On variations of the 1-2-3 Conjecture

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COATI

Introduction

Proper k-labellings

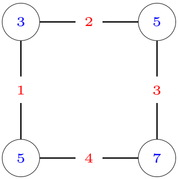
Given graph $G = (V, E)$, define:

- k-labelling of G: function $\ell : E \to \{1, \ldots, k\}$
- Colouring of G induced by ℓ: c_ℓ s.t. $\forall v \in V$, $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$

Proper labelling

The labelling ℓ is a **proper k-labelling** of G if c_ℓ is a proper vertex-colouring.

Example:



proper 4-labelling

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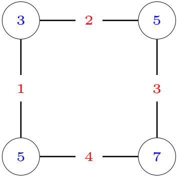

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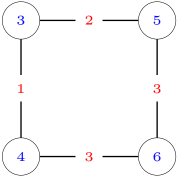

proper 3-labelling

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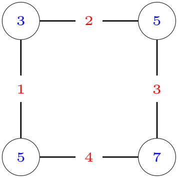

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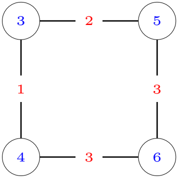

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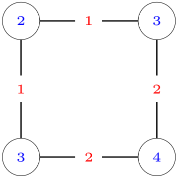


proper 2-labelling

Previous work

G nice (no component isomorphic to K_2) graph.
$\chi_\Sigma(G)$: minimum k s.t. G admits a proper k-labelling.

1-2-3 Conjecture: $\chi_\Sigma(G) \leq 3$ (2004, Karoński, Łuczak, Thomason)

- Verified for:
  - Complete graphs
  - Graphs with chromatic number ≤ 3
- In general: $\chi_\Sigma(G) \leq 5$ (2010, Kalkowski, Karoński, Pfender)
- G is regular $\Rightarrow \chi_\Sigma(G) \leq 4$ (2019, Przybyło)

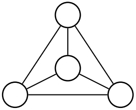
2-Labelling: decide if $\chi_\Sigma(G) = 2$

- $\mathcal{NP}$-Complete in general (2011, Dudek, Wajc)
- Easy for bipartite graphs (2016, Thomassen, Wu, Zhang)

Equivalent definition

Turning a graph into a **locally irregular** (adjacent vertices different degree) multigraph.

Inspired by (1988, Chartrand, Erdős, Oellermann)



K_4

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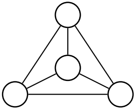

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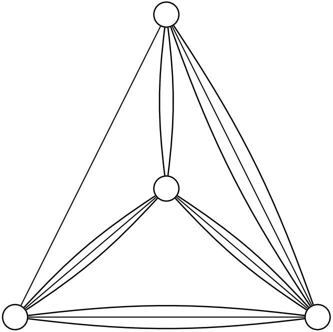

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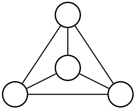

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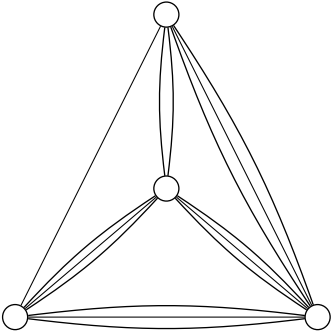

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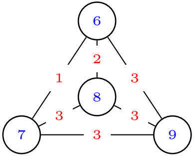

Proper labelling of K_4

1-2-3 Conjecture $\Rightarrow$ multiply each edge by at most 3.

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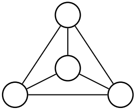

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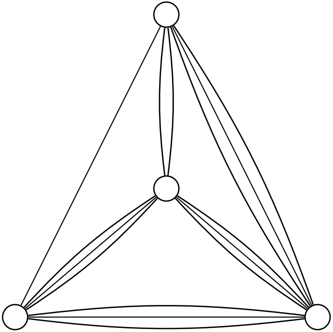

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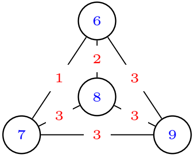

Proper labelling of K_4

1-2-3 Conjecture $\Rightarrow$ multiply each edge by at most 3.

