Solutions of linear ordinary differential equations in terms of special functions

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ABSTRACT

We describe a new algorithm for computing special function solutions of the form $y(x) = m(x)F(\xi(x))$ of second order linear ordinary differential equations, where $m(x)$ is an arbitrary Liouvillian function, $\xi(x)$ is an arbitrary rational function, and F satisfies a given second order linear ordinary differential equation. Our algorithm, which is based on finding an appropriate point transformation between the equation defining F and the one to solve, is able to find all rational transformations for a large class of functions F, in particular (but not only) the ${}_0F_1$ and ${}_1F_1$ special functions of mathematical physics, such as Airy, Bessel, Kummer and Whittaker functions. It is also able to identify the values of the parameters entering those special functions, and can be generalized to equations of higher order.

1. INTRODUCTION

Algorithms and software for computing closed form solutions of linear ordinary differential equations have improved significantly in the past decade, but mostly in the direction of computing their Liouvillian solutions (see e.g. [1, 8, 9]). In particular, computing the Liouvillian solutions of second order linear ordinary differential equations has become a routine task in recent versions of several computer algebra systems. The situation is different with respect to solving such equations in terms of non–Liouvillian special functions. While it is possible to detect whether the solutions of an equation can be expressed in terms of the solutions of equations of second order [7], there is no complete algorithm for deciding whether such solutions can be expressed in terms of the solutions of *specific* equations, usually the ones defining known special functions. This is a restricted instance of the equivalence problem for second-order linear ODEs [3]: given a target equation $y'' = uy$ with $u \in C(x)$ and a known fundamental solution set $\{F_1, F_2\}$ (for example the Airy equation $y'' = xy$), and an arbitrary input equation $y'' = vy$ with $v \in C(x)$, to find functions $m(x)$ and $\xi(x)$ such that $\{m(x)F_1(\xi(x)), m(x)F_2(\xi(x))\}$ is a fundamental solution set of $y'' = vy$. This is the equivalent to looking for a point transformation of the form

$$x \rightarrow \xi(x) \quad y \rightarrow m(x)y \tag{1}$$

that transforms $y'' = uy$ into $y'' = vy$. It is classically known that all second-order linear ODEs are equivalent under the group of transformations of the form (1), hence that an appropriate transformation always exists [4]. However, the functions $\xi(x)$ and $m(x)$ are given implicitely by differential equations themselves, so this does not provide explicit solutions in terms of F. We are interested in this paper in determining whether an *explicit* transformation of the form (1) exists, with $\xi \in C(x)$ and m a Liouvillian function, and to compute it when it exists. Applying the transformation (1) to $y'' = uy$ and matching the coefficients of the resulting equation with $y'' = vy$ (or equivalently, substituting $y = m(x)F(\xi(x))$ in $y'' = vy$) one obtains the equations $m = \xi'^{-1/2}$ and

$$3\xi''^2 - 2\xi'\xi''' + 4u(\xi)\xi'^4 - 4v\xi'^2 = 0\,, \tag{2}$$

so the remaining problem is to solve the above equation explicitely. Methods using that approach have appeared, in particular [10], who proceeds heuristically by trying various candidates functions ξ with undetermined constants parameters in (2). Each attempt yields systems of algebraic equations for the undetermined constants (and parameters of the special functions), and those equations can then be solved by existing computer algebra systems.

Our main contribution in this paper is an algorithm for computing all the solutions $\xi \in C(x)$ of (2). Our algorithm is applicable whenever the target equation $y'' = uy$ has an irregular singularity at infinity, in addition to any number of affine singularities of arbitrary type. This allows our algorithm to handle the ${}_0F_1$ and ${}_1F_1$ special functions of mathematical physics (*e.g.* the Airy, Bessel, Kummer and Whittaker functions) as well as non-hypergeometric ones. We also show that if the input equation has no Liouvillian solution, then our algorithm decides whether there is any solution of the form $m(x)F(\xi(x))$ for F any solution of the target equation. Our algorithm has been implemented in the computer algebra system MAPLE and our implementation can be tried interactively on the web[1]. While the abilities

[1] `http://www.inria.fr/cafe/Manuel.Bronstein/cathode/kovacic_demo.html`

of the MAPLE 7 differential equations solver have also been improved regarding solutions in terms of special functions[2] our algorithm is able to solve a larger class of examples, *e.g.*

$$4(x-1)^8 \frac{d^2y}{dx^2} = (3-50x+61x^2-60x^3+45x^4-18x^5+3x^6)y(x)\,,$$

whose solutions can be expressed in terms of Airy functions with rational functions as arguments (see examples below).

