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Symbolic Integration I

Transcendental Functions

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Foreword

This book brings together two streams of research in mathematics and computing that were begun in the nineteenth century and made possible through results brought to fruition in the twentieth century.

Methods for indefinite integration have been important ever since the invention of the calculus in the 1700s. In the 1800s Abel and Liouville began the earliest mathematical research on algorithmic methods on integration in finite terms leading to what might be considered today as an early mathematical vision of a complete algorithmic solution for integrating elementary functions. In an 1842 publication Lady Ada Augusta, Countess of Lovelace, describing the capabilities of Babbage's analytical engine put forth the vision that computational devices could do algebraic as well as numerical calculations when she said that "[Babbage's Analytical Engine] can arrange and combine its numerical quantities exactly as if they were *letters* or any other *general* symbols; and in fact it might bring out its results in algebraical *notation* were provisions made accordingly." Thus these two visions set the stage for a century and a half of research that partially culminates in this book.

Progress in the mathematical realm continued through out the nineteenth and twentieth centuries. The Russian mathematician Mordukhai-Boltovskoi wrote the first two books on this subject in 1910 and 1913[1].

With the invention of electronic computers in the late 1930s and early 1940s, a new impetus was given to both the mathematical and computational streams of work. In the meantime in the mathematical world important progress had been made on algebraic methods of research. Ritt began to apply the new algebraic techniques to the problem of integration in finite terms, an approach that has proven crucially important. In 1948 he published the results of his research in a little book, *Integration in Finite Terms*. The use of these algebraic ideas were brought to further fruition by Kolchin, Rosenlicht, and, particularly for problems of symbolic integration, by three of Rosenlicht's Ph.D. students — Risch, Singer, and Bronstein[2].

[1] *On the Integration in Finite Terms of Linear Differential Equations*. Warsaw, 1910 (in Russian) and *On the Integration of Transcendental Functions*. Warsaw, 1913 (in Russian).

[2] Let me hasten to add that there have been important contributions by many others and it is not my intention to give a complete history of the field in this short paragraph, but to indicate some of main streams of work that have led to the current book.

On the computational side, matters rested until 1953 when two early programs were written, one by Kahrimanian at Temple University and another by Nolan at Massachusetts Institute of Technology, to do analytic differentiation — the inverse of indefinite integration. There was active research in the late 1950s and early 1960s on list processing packages and languages that laid the implementation foundations for today's computer algebra systems. Slagle's 1961 thesis was an early effort to write a program, in LISP, to do symbolic integration. With the advent of general computer algebra systems, some kind of symbolic integration facility was implemented in most. These integration capabilities opened the eyes of many early users of symbolic mathematical computation to the amazing potential of this form of computation. But yet none of the systems had a complete implementation of the full algorithm that Risch had announced in barest outline in 1970. There were a number of reasons for this. First and foremost, no one had worked out the many aspects of the problem that Risch's announcement left incomplete.

Starting with his Ph.D. dissertation and continuing in a series of beautiful and important papers, Bronstein set out to fill in the missing components of Risch's 1970 announcement. Meanwhile working at the IBM T. J. Watson Research Center, he carried out an almost complete implementation of the integration algorithms for elementary functions. It is the most complete implementation of symbolic integration algorithms to date.

In this book, Bronstein brings these mathematical and computational streams of research together in a highly effective manner. He presents the algorithmic details in pseudo-code that is easy to implement in most of the general computer algebra systems. Indeed, my students and I have implemented and tested many of the algorithms in MAPLE and MACSYMA. Bronstein's style and appropriate level of detail makes this a straightforward task, and I expect this book to be the standard starting place for future implementers of symbolic integration algorithms. Along with the algorithms, he presents the mathematics necessary to show that the algorithms work correctly. This is a very interesting story in its own right and Bronstein tells it well. Nonetheless, for those primarily interested in the algorithms, much of the mathematics can be skipped at least in a first study. But the full beauty of the subject is to be most appreciated by studying both aspects.

The full treatment of the subject is a long one and it is not finished in this volume. The longer and more difficult part involving the integration of algebraic functions must await a second volume. This volume serves as a good foundation to the topic of symbolic integration and as a nice introduction to the literature for integration of algebraic functions and for other aspects such as integration involving non-elementary functions. Study, learn, implement, and enjoy!