Minimise the number of edges of the multigraph $\leftrightarrow$ the sum of the labels.

Other objectives

Minimise:

- number of distinct colours (2019, Baudon, Bensmail, Hocquard, Senhaji, Sopena)
- the maximum induced colour (2020, Bensmail, Li, Li, Nisse)
- sum of used labels (our work)

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Conjectures suspected to hold because:

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Conjectures suspected to hold because:

"proper 3-labellings do not require many edges labelled 3"

- minimise number of edges labelled 3 by a proper 3-labelling (our work)

Notation

Let G be a nice graph (no component isomorphic to K_2).

Minimise sum of labels

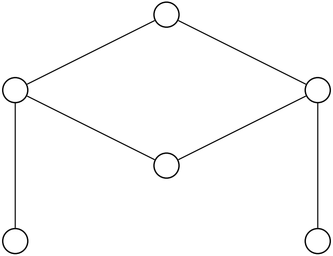
- $mE_k(G) =$ min sum of labels by any proper k-labelling of G
- $mE(G) =$ min sum of labels by any proper labelling of G

Minimise edges labelled 3

- $mT(G) =$ min number of edges labelled 3 by any proper 3-labelling of G
- $\mathcal{G}_p = \{G \mid mT(G) = p\}$

Warm-up

Example: locally irregular graphs



G

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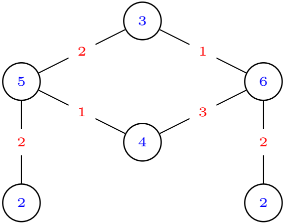

Sum of labels = 11,
number of 3's = 1

- $mE_k(G)$ = min sum of labels by any proper k-labelling of G
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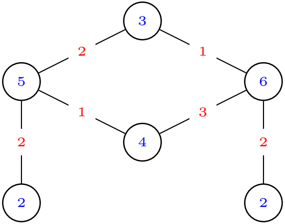

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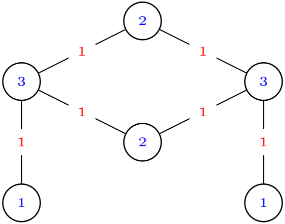

$mE(G) = 6, mT(G) = 0$
$G \in \mathcal{G}_0$

$mE(G) = |E| \Leftrightarrow G$ is locally irregular.

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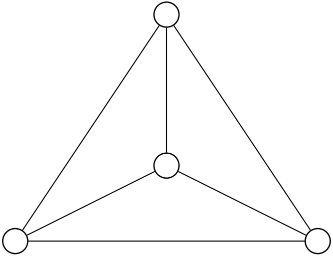
Example: a complete graph



K_4

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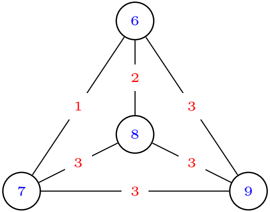

Sum of labels = 15,
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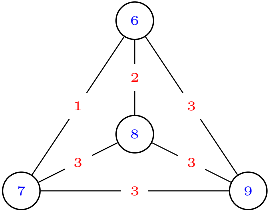

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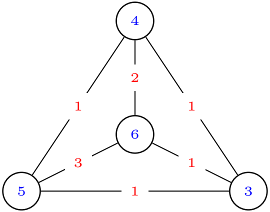

$mE(K_4) = 9$, $mT(K_4) = 1$
$K_4 \in \mathcal{G}_1$

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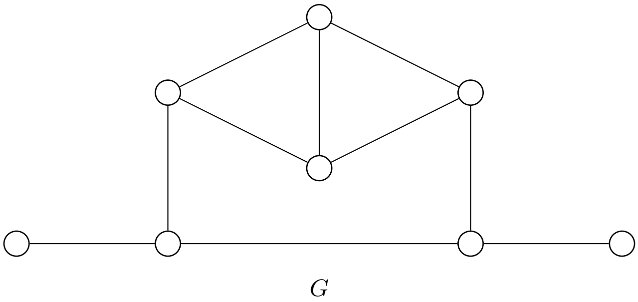
Example: "Maison" graph



