Factoring Systems of Linear Differential and Difference Equations with Finite-Dimensional Solution Spaces

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Problem statement

Finite-rank (linear functional) system:

- $S =$ a finite set of linear homogeneous differential and difference equations in one unknown

- $\mathrm{sol}(S)$ finite-dim over the constant field C; $\mathrm{rank}(S) = \dim_C \mathrm{sol}(S)$

Goals:

- factoring finite-rank systems

- decomposing finite-rank systems

- finding closed-form sols. : e.g. rational, hyperexponential sols.

Factoring finite-rank systems

Given a finite-rank system S, find finite-rank systems L (*factors*) such that

$$\mathrm{sol}(L) \subsetneq \mathrm{sol}(S).$$

Example 1:

$$S = \Big\{\partial_x^3 + \tfrac{x^3-3x}{x^2-1}\,\partial_x^2,\ \partial_y\partial_x + \tfrac{3yx^2}{x^2-1}\,\partial_x^2 + 3yx\,\partial_x + x\,\partial_y,$$

$$\partial_y^2 + \tfrac{9y^2x+3x+y}{x^2-1}\,\partial_x^2 + 3y\,\partial_y + (3+9y^2)\,\partial_x\Big\}.$$

$$\mathrm{sol}\left(\underbrace{\left\{\partial_x - \frac{2}{2x-3y^2},\ \partial_y + \frac{6y}{2x-3y^2}\right\}}_{L}\right) \subsetneq \mathrm{sol}(S)$$

Decomposing finite-rank systems

Given a finite-rank system S, find finite-rank systems $L_1, \ldots, L_r$, such that

$$\mathrm{sol}(S) = \mathrm{sol}(L_1) \oplus \cdots \oplus \mathrm{sol}(L_r).$$

Note: This is not always possible!

Example 2: S in Example 1

$$\mathrm{sol}(S) \;=\; \mathrm{sol}\left(\left\{\partial_x + x,\ \partial_y^2 + 3y\partial_y + y\right\}\right) \oplus \mathrm{sol}\left(\{\partial_x, \partial_y\}\right)$$

$$\oplus\ \mathrm{sol}\left(\left\{\partial_x - \frac{2}{2x-3y^2},\ \partial_y + \frac{6y}{2x-3y^2}\right\}\right)$$

A general sol is

$$c_1 \exp\left(-\frac{x^2}{2} - \frac{y}{3}\right) KummerU\left(\frac{1}{54}, \frac{1}{2}, -\frac{(9y-2)^2}{54}\right)$$

$$+\ c_2 \exp\left(-\frac{x^2}{2} - \frac{y}{3}\right) KummerM\left(\frac{1}{54}, \frac{1}{2}, \frac{(9y-2)^2}{54}\right)$$

$$+\ (2x - 3y^2)c_3 + c_4$$

Finding hyperexp. sols. of finite-rank systems

Let F be $\mathbb{C}(t_1, \ldots, t_m, k_1, \ldots, k_r)$ where t_i's continuous variables and k_j's discrete variables.

Let H be a function in $t_1, \ldots, t_m, k_1, \ldots, k_r$. Define

$$\partial_i(H) = \frac{\partial H}{\partial t_i}, \qquad E_j(H) = H(\cdots, k_j + 1, \cdots), \quad \text{for all } i, j.$$

H is *hyperexp.* if $\frac{\partial_i(H)}{H}, \frac{E_j(H)}{H} \in F$ for all i, j.

Example 3:

$$\left\{ \partial_x^2 - \frac{2((k-x)^2 + k)}{x(k-x)} \partial_x + \frac{((k-x)^3 - 3xk + 3k^2 + 2k)}{x^2(k-x)}, \right.$$
$$\left. E_k^2 - \frac{2x(x-k-1)}{x-k-2} E_k + \frac{x^2(x-k)}{x-k-2} \right\}$$

Hyperexp. sols. : $\frac{c_1 + c_2 k + c_3 x}{x-k} \exp(-x) x^{k+1}$.

Some previous work

	Factoring	Decomposing
LODE's	Beke (1894), Schwarz (1989) Bronstein (1994), van Hoeij (1997) Giesbrecht & Zhang (03), Cluzeau (03)	Singer (1996)
LOΔE's	Bronstein & Petkovšek (1996) van Hoeij (1999) (first order) Abramov & Paule & Petkovšek (1998)	Bomboy (2001)
LPDE's (finite-rank)	Li & Schwarz & Tsarev (2003)	(?)
LP$\{D+\Delta\}$E's (finite-rank)	Labahn & Li (2004) (first order factors)	(?)