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2. FORMAL CHANGE OF VARIABLE

The differential equations for ξ and m that result from having (1) map a given operator to another given one can always be obtained by substituting $y = m(x)F(\xi(x))$ in the corresponding differential equation, and this is a classic construction. We describe it in this section using differential polynomials and linear algebra, in a way that is easily performed in a computer algebra system for linear operators of arbitrary order.

Let $(k,')$ be a differential field, $k[D;']$ be the ring of differential operators with coefficients in k, and $L = D^n + \sum_{i=0}^{n-1} a_i D^i \in k[D;']$ be an operator of order $n > 0$. Let M, Z be differential indeterminates over k, $G_0, \dots, G_{n-1}$ be algebraic indeterminates over $k\langle M, Z\rangle$ and extend the derivation $'$ to $k\{M,Z\}[G_0,\dots,G_{n-1}]$ via $G_i' = Z'G_{i+1}$ for $0 \le i < n-1$ and $G_{n-1}' = -Z'\sum_{i=0}^{n-1} a_i G_i$. Let $y = MG_0$. Since y is a linear form in $G_0,\dots,G_{n-1}$ and $'$ preserves the total degree in $G_0,\dots,G_{n-1}$, the successive derivatives of y are all linear forms in $G_0,\dots,G_{n-1}$, so $y, y', \dots, y^{(n)}$ are linearly dependent over $k\langle M,Z\rangle$. Since for $i < n$, G_i appears with the nonzero coefficient MZ'^i in $y^{(i)}$ but does not appear in $y^{(i-1)}$, the elements $y, y', \dots, y^{(n-1)}$ must be linearly independent over $k\langle M,Z\rangle$, so there is a unique linear dependence of the form $y^{(n)} + \sum_{i=0}^{n-1} b_i y^{(i)} = 0$, which can be computed by linear algebra over $k\langle M,Z\rangle$. Define then

$$L_{M,Z} = D^n + \sum_{i=0}^{n-1} b_i D^i \quad \in k\langle M,Z\rangle[D;']$$

to be the *generic $M - Z$ associate of L*.

Given a differential extension K of k and any $m, \xi \in K$ such that $m\xi' \neq 0$, we can specialize $L_{M,Z}$ at $M = m$ and $Z = \xi$, and we denote the resulting operator $L_{m,\xi}$. If k contains an element x such that $x' = 1$ and if the elements of k can be viewed as functions[3] in x, then for any $f \in k$, we write $f(\xi)$ for the result of evaluating f at $x = \xi$. Replacing each a_i by $a_i(\xi)$ in $L_{m,\xi}$, we obtain a new operator, which we denote $L_{x\to\xi,y\to my}$. By construction, it has the following property: if $L(y) = 0$ for some y in a differential extension of k, then $L_{x\to\xi,y\to my}(my(\xi)) = 0$. So if $F_1,\dots,F_n$ is a fundamental solution set of L, then $mF_1(\xi),\dots,mF_n(\xi)$ are solutions of $L_{x\to\xi,y\to my}$. Since,

$$\mathrm{Wr}(mF_1(\xi),\dots,mF_n(\xi)) = m^n \xi'^N \mathrm{Wr}(F_1,\dots,F_n)(\xi)$$

for some integer $N > 0$, it follows that $mF_1(\xi),\dots,mF_n(\xi)$ is a fundamental solution set of $L_{x\to\xi,y\to my}$ (in other words, the transformation (1) sends L into $L_{x\to\xi,y\to my}$).

Let now $R = D^n + \sum_{i=0}^{n-1} c_i D^i \in k[D;']$ be another operator and suppose that there exist m, ξ in a differential extension of k such that $m\xi' \neq 0$ and $mF_1(\xi),\dots,mF_n(\xi)$ are solutions of R. Then, $mF_1(\xi),\dots,mF_n(\xi)$ is a fundamental solution set of both R and of $L_{x\to\xi,y\to my}$. Since they are both monic and of order n, we must have $R = L_{x\to\xi,y\to my}$. Equating the coefficients of the same powers of D in R and $L_{x\to\xi,y\to my}$ yield a system of n nonlinear ordinary differential equations that m and ξ must satisfy. Finding a fundamental solution set of the form $mF_1(\xi),\dots,mF_n(\xi)$ of R is thus reduced to solving those equations.