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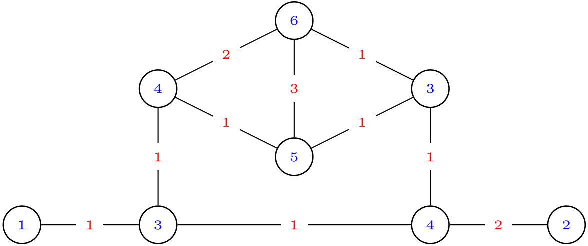

$mE(G) = 14$, $mT(G) = 1$
$G \in \mathcal{G}_1$

- $mE_k(G)$ = min sum of labels by any proper k-labelling of G
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Our results

Conjectures

Let $G = (V, E)$ be a nice graph.

Min sum labels

- Conjecture: $mE(G) \leq 2|E|$.
  Shown for:
  - Graphs with particular chromatic number
  - Bipartite graphs
- Conjecture: if G is bipartite, $mE(G) \leq \frac{3}{2}|E| + \mathcal{O}(1)$.
  Shown for:
  - Trees

Minimise 3's

Conjecture: $mT(G) \leq \frac{1}{3}|E|$.

There exist graphs that reach these values (e.g. $mE(C_3) = 2|E|, mT(C_6) = \frac{1}{3}|E|$).

Minimise the label sum

Tight results for:

- $mE(K_n) = \frac{3}{2}|E| + \mathcal{O}(1)$
- $mE(K_{n,n}) = |E| + \sqrt{|V|}$
- $mE(C_n) = \frac{3}{2}|E| + \mathcal{O}(1)$

Peculiar behaviour

Larger labels may help

Complexity

- For $k \in \mathbb{N}$, finding $mE_k(G)$ is $\mathcal{NP}$-hard, G: planar bipartite graph
- Polynomial algorithm, k fixed, G: bounded treewidth

Minimise edges labelled 3

- For each $p \geq 1$, $\mathcal{G}_p$ is well populated ($\mathcal{G}_p = \{G \mid mT(G) = p\}$)
- G is bipartite $\Rightarrow mT(G) \leq 2$

Lower/upper bounds on $mT(G)$

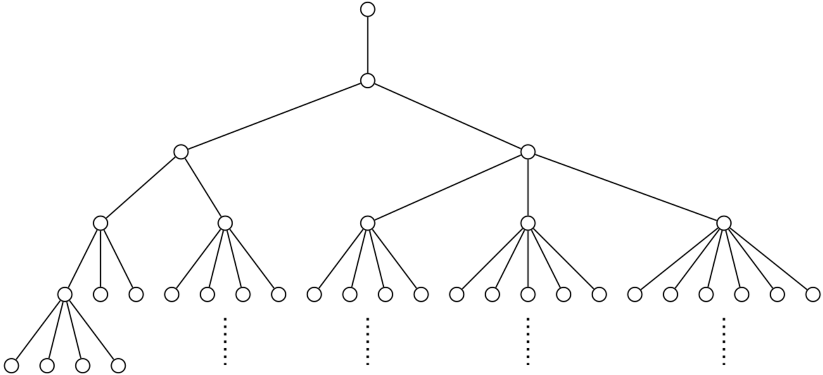
Family $\mathcal{F}$	$\exists$ arbitrarily large $G = (V, E) \in \mathcal{F}$: $mT(G) \geq$	$\forall G = (V, E) \in \mathcal{F}$: $mT(G) \leq$
$\mathcal{G}_p$	$\frac{1}{10}\lvert E\rvert$	?
$\chi(G) = 3$	$\frac{1}{10}\lvert E\rvert$	$\frac{\lvert V\rvert}{\lvert E\rvert}\lvert E\rvert$
Cubic other than K_4	$\frac{1}{10}\lvert E\rvert$	$\frac{1}{3}\lvert E\rvert$
Cactus	$\frac{1}{12}\lvert E\rvert$	$\frac{1}{3}\lvert E\rvert$
Planar	$\frac{1}{10}\lvert E\rvert$	?
Planar girth $g \geq 5k+1$, $k \geq 7$	$\frac{1}{g(g+1)}\lvert E\rvert$	$\frac{2}{k-1}\lvert E\rvert$
Outerplanar 1-connected	$\frac{1}{10}\lvert E\rvert$	?
Outerplanar 2-connected	?	$\frac{1}{3}\lvert E\rvert$
Halin graphs	?	$\frac{1}{3}\lvert E\rvert$

Min label sum using larger labels

Using larger labels to minimise sum of labels

Theorem

For any $k \geq 2$, there exists a tree T_k s.t. $mE_{k+1}(T_k) < mE_k(T_k)$.