B. F. Caviness

Preface

The integration problem, which is as old as calculus and differentiation, can be informally stated very concisely: given a formula for a function $f(x)$, determine whether there is a formula for a differentiable function $F(x)$ satisfying

$$\frac{dF}{dx} = f(x)$$

and compute such an $F(x)$, which is called an antiderivative of $f(x)$ and is denoted

$$F(x) = \int f(x)dx$$

if it exists. Yet, while symbolic differentiation is a rather simple mechanical process, suitable as an exercise in a first course in analysis or computer programming, the inverse problem has been challenging scientists since the time of Leibniz and Newton, and is still a challenge for mathematicians and computer scientists today. Despite the many great strides made since the 19^{th} century in showing that integration is in essence a mechanical process, although quite more complicated than differentiation, most calculus and analysis textbooks give students the impression that integration is at best a mixture of art and science, with flair in choosing the right change of variable or approach being an essential ingredient, as well as a comprehensive table of integrals.

The goal of this book is to show that computing symbolic antiderivatives is in fact an algorithmic process, and that the integration procedure for transcendental functions can be carried out by anyone with some familiarity with polynomial arithmetic. The integration procedure we describe is also capable of deciding when antiderivatives are not elementary, and proving it as a byproduct of its calculations. For example the following classical nonelementary integrals

$$\int e^{x^2} dx\ , \qquad \int \frac{dx}{\log(x)}\ , \qquad \int \frac{\sin(x)dx}{x}\ ,$$

can be proven nonelementary with minimal calculations.

The algorithmic approach, pioneered by Abel and Liouville in the past century, eventually succeeded in producing a mechanical procedure for deciding whether an elementary function has an elementary antiderivative, and for computing one if so. This procedure, which Risch described in a series of reports [58, 59, 60, 61], unfortunately not all of them published, forms the basis of most of the symbolic integration algorithms of the past 20 years, all of them loosely grouped under the appellation *Risch algorithm*. The procedure which we describe in this book also has its roots in the original Risch algorithm [60] and its improvements, our main sources besides Risch being [11, 12, 68, 74].

We have tried to keep the presentation as elementary as possible, with the minimal background for understanding the algorithm being an introductory course in algebra, where the topics rings and fields, polynomial greatest common divisors, irreducible polynomials and resultants are covered[3]. Some additional background in field theory, essentially algebraic and transcendental extensions, is occasionally used in the proofs associated with the algorithm. The reader willing to accept the algorithm without proof can skip those sections while learning the algorithm.

We have also generalized and extended the original Risch algorithm to a wider class of functions, thereby offering the following features, some of them new, to the reader already familiar with symbolic integration:

- The algorithms in this book use only rational operations, avoiding factorization of polynomials into irreducibles.
- Extensions by tangents and arc-tangents are treated directly, thereby real trigonometric functions are integrated without introducing complex exponentials and logarithms in the computations.
- Antiderivatives in elementary extensions can still be computed when arbitrary primitives are allowed in the integrand, *e.g.* $\mathrm{Erf}(x)$, rather than logarithms.
- Several subalgorithms are applicable to a large class of non-Liouvillian extensions, thereby allowing integrals to be computed for such functions.

The material in this book has been used in several courses for advanced undergraduates in mathematics or computer science at the Swiss Federal Institute of Technology in Zurich:

- In a one-semester course on symbolic integration, emphasizing the algorithmic and implementation aspects. This course covers Chap. 2 in depth, Chap. 3 and 4 superficially, then concentrates on Chap. 5, 6, 7 and 8.
- In the first part of a one-semester course on differential algebra. This course covers Chap. 3, 4 and 5 in depth, turning after Liouville's Theorem to other topics (e.g. differential Galois theory).
- In the last part of a one-semester introductory course in computer algebra, where some algorithms from Chap. 2 and 5 are presented, usually without proofs.

In all those courses, the material of Chap. 1 is covered as and when needed, depending on the background of the students. Chap. 9 contains complete

[3] Those topics are reviewed in Chap. 1.

proofs of several structure theorems and can be presented independently of the rest of this book.

By presenting the algorithm in pseudocode in various "algorithm boxes" throughout the text, we also hope to make this book useful for programmers implementing symbolic integrators: by following the pseudocode, they should be able to write an integrator without studying in detail the associated theory.

The reader will notice that several topics in symbolic integration are missing from this book, the main one being the integration of algebraic functions. Including algorithms for integrating algebraic and mixed algebraic-transcendental functions would however easily double the size of this book, as well as increase the mathematical prerequisites, since those algorithms require prior familiarity with algebraic curves and functions. We have thus decided to cover algebraic functions in a second volume, which will hopefully appear in the near future. In the meantime, this book is an adequate preparation to the extensive literature on the integration of algebraic functions [8, 9, 13, 14, 26, 58, 59, 61, 76]. Another related topic is integration in nonelementary terms, *i.e.* with new special functions allowed in the antiderivatives. Here also, the reader should have no difficulty moving on to the research literature [5, 6, 20, 21, 38, 39, 55, 79] after completing this book.

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M. Bronstein

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