New Algorithms for Mixed Dominating Set

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Summary

1. Mixed Dominating Set
2. Nice mds
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4. Construction
Mixed Dominating Set

1. Mixed Dominating Set
   - Definition
   - State of the Art
   - Our Results
   - Accepted Paper

2. Nice mds

3. Ideas of the Theorem

4. Construction
Mixed Dominating Set

- Graph $G = (V, E)$.

- A vertex $u \in V$ dominates itself, its incident edges and its neighbors.

- An edge $e \in E$ dominates itself, its two endpoints, and its adjacent edges.

- Mixed dominating set: Set of vertices $D \subseteq V$ and set of edges $M \subseteq E$ which dominates all vertices and edges of the graph $G$.

- Goal: A mixed dominating set (mds) of minimum size.
Mixed Dominating Set

Nice mds
Ideas of the Theorem
Construction

Definition
State of the Art
Our Results
Accepted Paper

Louis Dublois

Mixed Dominating Set
State of the Art

- NP-complete problem (Majumbar, 1992).

- 2-approximation in polynomial time (Hatami, 2010).

- $O^*(2^n)$ exact algorithm (and exponential space) (Madathil et al., 2019).

- FPT algorithm parameterized by the solution size in $O^*(4.172^k)$ (Xiao, Sheng, 2019).

- FPT algorithm parameterized by the treewidth in $O^*(6^{tw})$ and by the pathwidth in $O^*(5^{pw})$ (Jain et al., 2017).
Our Results

- $O^*(2^n)$ exact algorithm (and exponential space) (Madathil et al., 2019).
  - $O^*(1.912^n)$ and polynomial space.

- FPT algorithm parameterized by the solution size in $O^*(4.172^k)$ (Xiao, Sheng, 2019).
  - $O^*(3.510^k)$ and polynomial space.

- FPT algorithm parameterized by the treewidth in $O^*(6^{tw})$ and by the pathwidth in $O^*(5^{pw})$ (Jain et al., 2017).
  - $O^*(5^{tw})$ parameterized by the treewidth.
  - Under SETH, for any $\varepsilon > 0$, no algorithm in time $O^*((5 - \varepsilon)^{pw})$. 
Accepted Paper


Nice mds

1. Mixed Dominating Set

2. Nice mds
   - Definition of nice mds
   - Existence of nice mds
   - Implications

3. Ideas of the Theorem

4. Construction
Definition of nice mds

A *nice* mixed dominating set of a graph \( G = (V, E) \) is a mixed dominating set \( D \cup M \) which satisfies the following:

(i) \( D \cap V(M) = \emptyset \);

(ii) \( M \) is a matching;

(iii) For all \( u \in D \) there exists at least two private neighbors of \( u \), that is, two vertices \( v_1, v_2 \in V \setminus (D \cup V(M)) \) with \( N(v_1) \cap D = N(v_2) \cap D = \{u\} \).
Existence of nice mds

**Lemma**

For any graph $G = (V, E)$ without isolated vertices, $G$ has an mds $D \cup M$ of size at most $k$ if and only if $G$ admits a nice mds $D' \cup M'$ of size at most $k$. 
Existence of nice mds

- Let $G = (V, E)$ be a graph without isolated vertices, and $D \cup M$ an mds of $G$.

- By a result of (Madathil et al., 2019), we know:
  - If a graph has an mds of size $k$, then it also has an mds that satisfies the first two properties (i.e. (i) $D \cap V(M) = \emptyset$ and (ii) $M$ is a matching).

- From this mds $D \cup M$, we will edit it to obtain the third property.

- Let $I = V \setminus (D \cup V(M))$. 
Existence of nice mds

If there exists $u \in D$ with exactly one private neighbor $v$ : its other neighbors are dominated by $(D \cup M) \setminus \{u\}$. 
Existence of nice mds

- If there exists $u \in D$ with no private neighbor: its neighborhood is dominated by $(D \cup M) \setminus \{u\}$.
  - If there exists $v \in N(u) \cap I$.

\[
\begin{align*}
  u & \quad v \\
\end{align*}
\]

\[
\begin{align*}
  u & \quad v \\
\end{align*}
\]
Existence of nice mds

- If there exists \( u \in D \) with no private neighbor and \( N(u) \cap I = \emptyset : N(u) \subseteq D \cup V(M) \):
  - If there exists \( v \in N(u) \cap D \).
Existence of nice mds

- If there exists \( u \in D \) with no private neighbor and \( N(u) \cap (D \cup I) = \emptyset : N(u) \subseteq V(M) \).
- If there exists \( v \in N(u) \cap V(M) \) with \((v, w) \in M\) such that there exists \( z \in N(w) \cap I \).
If there exists \( u \in D \) with no private neighbor, \( N(u) \subseteq V(M) \), and there does not exist \( v \in N(u) \cap V(M) \) with \( (v, w) \in M \) such that there exists \( z \in N(w) \cap I \) : for all \( v \in N(u) \) with \( (v, w) \in M \), \( N(w) \subseteq D \cup V(M) \).
Implications

- Speed-up the branching rules on low-degree vertices (a vertex in $D$ must have two private neighbors).

- Faster FPT and exact branching algorithms.
Ideas of the Theorem

1. Mixed Dominating Set

2. Nice mds

3. Ideas of the Theorem
   - $O^*(5^{tw})$ Algorithm
   - Goal
   - Method
   - Why 5?

4. Construction
**$O^*(5^{tw})$ Algorithm**

- Incidence graph $G' = (V', E')$ of $G = (V, E)$:
  - $V' = V \cup E$
  - $E' = E \cup \{(u, e), (e, v) : e \in (u, v) \in E\}$

- **Mixed Dominating Set** on a graph $G$ is equivalent to **Distance-2-Dominating Set** on the incidence graph of $G$.