T_2

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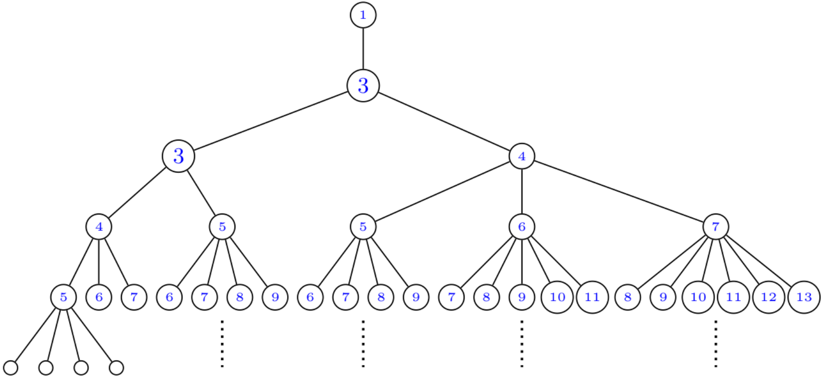
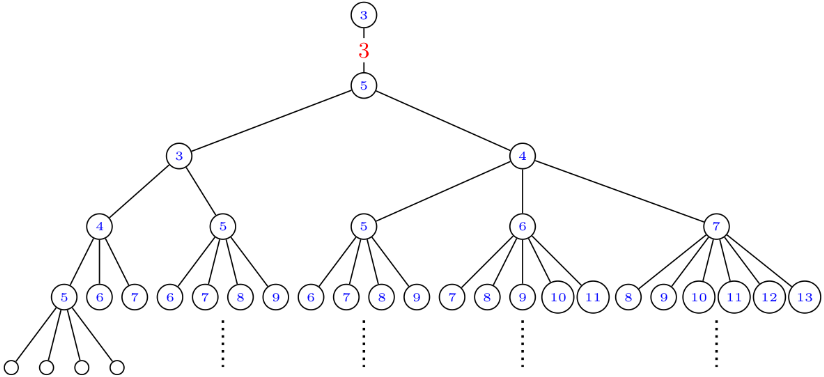


T_2, all labelled 1

Using larger labels to minimise sum of labels

T_2 s.t. $mE_3(T_2) < mE_2(T_2)$



$mE_3(T_2) = |E(T_2)| + 2 \Rightarrow$ better than any 2-labelling that uses label 2 more than two times.

Using larger labels to minimise sum of labels

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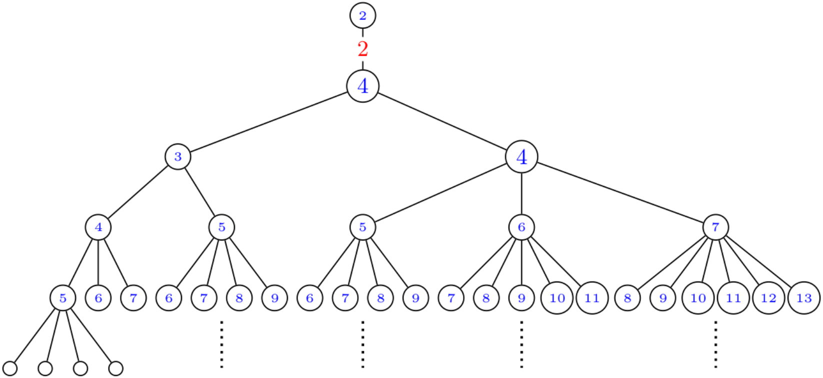