Basic definitions

$F = C(x_1, \cdots, x_n)$ with char $C = 0$

$A = F[\partial_1, \ldots, \partial_n]$ with multiplicative rules:

$$\partial_i \partial_j = \partial_j \partial_i, \quad \partial_i a = a \partial_i + \frac{\partial a}{\partial x_i}, \quad \forall a \in F, \forall i, j.$$

is a *ring of differential operators*.

A is noncommutative.

Note: The ideas to be presented carry over to difference and differential-difference cases as well.

Connection between systems and (left) ideals

An ideal I of A is *finite-rank* if $\dim_F(A/I) < +\infty$.

Define $\mathrm{rank}(I) := \dim_F(A/I) = \dim_C(\mathrm{sol}(I))$

Finite-rank systems $S \Longleftrightarrow$ finite-rank ideals (S) of A.

Factors L of $S \Longleftrightarrow$ Finite-rank ideals (L) of A s. t. $(L) \supsetneq (S)$.

Connections between ideals and A-modules

(Left) ideal I of A $\Longleftrightarrow$ (left) A-module A/I (not a ring!)

Finite-rank ideal I $\Longleftrightarrow$ A-module A/I with $\dim_F(A/I) < +\infty$

Factor $J \supsetneq I$ $\Longleftrightarrow$ A-submodule J/I of A/I

Advantages of module language:

1. more concise: e.g. bases for vector spaces instead of bases for ideals

2. more general: e.g. applicable to systems in several unknowns

3. more powerful: e.g. convenient for using multi-linear algebra

∂-finite modules

Let $A = F[\partial_1, \ldots, \partial_n]$ be a ring of differential operators.

$(M, \partial_1, \ldots, \partial_n)$ is a *∂-finite module* if M is an A-module that is an F-vector space of finite dimension.

Factor finite-rank systems $\Longleftrightarrow$ Find submodules of M

Decompose finite-rank systems $\Longleftrightarrow$ $M = M_1 \oplus \cdots \oplus M_r$

Find hyperexp. sols. $\Longleftrightarrow$ Find 1-dim submodules of M^* (dual)

Idea of factorization: use exterior power

$(M, \partial_1, \ldots, \partial_n)$ $\partial_i(e_1, \ldots, e_s) = (e_1, \ldots, e_s)A_i$		$(\wedge^d M, \partial_1, \ldots, \partial_n)$ $\partial_i(f_1, \ldots, f_t) = (f_1, \ldots, f_t)B_i$
$N \subsetneq M$ $(\dim_F(N) = d)$		$\wedge^d N \subsetneq \wedge^d M$ $(\dim_F(\wedge^d N) = 1)$
$N := \{\sum_{i=1}^d a_i w_i \mid a_i \in F\}$	$\Longleftarrow$	$\wedge^d N = Fw$ with $w = w_1 \wedge \cdots \wedge w_d$ $\updownarrow$
		find hyperexp. sols. of integrable system associ. with $\wedge^d M$ w.r.t. $\{f_1, \ldots, f_t\}$: $S^{\wedge^d} = \{\partial_i Z = -B_i Z, \; i = 1, \ldots, n\}$

1-dim submodules of $\wedge^d M$

By the algorithm developed by Li, Schwarz and Tsarev (2003), get all hyperexp. sols. of $S^{\wedge^d}$ of form

$$e^{\int u} w \quad \text{where } u \in F \text{ and } w \in \wedge^d M.$$

Note: All 1-dim submodules of $\wedge^d M$ are in form $\{aw \mid a \in F\}$.

To determine if w is decomposable, consider a map

$$\varphi_w : M \longrightarrow \wedge^{d+1} M, \quad v \mapsto v \wedge w.$$

Thm. $\quad w = m_1 \wedge \cdots \wedge m_d \iff \dim_F(\ker(\varphi_w)) = d$

Note: When computing a basis of the kernel, we may encounter a nonlinear algebraic system in unspecified consts.

Example 4

Set $F = \mathbb{C}(x,y)$ and $A = F[\partial_x, \partial_y]$. Find all first order factors of

$$L = \Big\{ L_1 = \partial_x^2 + \tfrac{y^2}{4x}\partial_y + \tfrac{2-xy}{4x}\partial_x - \tfrac{y}{4x},\ L_2 = \partial_x\partial_y + \tfrac{y}{4}\partial_y - \tfrac{2+xy}{4y}\partial_x - \tfrac{1}{4},$$
$$L_3 = \partial_y^2 + \tfrac{x}{4}\partial_y + \tfrac{2x-x^2y}{4y^2}\partial_x - \tfrac{x}{4y} \Big\}.$$