- The incidence graph of $G$ has the same treewidth as $G$.

- **Distance-2-Dominating Set** can be solved in time $O^*(5^{tw})$ (Borradaile, Le, 2016).

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**Theorem**

*There is an $O^*(5^{tw})$-time algorithm for Mixed Dominating Set in graphs of treewidth $tw$.*
Goal

Theorem

*Under SETH*, for all $\varepsilon > 0$, no algorithm solves *Mixed Dominating Set* in time $O^*((5 - \varepsilon)^{pw})$, where $pw$ is the input graph’s pathwidth.*
Method

**Definition**

A $q$-CSP-5 instance $\varphi$ is a Constraint Satisfaction Problem (CSP) instance with $n$ variables $x_1, \ldots, x_n$ taking values over the $\{0, 1, 2, 3, 4\}$, and $m$ constraints $c_0, \ldots, c_{m-1}$, each containing exactly $q$ variables and exactly $C = 5^q - 1$ possible assignments (given as a list) over the $q$ variables, for $j \in \{0, \ldots, m - 1\}$.

**Lemma (Theorem 2 from (Lampis, 2018))**

*For any $\varepsilon > 0$, under the SETH, there exists a $q$ such that $q$-CSP-5 with $n$ variables cannot be solved in time $O^*(5^{q - \varepsilon}n)$.***
Method

- Reduction from an instance $\varphi$ of $q$-CSP-5 to an instance $(G = (V, E), k)$ of Mixed Dominating Set such that $\varphi$ is satisfiable if and only if $G$ admits an mds of size at most $k$.

- The pathwidth $pw(G)$ of $G$ is upper-bounded by $n + O(1)$.

- If, for any $\varepsilon > 0$, Mixed Dominating Set can be solved in time $O^*((5 - \varepsilon)^{pw})$, then $q$-CSP-5 can be solved in time $O^*((5 - \varepsilon)^n)$, contradicting the Theorem of (Lampis, 2018) and the SETH.
Why 5?

- It corresponds to the base of our target lower bound.

- In our reduction, we will represent the 5 different values a variable can take with a path of 5 vertices in which there is exactly 5 different ways of selecting one vertex and one edge among these 5 vertices.
Construction

1. Mixed Dominating Set
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4. Construction
   - Main Part
   - Verification Gadget
   - Details
   - Theorem
Main Part

- The graph $G$ consists of a *main part* of $n$ paths $P_i$ ($1 \leq i \leq n$) of $5m$ vertices, each divided into $m$ sections:
  - Each path represents a variable.
  - Each section represents a constraint.

- An optimal solution in $G$ will verify, for each path, a specific pattern:
  - For 5 consecutive vertices, there are exactly 5 ways of taking one vertex and one edge to dominate the 5 vertices and the edges between.

- These 5 *configurations* for each path will represent all possible assignments for the variables.
Main Part

section $j$

0

1

2

3

4
Verification Gadget

- For each $j \in \{0, \ldots, m\}$, we add a verification gadget $H_j$:
  - Only connected to the main part to the 5 vertices of all variables $x_i$ appearing in the constraint $c_j$.

- An optimal solution in $G$ will verify a specific form in the gadget $H_j$:
  - The solution has this form in $H_j$ if and only if the constraint is satisfied.
Verification Gadget

\[ H_j \]

\[ Z_{C,j} \]

\[ Z_{1,j} \]

\[ Z_{2,j} \]
Verification Gadget

\[ s_2 \rightarrow s \rightarrow s_1 \]

\[ H_j \]

\[ Z_{C,j} \]

\[ Z_{1,j} \]

\[ Z_{2,j} \]

\[ x_1 \]

\[ x_2 \]
Verification Gadget

\[ H_j \]

\[ Z_{C,j} \]

\[ Z_{1,j} \]

\[ Z_{2,j} \]

\[ s_2 \rightarrow s \rightarrow s_1 \]
Verification Gadget

![Diagram of Verification Gadget]

The diagram illustrates the construction of a Verification Gadget, involving sets $H_j$, $W_j$, $Z_{C,j}$, $Z_{1,j}$, and $Z_{2,j}$, with nodes $s_2$, $s$, and $s_1$ connected accordingly.
Verification Gadget

\[ s \]

\[ s_2 \quad s_1 \]

\[ H_j \]

\[ W_j \]

\[ Z_{C,j} \]

\[ Z_{1,j} \]

\[ Z_{2,j} \]

\[ x_1 \]

\[ x_2 \]
Verification Gadget
Details

- Add *consistency* gadgets connected to each path and each section in order to force an optimal solution to follow one of the five configurations for each path.

- Make $F = (3n + 1)(2n + 1)$ copies of $G$ and glue them together one after the other.

- $k = 8AFmn + 2Fmn + 2Fmq(C - 1) + n + 1$.

- $pw(G) \leq n + O(q5^q)$. 
Theorem

Lemma

\( \varphi \) is satisfiable if and only if there exists an mds in \( G \) of size at most \( k \).

Lemma

The pathwidth of \( G \) is at most \( n + O(1) \).

Theorem

Under SETH, for all \( \varepsilon > 0 \), no algorithm solves Mixed Dominating Set in time \( O^*((5 - \varepsilon)^{pw}) \), where \( pw \) is the input graph’s pathwidth.
Merci