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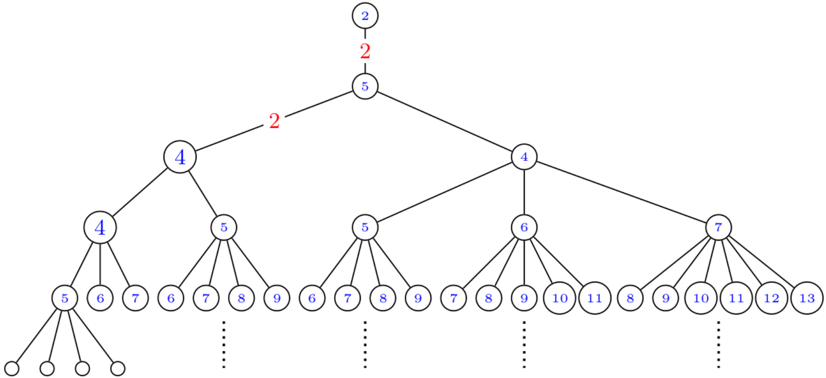

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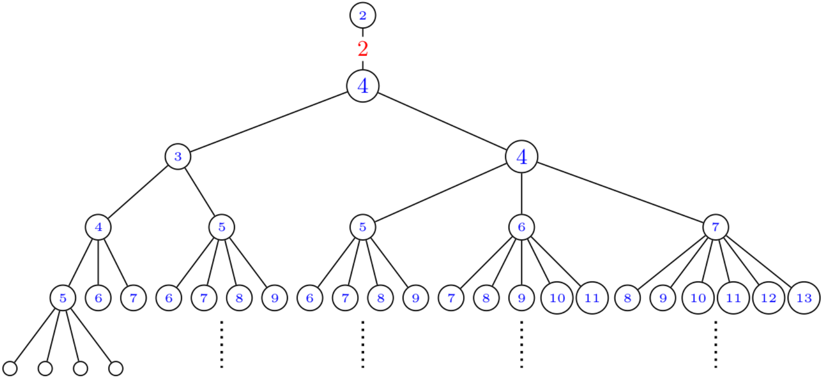

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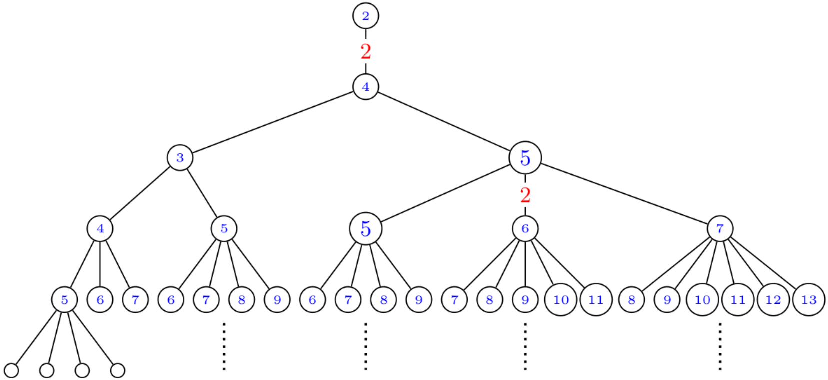

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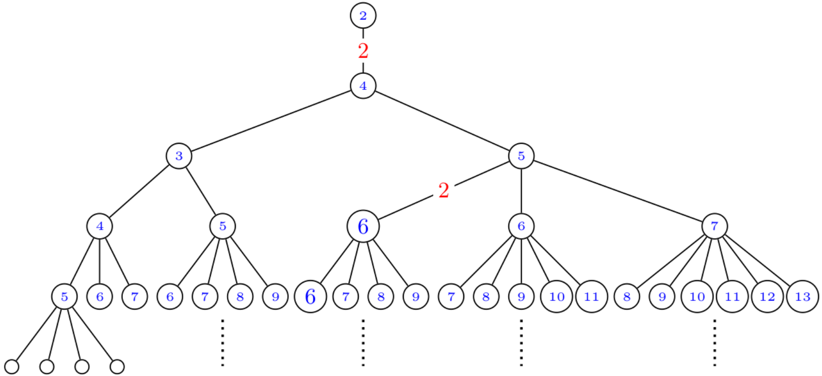

