A mixed finite element method for deformed cubic meshes
Inria Project Lab C2S@Exa, technical meeting at ANDRA, Châtenay-Malabry

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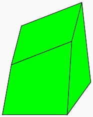
Goals

Numerical method

- Define a mixed finite element method for deformed cubes
- 1 pressure per cell
- 1 flux per face



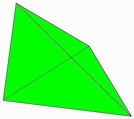
Deformed cube

High performance computing

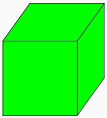
- Implementation in Traces (ANDRA)
- Parallelism and optimization

Mixed finite element methods

Works well with tetrahedral and cubic meshes (Raviart-Thomas, Nédélec)

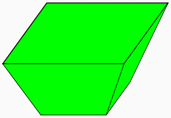


Tetrahedron

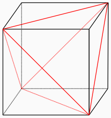


Cube

Composite element for hexahedron (Sboui-Jaffré-Roberts)



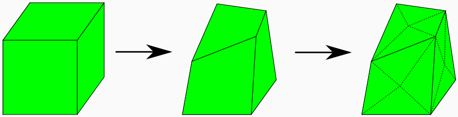
Hexahedron



Hexahedron split into 5 tetrahedrons

Approximation of a deformed cube

Approximation of a deformed cube $\begin{cases} \text{Face divided into 4 triangles} \\ \text{Cell divided into 24 tetrahedrons} \end{cases}$

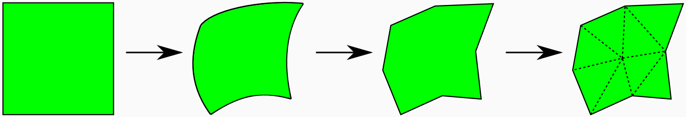


Deformed cube approximation

The alternatives :

1. Build a composite element
2. Use the static condensation

In two dimensions $\begin{cases} \text{Face divided into 2 segments} \\ \text{Cell divided into 8 triangles} \end{cases}$



Deformed square approximation

Mixed finite element formulation

Incompressible Darcy flow

$$
\begin{aligned}
\mathbf{u} &= -\mathbb{K}\,\nabla\, p && \text{in } \Omega \\
\nabla\cdot \mathbf{u} &= g && \text{in } \Omega \\
p &= f && \text{on } \partial\Omega
\end{aligned}
$$

Find $\mathbf{u} \in \mathbf{H}(\text{div};\Omega)$ and $p \in L^2(\Omega)$ such that

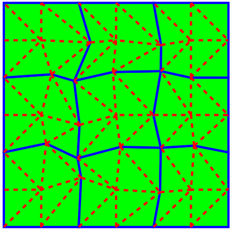
$$
\begin{aligned}
(\mathbb{K}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot \mathbf{v}) &= (f, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{H}(\text{div};\Omega) \\
(\nabla\cdot \mathbf{u}, \phi) &= (g, \phi) && \forall \phi \in L^2(\Omega)
\end{aligned}
$$

Find $\mathbf{u}_h \in \mathbf{RTN}_0(\Omega_h)$ and $p_h \in L^2(\Omega_h)$ such that

$$
\begin{aligned}
(\mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot \mathbf{v}_h) &= (f, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{RTN}_0\,(\Omega_h) \\
(\nabla\cdot \mathbf{u}_h, \phi_h) &= (g, \phi_h) && \forall \phi_h \in L^2(\Omega_h)
\end{aligned}
$$

Static condensation in two steps

Standard mixed finite element on $\mathcal{T}_h$ gives



The mesh $\mathcal{C}_h$ and the triangular sub-mesh $\mathcal{T}_h$

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}$$

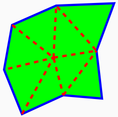
Reduction of the dimension of the system in two steps :

1. Eliminate internal d.o.f.s
   - 1 pressure per cell $C \in \mathcal{C}_h$
   - 2 fluxes per face
2. Reduction to 1 flux per face
   - 1 pressure per cell $C \in \mathcal{C}_h$
   - 1 flux per face

The global system in terms of local matrices

The linear system

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}$$



Cell C divided into 8 triangles T

γ's and θ's are local to global matrix extensions.

$$\mathbb{A} = \sum_{T\in\mathcal{T}_h} \gamma_T(\mathbb{A}_T) = \sum_{C\in\mathcal{C}_h} \gamma_C\left(\mathbb{A}_C\right) \qquad \mathbb{A}_C = \sum_{T\in\mathcal{T}_h} \gamma_{CT}\left(\mathbb{A}_T\right)$$

$$\mathbb{B} = \sum_{T\in\mathcal{T}_h} \gamma_T\left(\mathbb{B}_T\right) = \sum_{C\in\mathcal{C}_h} \gamma_C\left(\mathbb{B}_C\right) \qquad \mathbb{B}_C = \sum_{T\in\mathcal{T}_h} \gamma_{CT}\left(\mathbb{B}_T\right)$$