We can also ask a weaker question, namely does R admit some solution of the form $mF(\xi)$ where F is a nonzero solution of L and $m\xi' \neq 0$. In that case, we can only say that R and $L_{x\to\xi,y\to my}$ have a nontrivial right factor in $k\langle m,\xi\rangle[D;']$, so we cannot generate equations for m and ξ. However, if we request in addition that R be irreducible in $k\langle m,\xi\rangle[D;']$, then the existence of such a solution implies that $R = L_{x\to\xi,y\to my}$, hence that m and ξ satisfy the n equations generated. In particular, a second order equation with no Liouvillian solution over k must be irreducible over any Liouvillian extension of k, so if such an equation has a solution of the form $mF(\xi)$ with $m\xi' \neq 0$ and m and ξ Liouvillian over k, then $R = L_{x\to\xi,y\to my}$.

3. SECOND ORDER EQUATIONS

We carry out explicitly in this section the derivation of the above nonlinear differential equations in the case of second-order operators. Computing the generic $M - Z$ associate of $L = D^2 + a_1 D + a_0 \in k[D;']$, we get

$$y = MG_0, \qquad y' = M'G_0 + MG_0' = M'G_0 + MZ'G_1\,,$$

and

$$\begin{aligned} y'' &= M''G_0 + M'G_0' + M'Z'G_1 + MZ''G_1 + MZ'G_1' \\ &= M''G_0 + (2M'Z' + MZ'')G_1 - MZ'^2(a_0G_0 + a_1G_1) \\ &= (M'' - a_0MZ'^2)G_0 + (2M'Z' + MZ'' - a_1MZ'^2)G_1\,. \end{aligned}$$

A calculation of the linear dependence between y, y' and y'' shows that

$$\begin{aligned} L_{M,Z} &= D^2 - \left(2\frac{M'}{M} + \frac{Z''}{Z'} - a_1Z'\right)D \qquad (3) \\ &\quad -\left(\left(\frac{M'}{M}\right)' - \frac{M'^2}{M^2} - \frac{M'}{M}\frac{Z''}{Z'} + a_1Z'\frac{M'}{M} - a_0Z'^2\right). \end{aligned}$$

Let now $v \in k$ be given. As explained in the previous section, if there are m and ξ in a differential extension of k such that $m\xi' \neq 0$ and either

- $mF_1(\xi)$ and $mF_2(\xi)$ are solutions of $y'' = vy$, where and F_1, F_2 is a fundamental solution set of L, or
- $mF(\xi)$ is a solution of $y'' = vy$, where F is some solution of L, m and ξ are Liouvillian over k and $y'' = vy$ has no Liouvillian solution,

[2]See *e.g.* `http://lie.uwaterloo.ca/odetools/hyper3.htm` where the candidate $\xi = (ax^k + b)/(cx^k + d)$ is tried.

[3]This is obviously the case when $k = C(x)$ for some constant field C, and Seidenberg's Embedding Theorem [5, 6] implies that is is also the case when k is a finitely generated differential extension of $\mathbb{Q}(x)$.

then $D^2 - v = L_{x\to\xi, y\to my}$. Using (3) and equating the coefficients of D^1 and D^0 on both sides, we get

$$2\frac{m'}{m} + \frac{\xi''}{\xi'} - a_1(\xi)\xi' = 0 \tag{4}$$

and

$$\left(\frac{m'}{m}\right)' - \frac{m'^2}{m^2} - \frac{m'}{m}\frac{\xi''}{\xi'} + a_1(\xi)\xi'\frac{m'}{m} - a_0(\xi)\xi'^2 = v\,. \tag{5}$$

Equation (4) implies that

$$\frac{m'}{m} = \frac{1}{2}\left(a_1(\xi)\xi' - \frac{\xi''}{\xi'}\right) \tag{6}$$

and using that to eliminate m'/m from (5) we obtain

$$3\xi''^2 - 2\xi'\xi''' + \left(a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi)\right)\xi'^4 - 4v\xi'^2 = 0\,, \tag{7}$$

which is equation (2) when $a_1 = 0$ and $a_0 = -u$.

4. RATIONAL SOLUTIONS FOR ξ

We now proceed to show that for a large class of target operators L, there is an algorithm for computing all the rational solutions ξ of (7). Suppose from now on that our differential field k is a rational function field $k = C(x)$ where $x' = 1$ and $c' = 0$ for all $c \in C$. Recall that the *order at* ∞ is the function $\nu_\infty(q) = -\deg(q)$ for $q \in C[x] \setminus \{0\}$, and given an irreducible $p \in C[x]$, the *order at* p is the function

$$\nu_p(q) = \max\{n \in \mathbb{Z} \text{ such that } p^n | q\}$$

for $q \in C[x] \setminus \{0\}$. Both functions are extended to fractions via $\nu_\infty(a/b) = \nu_\infty(a) - \nu_\infty(b)$ and $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$. By convention, $\nu_\infty(0) = \nu_p(0) = +\infty$. Furthermore, for $a, b \in C(x)$, they satisfy the following properties (where ν stands for either ν_∞ or ν_p):

- $\nu(ab) = \nu(a) + \nu(b)$,
- $\nu(a+b) \ge \min(\nu(a), \nu(b))$
- $\nu(a) \ne \nu(b) \implies \nu(a+b) = \min(\nu(a), \nu(b))$,
- $\nu(a) < 0 \implies \nu(b(a)) = -\nu_\infty(b)\nu(a)$,
- $\nu_\infty(a) < 0 \implies \nu_\infty(a') = \nu_\infty(a) + 1$,
- $\nu_p(a) < 0 \implies \nu_p(a') = \nu_p(a) - 1$.