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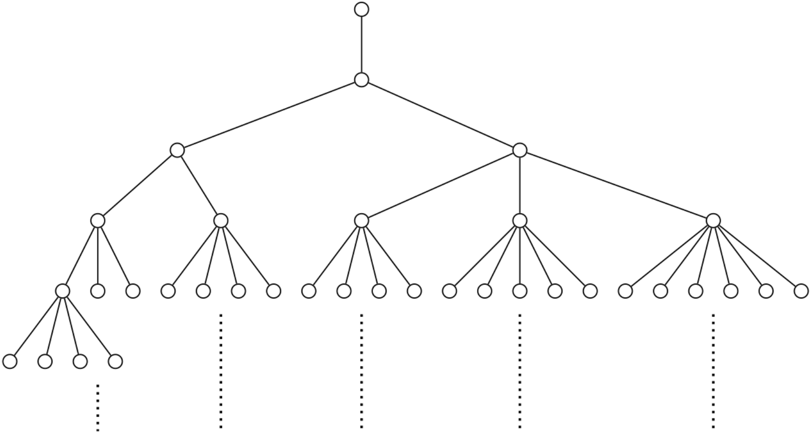
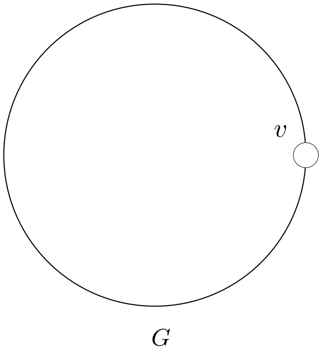
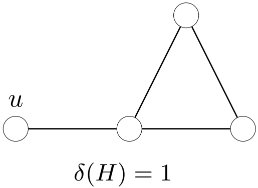


$k+1$ can be arbitrarily better than k

$\forall p, \exists G$ requiring p edges labelled 3

$\mathcal{G}_p$ well populated $(\mathcal{G}_p = \{G \mid mT(G) = p\})$





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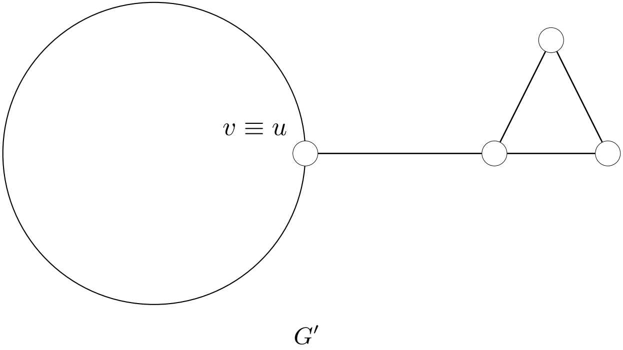

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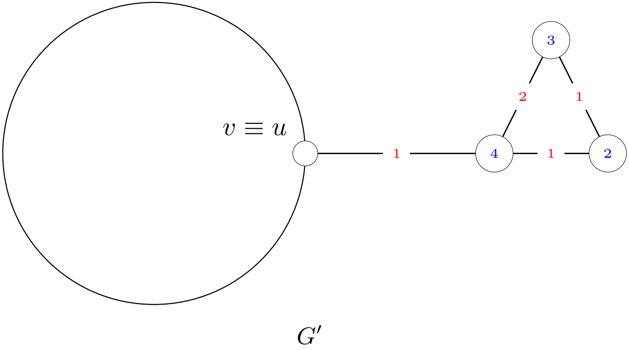

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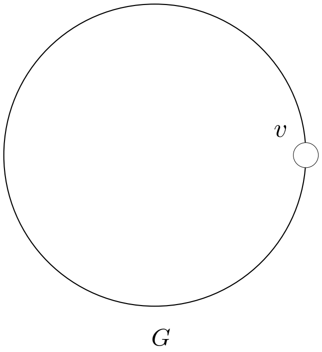