Step 1 : From systems to modules

Let $M = A/(L)$ with a basis $\{1, \partial_x, \partial_y\}$. The actions of ∂_x, ∂_y on M:

$$\partial_x(1,\partial_x,\partial_y) = (1,\partial_x,\partial_y)A_x, \quad \partial_y(1,\partial_x,\partial_y) = (1,\partial_x,\partial_y)A_y$$

where

$$A_x = \begin{pmatrix} 0 & \frac{y}{4x} & \frac{1}{4} \\ 1 & \frac{xy-2}{4x} & \frac{2+xy}{4y} \\ 0 & -\frac{y^2}{4x} & -\frac{y}{4} \end{pmatrix} \qquad A_y = \begin{pmatrix} 0 & \frac{1}{4} & \frac{x}{4y} \\ 0 & \frac{2+xy}{4y} & \frac{x^2y-2x}{4y^2} \\ 1 & -\frac{y}{4} & -\frac{x}{4} \end{pmatrix}.$$

Find 2-dim submodules of M.

Step 2: From M to $\wedge^2 M$

The integrable system associated with $\wedge^2 M$ is

$$S^{\wedge^2} = \{\, \partial_x(Z) = A_{xx} Z, \quad \partial_y(Z) = A_{yy} Z \,\} \quad \text{where}$$

$$A_{xx} = \begin{pmatrix} \frac{2-xy}{4x} & -\frac{2+xy}{4y} & \frac{1}{4} \\ \frac{y^2}{4x} & \frac{y}{4} & -\frac{y}{4x} \\ 0 & -1 & \frac{1}{2x} \end{pmatrix} \qquad A_{yy} = \begin{pmatrix} -\frac{2+xy}{4y} & \frac{2x-x^2y}{4y^2} & \frac{x}{4y} \\ \frac{y}{4} & \frac{x}{4} & -\frac{1}{4} \\ 1 & 0 & -\frac{1}{2y} \end{pmatrix}$$

Step 3: Compute 1-dim submodules of $\wedge^2 M$

$$Z = w\, e^{\int \frac{1}{2x}\, dx - \frac{1}{2y}\, dy} \qquad \text{where} \quad w = (c_1,\, 0,\, c_1 y)^\tau$$

where c_1 is an unspecified constant. There is only one 1-dim submodule $\{a\,(1 \wedge \partial_x + y\partial_x \wedge \partial_y) \mid a \in F\}$.

Step 4: w decomposable ?

$$\varphi_w : M \longrightarrow \wedge^3 M, \quad v \mapsto v \wedge w$$

The kernel of φ_w is generated by

$$h_1 = -\frac{1}{y} + \partial_y, \quad h_2 = \partial_x,$$

So $N = \{a_1 h_1 + a_2 h_2 \mid a_1, a_2 \in F\}$ is a 2-dim submodule of M.

Step 5: From modules to ideals (systems)

$N = J/(L)$ for some $J \supsetneq (L)$. A Groebner basis of $J = [h_1, h_2, L_1, L_2, L_3]$:

$$\{-1 + y\partial_y,\ \partial_x\}$$

is a first order factor of L.

Example 5

L is same as in Example 4. Find second order factors of L.

Step 1 : From systems to modules

Let $M = A/(L)$. $\dim_F M = 3$. Find 1-dim submodules of M.

Step 2: From M to $\wedge^1 M = M$:

The integrable system associated with M w.r.t. $\{1, \partial_x, \partial_y\}$:

$$S = \{\, \partial_x(Z) = -A_x Z, \quad \partial_y(Z) = -A_y Z \,\}.$$

Step 3: Compute 1-dim submodules of M

Compute all hyperexp. sols. of S:

$$Z = w = (0, -c_1 \frac{x}{y}, \; c_1)^\tau$$

where c_1 is an unspecified constant. There is only one 1-dim submodule $\{a\,(-\frac{x}{y}\partial_x + \partial_y) \mid a \in F\}$.

Step 4: w decomposable ?

$$\varphi_w : M \longrightarrow \wedge^2 M, \quad v \mapsto v \wedge w$$

The kernel of φ_w is generated by

$$h = \partial_y - \frac{x}{y}\partial_x.$$

So, $\{a h \mid a \in F\}$ is a 1-dim submodule of M.

Step 5: From modules to ideals (systems)

$N = J/(L)$ for some $J \supsetneq (L)$. A Groebner basis of $J = [h, L_1, L_2, L_3]$:

$$\{-y\partial_y + x\partial_x, 4y\partial_y^2 + 2\partial_y - x\}$$

is a second order factor of L.

Future Study

	Factoring	Decomposing
LPDE's (finite-rank)	improve efficiency	eigenring
LP$\{D+\Delta\}$E's (finite-rank)	find any order factors	eigenring

Improve efficiency of computing hyperexp. sols. of integrable systems