Given an hypothesis on the pair (a_0, a_1), the following gives an ansatz with a finite number of undetermined constants for the rational solutions of (7).

THEOREM 1. *Let $\prod_i Q_i^i$ be the squarefree decomposition of the denominator of $v \in C(x)$. If $\nu_\infty(a_1^2 + 2a_1' - 4a_0) < 2$, then any solution $\xi \in C(x)$ of (7) can be written as $\xi = P/Q$ where*

$$Q = \prod_i Q^i_{(2-\nu_\infty(a_1^2+2a_1'-4a_0))i+2} \quad \in C[x]\,, \tag{8}$$

and $P \in C[x]$ is such that either $\deg(P) \le \deg(Q) + 1$ *or*

$$\deg(P) = \deg(Q) + \frac{2 - \nu_\infty(v)}{2 - \nu_\infty(a_1^2 + 2a_1' - 4a_0)} \tag{9}$$

PROOF. Write

$$\Delta = a_1^2 + 2a_1' - 4a_0\,, \quad \delta = \nu_\infty(\Delta)$$

and suppose that $\delta < 2$. The solution $\xi = 0$ can certainly be written in the above form, so let $\xi \in C(x)^*$ be a nonzero solution of (7), and $p \in C[x]$ be an irreducible such that $\nu_p(\xi) < 0$. Then, $\nu_p(\xi''^2) = \nu_p(\xi'\xi''') = 2\nu_p(\xi) - 4$ and $\nu_p(\xi'^4) = 4\nu_p(\xi) - 4$. In addition,

$$\nu_p(a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi)) = \nu_p(\Delta(\xi)) = -\delta\nu_p(\xi)\,,$$

so

$$\nu_p\left((a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = (4-\delta)\nu_p(\xi) - 4\,.$$

Since $\delta < 2$, $(4-\delta)\nu_p(\xi) - 4 < 2\nu_p(\xi) - 4$, so

$$\nu_p\left(3\xi''^2 - 2\xi'\xi''' + (a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = (4-\delta)\nu_p(\xi) - 4\,.$$

Thus, we must have $\nu_p(4v\xi'^2) = (4-\delta)\nu_p(\xi) - 4$. Since $\nu_p(4v\xi'^2) = \nu_p(v) + 2\nu_p(\xi) - 2$, we get

$$\nu_p(v) = (2-\delta)\nu_p(\xi) - 2 \le -3\,.$$

This implies that the affine poles of ξ are among the poles of v of multiplicity 3 or more. Furthermore,

$$\nu_p(\xi) = \frac{\nu_p(v) + 2}{2 - \delta} \tag{10}$$

so ξ must be of the form $\xi = P/Q$ where $P \in C[x]$ and

$$Q = \prod_i Q^i_{(2-\delta)i+2}$$

Suppose now that $\deg(P) > \deg(Q) + 1$. Then, $\nu_\infty(\xi) < -1$, so $\nu_\infty(\xi'^4) = 4\nu_\infty(\xi) + 4$ and

$$\nu_\infty(a_1(\xi)^2 - 4a_0(\xi)) = \nu_\infty(\Delta(\xi)) = -\delta\nu_\infty(\xi)\,,$$

which implies that

$$\nu_\infty\left((a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = (4-\delta)\nu_\infty(\xi) + 4\,.$$

