet, Gwenaël Joret, William Lochet, François Pirot

Supported by the National Research Fund Luxembourg (FNR)

Fonds National de la Recherche Luxembourg

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Packing and covering balls in graphs excluding a minor

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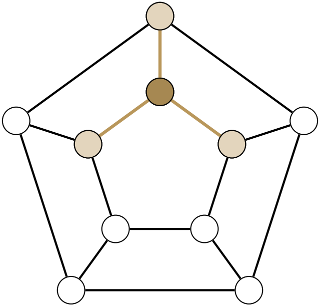
Balls in graphs and ball hypergraphs

Given graph G and $r > 0$

- *r-ball* : $B_r(v) = \{u \in V \mid \mathrm{dist}(u, v) \leqslant r\}$
- *ball hypergraph* : $\mathcal{H}(G)$ with $V(\mathcal{H}) = V(G)$ and $E(\mathcal{H}) \subseteq \{B_r(v) \mid v \in V(G), r > 0\}$



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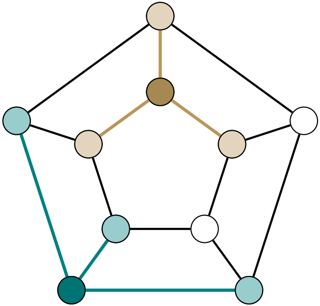

Packing

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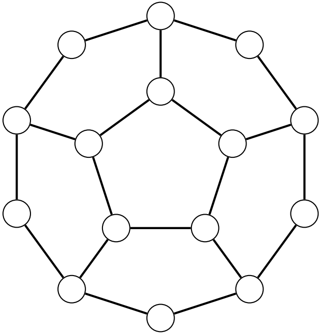
  Packing : set of mutually disjoint B_i
- $\nu(\mathcal{H})$ = maximum size of packing



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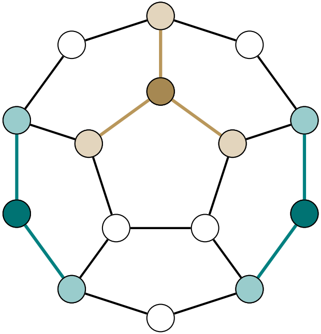

Covering

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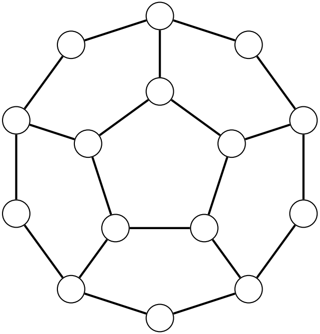

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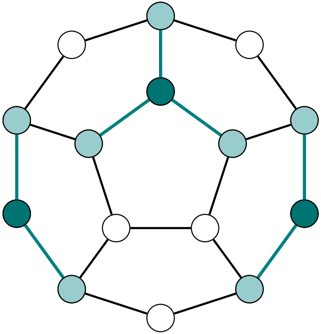

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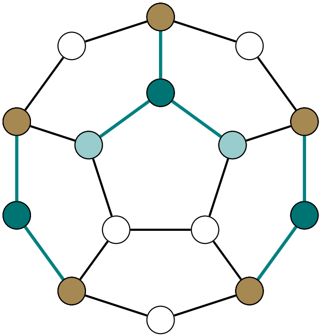

Packing-covering duality

Given $\mathcal{H}$ any hypergraph

$$\underset{\text{packing}}{\nu(\mathcal{H})} \leqslant \underset{\text{covering}}{\tau(\mathcal{H})}$$

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- *Erdős-Pósa property* : Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ s.t. $\tau(\mathcal{H}) \leqslant f(\nu(\mathcal{H}))$?
- *linear EP property* : Does there exist a constant c s.t. $\tau(\mathcal{H}) \leqslant c \cdot \nu(\mathcal{H})$?

Planar graphs

Conjecture (Gavoille, Peleg, Raspaud, Sopena, 2001)

There exists a constant ρ *such that for every planar graph* G *with* r*-ball packing number* 1*, the vertices of* G *can be covered with at most* ρ *balls of radius* r*.*

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Conjecture (Chepoi, Estellon, Vaxès, 2007)

There exists a constant ρ such that for every integer $r > 0$, every planar graph G, and every r-ball hypergraph $\mathcal{H}$ of G, we have $\tau(\mathcal{H}) \leqslant \rho \cdot \nu(\mathcal{H})$.