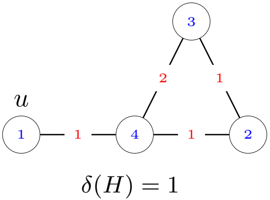

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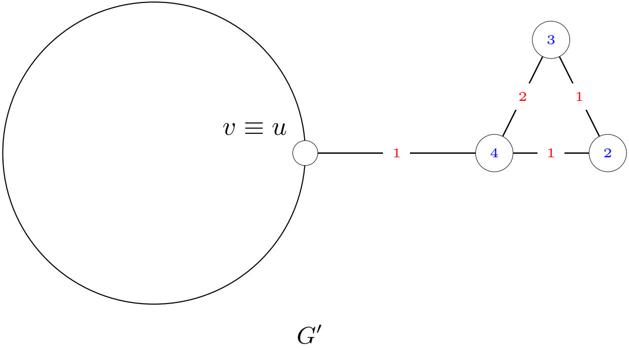

Lemma

ℓ proper 3-labelling of $G' \Rightarrow \ell|_H$ is proper 3-labelling of $H \Rightarrow mT(G') \geq mT(H)$.

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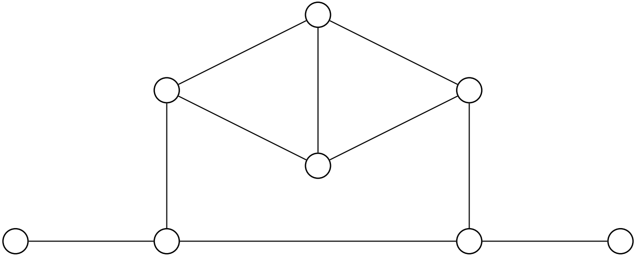
Do we have graphs with $\delta(G) = 1$ and $mT(G) \geq 1$?

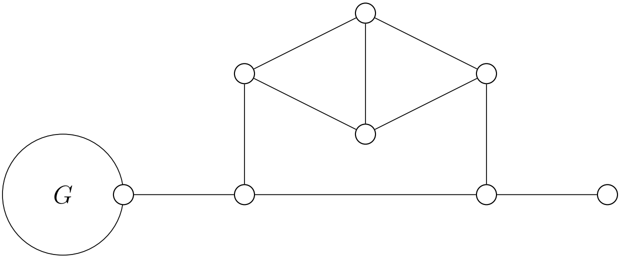
$\mathcal{G}_p$ well populated $(\mathcal{G}_p = \{G \mid mT(G) = p\})$

Do we have graphs with $\delta(G) = 1$ and $mT(G) \geq 1$? The "maison" graph!



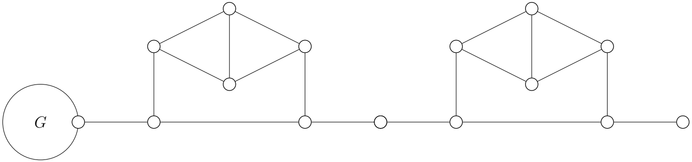
$mT(H) = 1$

$\mathcal{G}_p$ well populated $(\mathcal{G}_p = \{G \mid mT(G) = p\})$



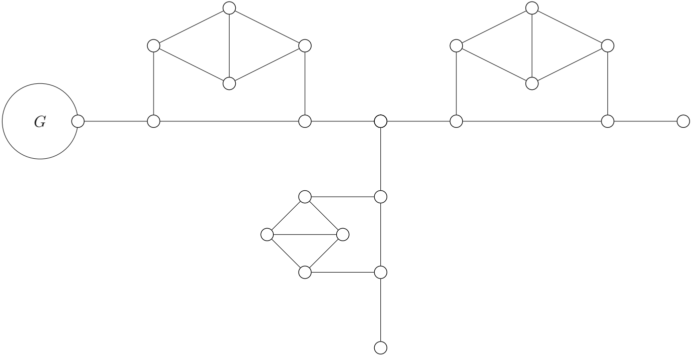
$$mT(G') \geq 1$$

$\mathcal{G}_p$ well populated ($\mathcal{G}_p = \{G \mid mT(G) = p\}$)