In addition, $\nu_\infty(\xi''^2) = 2\nu_\infty(\xi) + 4$ and either $\nu_\infty(\xi'\xi''') = 2\nu_\infty(\xi) + 4$ when $\nu_\infty(\xi) < -2$, or $\nu_\infty(\xi'\xi''') \ge -1$ when $\nu_\infty(\xi) = -2$. Since $\delta < 2$, $(4-\delta)\nu_\infty(\xi) + 4 < 2\nu_\infty(\xi) + 4$, and $(4-\delta)\nu_\infty(\xi) + 4 = 2\delta - 4 < -1$ when $\nu_\infty(\xi) = -2$, so

$$\nu_\infty\left(3\xi''^2 - 2\xi'\xi''' + (a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = (4-\delta)\nu_\infty(\xi) + 4$$

in any case. We must then have $\nu_\infty(4v\xi'^2) = (4-\delta)\nu_\infty(\xi) + 4$. Since $\nu_\infty(4v\xi'^2) = \nu_\infty(v) + 2\nu_\infty(\xi) + 2$, we get

$$\nu_\infty(v) = (2-\delta)\nu_\infty(\xi) + 2 \tag{11}$$

and the theorem follows. $\square$

We note that the upper bound $\deg(P) \le \deg(Q) + 1$ can be improved when $\delta < 0$. In that case, if $\deg(P) = \deg(Q) + 1$, then $\nu_\infty(\xi) = -1$, so an argument similar to the above shows that

$$\nu_\infty\left(3\xi''^2 - 2\xi'\xi''' + (a_1(\xi)^2 + 2a_1'(\xi) - 4a_0(\xi))\xi'^4\right) = \delta < 0\,.$$

We must then have $\nu_\infty(4v\xi'^2) = \delta$, so $\nu_\infty(v) = \delta$ and (9) holds. Therefore, when $\nu_\infty(a_1^2 + 2a_1' - 4a_0) < 0$, either $\deg(P) \le \deg(Q)$ or $\deg(P)$ is given by (9).

When it is applicable, Theorem 1 yields an immediate algorithm for computing all the solutions $\xi \in C(x)$ of (7) given $v \in C(x)$ as input: we substitute $\sum_{j=0}^{n} c_j x^j / Q$ for ξ in (7), where Q is given by (8), n is the upper bound on $\deg(P)$ given by Theorem 1 and the c_j are undetermined constants. This yields a nonlinear system Σ of algebraic equations for the c_j, whose solutions correspond to all the solutions $\xi \in C(x)$ of (7). Since any constant satisfies (7), Σ always has the line of solutions $(c_0, \dots, c_n) = \lambda(q_0, \dots, q_n)$ where $Q = q_0 + q_1 x + \cdots + q_n x^n$ (note that n is always at least $\deg(Q)$). Those solutions do not satisfy the condition $m\xi' \neq 0$, so we adjoin to Σ the additional equation

$$\sum_{j=0}^{n} (q_j c_N - q_N c_j) w_j = 1 \tag{12}$$

where $w_0, \dots, w_n$ are new indeterminates and N is chosen such that $q_N \neq 0$. Any solution of this augmented system must satisfy $q_j c_N \neq q_N c_j$ for some j, which implies that the corresponding $\xi \in C(x)$ is a nonconstant solution of (7). In addition, when a_0 and a_1 contain parameters (as in the case of families of special functions, *e.g.* Bessel functions), considering them as unknowns in Σ allows the values of those parameters to be found also (this is illustrated in the examples below). Essentially all the computation time of our algorithm is spent finding a solution of Σ, a problem whose complexity is exponential in $\deg(P)$.

Our approach can obviously be used to find all the rational solutions $\xi \in \overline{C}(x)$ of (7), it just means searching for solutions of Σ in $\overline{C}$ rather than C. Of more interest, it can also be used to find some algebraic function solutions of (7). Indeed, equations (10) and (11) provide the ramifications of ξ at the singularities of the equation and at infinity, so it is natural to look for solutions of the form

$$\xi = P\left(x^{1/(2-\nu)}\right) \prod_{i>2} Q_i^{(i-2)/(2-\nu)} \tag{13}$$

where $\nu = \nu_\infty(a_1^2 + 2a_1' - 4a_0)$ and $P \in C[x]$. To bound $\deg(P)$, we note that (11) is valid for $\nu_\infty(\xi) \leq -2$ only, so either

$$\deg(P) < (2 - \nu_\infty(a_1^2 + 2a_1' - 4a_0))(\deg(Q) + 2)$$

or

$$\deg(P) = (2 - \nu_\infty(a_1^2 + 2a_1' - 4a_0)) \deg(Q) + 2 - \nu_\infty(v)\,.$$

As for rational functions, substituting a candidate with undetermined constant coefficients for ξ yields a nonlinear algebraic system for those coefficients. This method does not yield all the algebraic functions solutions of (7) however.

Once a nonconstant solution ξ is found (rational or otherwise), the corresponding m is given by (6), which can be integrated yielding

$$m = \xi'^{-\frac{1}{2}} e^{\frac{1}{2}\int a_1(\xi)\xi'} \tag{14}$$

5. CLASSICAL SPECIAL FUNCTIONS

We now apply the algorithm of the previous section to classical classes of ${}_0F_1$ and ${}_1F_1$ special functions, all satisfying the hypothesis of Theorem 1. Although Kummer and Whittaker functions are rationally equivalent, we explicit the solving algorithm for both of them, allowing users to choose one over the other.