Polynomial bound

Theorem (Bousquet, Thomassé, 2015)

There exists a constant c such that for every integer $r \geqslant 0$, every planar graph G, and every r-ball hypergraph $\mathcal{H}$ of G, we have $\tau(\mathcal{H}) \leqslant c \cdot \nu(\mathcal{H})^{11}$.

Polynomial bound

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Theorem (Bousquet, Thomassé, 2015)

For every integer $t \geqslant 1$, there exists a constant c_t such that for every integer $r \geqslant 0$, every K_t-minor free graph G, and every r-ball hypergraph $\mathcal{H}$ of G, we have $\tau(\mathcal{H}) \leqslant c_t \cdot \nu(\mathcal{H})^{2t+1}$.

Linear bound

Theorem (Bousquet, Cames van Batenburg, Esperet, Joret, Lochet, M, Pirot, 2020)

For every integer $t \geqslant 1$, there is a constant c_t such that

$$\tau(\mathcal{H}) \leqslant c_t \cdot \nu(\mathcal{H})$$

for every K_t-minor free graph G and every ball hypergraph $\mathcal{H}$ of G.

Proof tools

- Bounding function for ball hypergraphs with *different radii*:

$$f_t : \mathbb{N} \to \mathbb{R} \text{ such that } \tau(\mathcal{H}) \leqslant f_t(\nu(\mathcal{H}))$$

- Bootstrapping:
  - If $\nu(\mathcal{H})$ small, use f_t
  - Otherwise, shrink $\nu(\mathcal{H})$ by constant factor without spending too much

Global bounding function

Show $\tau(\mathcal{H}) \leqslant f_t(\nu(\mathcal{H}))$ with $f_t(n) = O(n \log n))$

Ingredients :

- Linear programming relaxations :

$$\nu(\mathcal{H}) \leqslant \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leqslant \tau(\mathcal{H})$$

- Ding-Seymour-Winkler (1994) :
  $\tau(\mathcal{H}) = O(d \cdot \tau^*(\mathcal{H}) \log \tau^*(\mathcal{H}))$ if $\mathcal{H}$ has VC-dimension d
- Bousquet-Thomassé (2015) : *$\mathcal{H}$ has VC-dimension $\leqslant t - 1$*
- We show : $\nu^*(\mathcal{H}) = O(\nu(\mathcal{H}))$

Bootstrap

- Graphs with "small" packing number:
  $\tau(\mathcal{H}) = O(\nu(\mathcal{H}) \log \nu(\mathcal{H}))$ for each ball hypergraph with fixed radius

Bootstrap

- Graphs with "small" packing number: $\tau(\mathcal{H}) = O(\nu(\mathcal{H}) \log \nu(\mathcal{H}))$ for each ball hypergraph with fixed radius
- Graphs with "big" packing number:
  - Fix a maximum matching $\mathcal{B}$ in $\mathcal{H}$
  - $\mathcal{E}_1$ set of edges of $\mathcal{H}$ intersecting "few" balls of $\mathcal{B}$
  - $\mathcal{E}_2$ set of edges of $\mathcal{H}$ intersecting "many" balls of $\mathcal{B}$
  - For $\mathcal{E}_1$, $\mathcal{E}_2$ find a suitable auxilary hypergraph $\mathcal{H}_1$, $\mathcal{H}_2$ s.t. $\tau(\mathcal{H}) \leqslant \tau(\mathcal{H}_1) + \tau(\mathcal{H}_2)$
  - Bound $\tau(\mathcal{H}_1)$ and $\tau(\mathcal{H}_2)$ separately

Further research

- Can we improve the constant in our main Theorem, which is exponential in t?

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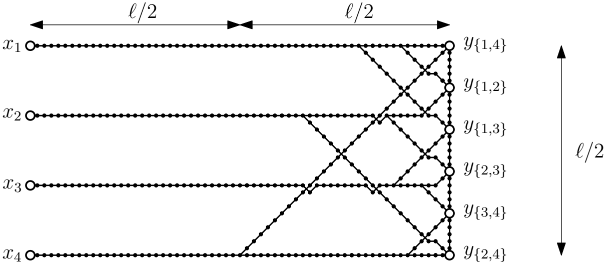

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Questions ?