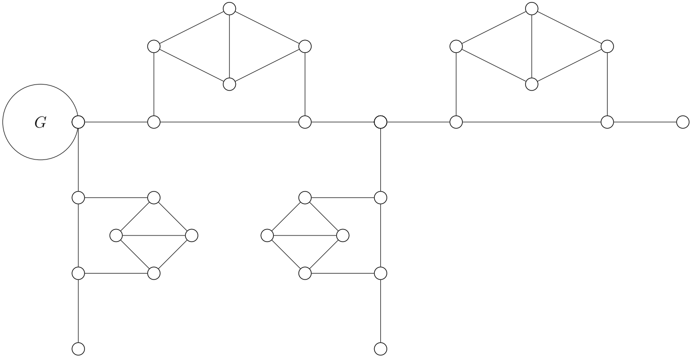
$$mT(G') \geq 2$$

$\mathcal{G}_p$ well populated ($\mathcal{G}_p = \{G \mid mT(G) = p\}$)



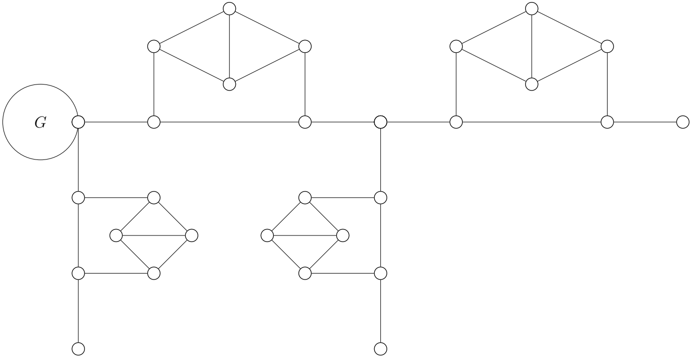
$$mT(G') \geq 3$$

$\mathcal{G}_p$ well populated ($\mathcal{G}_p = \{G \mid mT(G) = p\}$)



$$mT(G') \geq 4$$

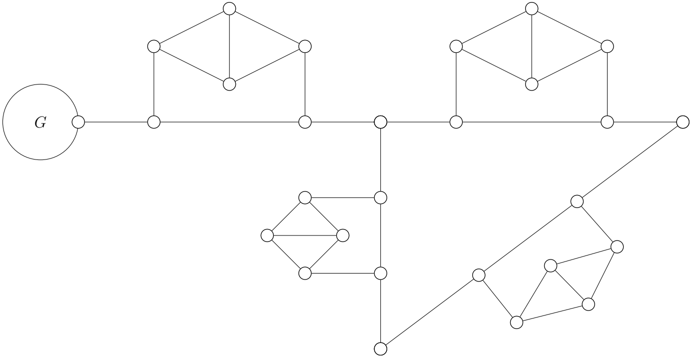
$\mathcal{G}_p$ well populated $(\mathcal{G}_p = \{G \mid mT(G) = p\})$



$$mT(G') \geq 4$$

For each p, there exists a graph $G' \in \mathcal{G}_p$.

$\mathcal{G}_p$ well populated ($\mathcal{G}_p = \{G \mid mT(G) = p\}$)



$mT(G') \geq 4$

For each p, there exists a graph $G' \in \mathcal{G}_p$.

Conclusion

Open questions

Min sum labels:

- For $k \in \mathbb{N}$, determining $mE_k(G)$ is $\mathcal{NP}$-Hard. But what about $mE(G)$? No general bound on k even for trees!
- Conjecture: $mE(G) \leq 2|E|$
- Conjecture: if G is bipartite, $mE(G) \leq \frac{3}{2}|E| + \mathcal{O}(1)$

Min number of 3's:

- Complete graphs:
  - $K_3, K_4, K_5 \in \mathcal{G}_1$
  - $K_6, K_7, K_8, K_9 \in \mathcal{G}_2$
  - $K_{10}, K_{11}, K_{12} \in \mathcal{G}_3$
  - $mT(K_n) \leq \frac{1}{4}|V|$
  - $mT(K_n) =?$
- Conjecture $mT(G) \leq \frac{1}{3}|E|$

Open questions

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  - $mT(K_n) =?$
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Merci