5.1 Airy functions

The operator defining the Airy functions is $L = D^2 - x$, so $a_1 = 0$, $a_0 = -x$ and equation (7) becomes

$$3\xi''^2 - 2\xi'\xi''' + 4\xi\xi'^4 - 4v\xi'^2 = 0\,. \tag{15}$$

Since $a_1^2 + 2a_1' - 4a_0 = 4x$, $\nu_\infty(a_1^2 + 2a_1' - 4a_0) = -1 < 0$, so by Theorem 1 and the remark following it, any solution of (15) must be of the form $\xi = P/Q$ where

$$Q = \prod_i Q_{3i+2}^i \quad \in C[x]\,,$$

and $P \in C[x]$ is such that either $\deg(P) \leq \deg(Q)$ or

$$\deg(P) = \deg(Q) + \frac{2 - \nu_\infty(v)}{3}\,.$$

Finally, since $a_1 = 0$, equation (14) becomes

$$m = \sqrt{\frac{1}{\xi'}} \tag{16}$$

5.2 Bessel functions

The operator defining the Bessel and modified Bessel functions is

$$L = D^2 + \frac{1}{x}D + \epsilon - \frac{\nu^2}{x^2}$$

where $\epsilon = 1$ for the Bessel functions and $\epsilon = -1$ for the modified Bessel functions. Therefore, $a_1 = 1/x$ and $a_0 = \epsilon - \nu^2/x^2$, so equation (7) becomes

$$3\xi''^2 - 2\xi'\xi''' + (4\nu^2 - 1)\frac{\xi'^4}{\xi^2} - 4\epsilon\xi'^4 - 4v\xi'^2 = 0\,. \tag{17}$$

Since

$$a_1^2 + 2a_1' - 4a_0 = \frac{4\nu^2 - 1}{x^2} - 4\epsilon\,,$$

$\nu_\infty(a_1^2 + 2a_1' - 4a_0) = 0 < 2$, so by Theorem 1, any solution of (17) must be of the form $\xi = P/Q$ where

$$Q = \prod_i Q_{2i+2}^i \quad \in C[x]\,,$$

and $P \in C[x]$ is such that either $\deg(P) \leq \deg(Q) + 1$ or

$$\deg(P) = \deg(Q) + 1 - \frac{\nu_\infty(v)}{2}\,.$$

Finally, since $a_1 = 1/x$, equation (14) becomes

$$m = \sqrt{\frac{\xi}{\xi'}} \tag{18}$$

5.3 Kummer functions

The operator defining the Kummer functions is

$$L = D^2 + \left(\frac{\nu}{x} - 1\right)D - \frac{\mu}{x}\,,$$

so $a_1 = \nu/x - 1$, $a_0 = -\mu/x$ and equation (7) becomes

$$3\xi''^2 - 2\xi'\xi''' + (\nu^2 - 2\nu)\frac{\xi'^4}{\xi^2} + (4\mu - 2\nu)\frac{\xi'^4}{\xi} + \xi'^4 - 4v\xi'^2 = 0\,. \tag{19}$$

Since

$$a_1^2 + 2a_1' - 4a_0 = 1 + \frac{4\mu - 2\nu}{x} + \frac{\nu^2 - 2\nu}{x^2}\,,$$

$\nu_\infty(a_1^2 + 2a_1' - 4a_0) = 0 < 2$, so by Theorem 1, any solution of (19) must be of the form $\xi = P/Q$ where

$$Q = \prod_i Q_{2i+2}^i \quad \in C[x]\,,$$

and $P \in C[x]$ is such that either $\deg(P) \le \deg(Q) + 1$ or

$$\deg(P) = \deg(Q) + 1 - \frac{\nu_\infty(v)}{2}\,.$$

Finally, since $a_1 = \nu/x - 1$, equation (14) becomes

$$m = e^{-\frac{1}{2}\int \xi} \sqrt{\frac{\xi\nu}{\xi'}}$$

5.4 Whittaker functions

The operator defining the Whittaker functions is

$$L = D^2 - \left(\frac{1}{4} - \frac{\mu}{x} - \frac{1/4 - \nu^2}{x^2}\right),$$

so $a_1 = 0$, $a_0 = -1/4 + \mu/x + (1/4 - \nu^2)/x^2$ and equation (7) becomes

$$3\xi''^2 - 2\xi'\xi''' + \left(1 - \frac{4\mu}{\xi} - \frac{1-4\nu^2}{\xi^2}\right)\xi'^4 - 4v\xi'^2 = 0\,. \quad (20)$$

Since

$$a_1^2 + 2a_1' - 4a_0 = 1 - \frac{4\mu}{x} - \frac{1-4\nu^2}{x^2}\,,$$

$\nu_\infty(a_1^2 + 2a_1' - 4a_0) = 0 < 2$, so by Theorem 1, any solution of (17) must be of the form $\xi = P/Q$ where

$$Q = \prod_i Q_{2i+2}^i \quad \in C[x]\,,$$

and $P \in C[x]$ is such that either $\deg(P) \le \deg(Q) + 1$ or

$$\deg(P) = \deg(Q) + 1 - \frac{\nu_\infty(v)}{2}\,.$$

Finally, since $a_1 = 0$, equation (14) becomes

$$m = \sqrt{\frac{1}{\xi'}} \quad (21)$$

as in the case of Airy functions.

6. EXAMPLES

6.1 Airy functions

We start by solving the equation given at the end of the introduction in terms of Airy functions. The equation is $y'' = vy$ with

$$v = \frac{3 - 50x + 61x^2 - 60x^3 + 45x^4 - 18x^5 + 3x^6}{4(x-1)^8}\,,$$

so $\nu_\infty(v) = 2$ and its denominator is $4(x-1)^8$. Therefore, any solution $\xi \in C(x)$ of (17) must be of the form

$$\xi = \frac{P}{(x-1)^3}$$

where $P \in C[x]$ is of degree $0, 1, 2$ or 3. Substituting $\xi = (c_0 + c_1 x + c_2 x^2 + c_3 x^3)/(x-1)^3$ in (15) yields a system of 14 algebraic equations. The nonconstant condition (12) becomes

$$(3c_0 + c_1)w_1 + (-3c_0 + c_2)w_2 + (c_0 + c_3)w_3 - 1 = 0$$

and solving the resulting system for $c_0, c_1, c_2, c_3, w_1, w_2$ and w_3 yields the 3 solutions

$$\xi = \frac{x(x-2)}{(x-1)^2} \quad \text{and} \quad \xi = -\left(1 \pm \sqrt{-3}\right)\frac{x(x-2)}{2(x-1)^2}\,.$$

Using (16) we compute

$$m = \sqrt{\frac{1}{\xi'}} = c(x-1)^{3/2}$$

for some constant c. Therefore, a basis of the solutions of $y'' = vy$ is given by

$$(x-1)^{3/2} Ai\left(\frac{x(x-2)}{(x-1)^2}\right) \text{ and } (x-1)^{3/2} Bi\left(\frac{x(x-2)}{(x-1)^2}\right)$$

where Ai and Bi are Airy functions.

6.2 Bessel functions

We now look for solutions in terms of modified Bessel functions of

$$y'' - (v_0 + v_1 x)^n y = 0 \quad \text{where } n > 0 \text{ and } v_1 \neq 0\,. \quad (22)$$

Letting $v = (v_0 + v_1 x)^n$, $\nu_\infty(v) = -n$ and its denominator is 1, so any solution $\xi \in C(x)$ of (17) must be a polynomial of degree $0, 1$, or $1 + n/2$. Substituting $\xi = c_0 + c_1 x$ in (17) yields

$$\frac{c_1^4(1 + 4\epsilon c_0^2 - 4\nu^2) + 8\epsilon c_0 c_1^5 x + 4\epsilon c_1^6 x^2}{(c_0 + c_1 x)^2} = -4c_1^2(v_0 + v_1 x)^n\,,$$

whose only solution for $n > 0$ and $v_1 \neq 0$ is $c_1 = 0$. Therefore, any nonconstant solution must be a polynomial of degree exactly $1 + n/2$, which implies that there can be such solutions only when n is even. We proceed with $n = 4$, which is the smallest even value for which MAPLE 7 is unable to solve the above equation. Substituting $\xi = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ in (17) yields a system of 15 algebraic equations. The nonconstant condition (12) becomes

$$c_1 w_1 + c_2 w_2 + c_3 w_3 + 1 = 0$$

and solving the resulting system for $c_0, c_1, c_2, c_3, w_1, w_2, w_3$ and ν with parameters v_0 and v_1 and $\epsilon = -1$ yields the 4 solutions

$$\nu = \pm\frac{1}{6}\,, \quad \xi = \pm\frac{1}{3}\frac{(v_0 + v_1 x)^3}{v_1}\,.$$

Using (18) we compute

$$m = \sqrt{\frac{\xi}{\xi'}} = c\sqrt{v_0 + v_1 x}$$

for some constant c. Therefore, a basis of the solutions of (22) for $n = 4$ is given by

$$\sqrt{v_0 + v_1 x}\; I_{1/6}\left(\frac{1}{3}\frac{(v_0 + v_1 x)^3}{v_1}\right)$$

and

$$\sqrt{v_0 + v_1 x}\; K_{1/6}\left(\frac{1}{3}\frac{(v_0 + v_1 x)^3}{v_1}\right)$$

where I_ν and K_ν are the modified Bessel functions of the first and second kinds.

6.3 Whittaker functions

For an example with two parameters to identify, we look for solutions in terms of Whittaker functions of

$$y'' + (ax^4 + bx)y = 0 \quad \text{where } a \neq 0\,, \tag{23}$$

which is Kamke's example 2.16 [2] with a specific integer choice for c. Letting $v = -(ax^4 + bx)$, $\nu_\infty(v) = -4$ and its denominator is 1, so any solution $\xi \in C(x)$ of (20) must be a polynomial of degree $0, 1$, or 3. Substituting $\xi = c_0 + c_1 x$ in (20) yields

$$4ac_1^4 x^6 + \text{ lower terms } = 0\,,$$

which implies $c_1 = 0$ whenever $a \neq 0$. Therefore, any nonconstant solution must be a polynomial of degree exactly 3. Substituting $\xi = c_0+c_1x+c_2x^2+c_3x^3$ in (20) yields a system of 15 algebraic equations. The nonconstant condition (12) becomes

$$c_1w_1 + c_2w_2 + c_3w_3 + 1 = 0$$

and solving the resulting system for $c_0, c_1, c_2, c_3, w_1, w_2, w_3, \nu$ and μ with parameters a and b yields the 2 solutions

$$\mu = \frac{1}{6}\frac{b}{\sqrt{-a}}\,, \quad \nu = \pm\frac{1}{6}\,, \quad \xi = \frac{2}{3}x^3\sqrt{-a}\,.$$

Using (21) we compute

$$m = \sqrt{\frac{1}{\xi'}} = \frac{c}{x}$$

for some constant c. Therefore, a basis of the solutions of (23) is given by

$$\frac{1}{x}M_{\frac{b}{6\sqrt{-a}},\frac{1}{6}}\left(\frac{2}{3}x^3\sqrt{-a}\right) \text{ and } \frac{1}{x}W_{\frac{b}{6\sqrt{-a}},\frac{1}{6}}\left(\frac{2}{3}x^3\sqrt{-a}\right)$$

where $M_{\mu,\nu}$ and $W_{\mu,\nu}$ are Whittaker functions.

6.4 An algebraic transformation ξ

We illustrate the use of the algebraic candidate (13) by solving the Airy equation $y'' = xy$ in terms of modified Bessel functions, thereby recovering classical expressions of Airy functions as Bessel functions. Letting $v = x$, $\nu_\infty(v) = -1$ and its denominator is 1, so any solution $\xi \in C(x)$ of (17) must be a polynomial of degree 0 or 1. Substituting $\xi = c_0 + c_1 x$ in (17) yields

$$-4c_1^4 x^3 + \text{ lower terms } = 0\,,$$

which implies $c_1 = 0$, hence that (17) has no nonconstant rational solution. However, formula (13) yields the algebraic candidate $\xi = P(\sqrt{x})$ where P is a polynomial of degree $0, 1, 2$ or 3. Substituting $\xi = c_0+c_1x^{1/2}+c_2x+c_3x^{3/2}$ in (17) yields a system of 17 algebraic equations. The nonconstant condition (12) becomes

$$c_1w_1 + c_2w_2 + +c_3w_3 + 1 = 0$$

and solving the resulting system for $c_0, c_1, c_2, c_3, w_1, w_2, w_3$ and ν with $\epsilon = -1$ yields the 4 solutions

$$\nu = \pm\frac{1}{3}\,, \quad \xi = \pm\frac{2}{3}x^{3/2}\,.$$

Using (18) we compute

$$m = \sqrt{\frac{\xi}{\xi'}} = c\sqrt{x}$$

for some constant c. Therefore, a basis of the solutions of the Airy equation $y'' = xy$ is given by

$$\sqrt{x}\ I_{1/3}\left(\frac{2}{3}x^{3/2}\right) \quad \text{and} \quad \sqrt{x}\ K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$$

where I_ν and K_ν are the modified Bessel functions of the first and second kinds. It follows that the Airy functions Ai and Bi can be expressed as linear combinations

$$\sqrt{x}\left(c_1 I_{1/3}\left(\frac{2}{3}x^{3/2}\right) + c_2 K_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right)$$

and the constants c_1 and c_2 can be found by looking at their values at two points.

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