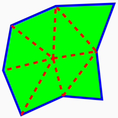
$$\mathbf{F} = \sum_{T\in\mathcal{T}_h} \theta_T\left(\mathbf{F}_T\right) = \sum_{C\in\mathcal{C}_h} \theta_C\left(\mathbf{F}_C\right) \qquad \mathbf{F}_C = \sum_{T\in\mathcal{T}_h} \theta_{CT}\left(\mathbf{F}_T\right)$$

$$\mathbf{G} = \sum_{T\in\mathcal{T}_h} \theta_T\left(\mathbf{G}_T\right) = \sum_{C\in\mathcal{C}_h} \theta_C\left(\mathbf{G}_C\right) \qquad \mathbf{G}_C = \sum_{T\in\mathcal{T}_h} \theta_{CT}\left(\mathbf{G}_T\right)$$

Eliminate internal d.o.f.s for a cell $C \in \mathcal{C}_h$

Internal flux $\mathbf{U}_C^{int}$ External flux $\mathbf{U}_C^{ext}$

Pressure variation $\mathbf{P}_C^0$ Pressure mean $\mathbf{P}_C^1$



The cell C

Local equations on the cell C

$$
\begin{aligned}
\mathbb{A}_C^{int,int}\mathbf{U}_C^{int} + \mathbb{A}_C^{int,ext}\mathbf{U}_C^{ext} + \left(\mathbb{B}_C^{0,int}\right)^t \mathbf{P}_C^0 &= \mathbf{F}^{int} \quad (1)\\
\mathbb{B}_C^{0,int}\mathbf{U}_C^{int} + \mathbb{B}_C^{0,ext}\mathbf{U}_C^{ext} &= \mathbf{G}^0 \quad (2)\\
\mathbb{A}_C^{ext,int}\mathbf{U}_C^{int} + \mathbb{A}_C^{ext,ext}\mathbf{U}_C^{ext} + \left(\mathbb{B}_C^{0,ext}\right)^t \mathbf{P}_C^0 + \left(\mathbb{B}_C^{1,ext}\right)^t \mathbf{P}_C^1 &= \mathbf{F}^{ext} \quad (3)\\
\mathbb{B}_C^{1,ext}\mathbf{U}_C^{ext} &= \mathbf{G}^1 \quad (4)
\end{aligned}
$$

Eliminate $\mathbf{U}_C^{int}$, $\mathbf{P}_C^0$ from eqs (1),(2) to obtain local eqs for $\mathbf{U}_C^{ext}$, $\mathbf{P}_C^1$.

Thus one obtains the global intermediary system

$$
\begin{pmatrix} \bar{\mathbb{A}} & \bar{\mathbb{B}}^t \\ \bar{\mathbb{B}} & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{P}} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{F}} \\ \bar{\mathbf{G}} \end{pmatrix}
$$

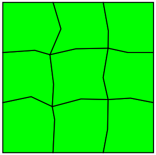
$\bar{\mathbf{U}}$: 2 fluxes per face of C
$\bar{\mathbf{P}}$: 1 pressure per cell C

Static condensation : second step

We start from the intermediary system

$$\begin{pmatrix} \bar{\mathbb{A}} & \bar{\mathbb{B}}^t \\ \bar{\mathbb{B}} & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{P}} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{F}} \\ \bar{\mathbf{G}} \end{pmatrix}$$

$\bar{\mathbf{U}}$: 2 fluxes per face of $C \in \mathcal{C}_h$
$\bar{\mathbf{P}}$: 1 pressure per cell $C \in \mathcal{C}_h$



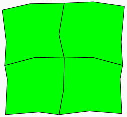
The mesh $\mathcal{C}_h$

How to reduce the system to only

- 1 pressure per cell $C \in \mathcal{C}_h$ and
- 1 flux per face ?

It is not possible when considering a single cell.

We need to consider patches P of 4 cells :



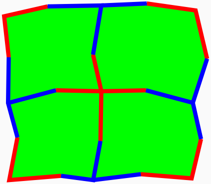
A patch P

Reduction to 1 flux per face

On the patch P define

red subface's flux $\bar{\mathbf{U}}_P^r$

blue subface's flux $\bar{\mathbf{U}}_P^b$



A patch P

For each face e with subfaces define

$$\mathbf{U}_e^+ = n_{e,b}\bar{\mathbf{U}}_e^b + n_{e,r}\bar{\mathbf{U}}_e^r$$

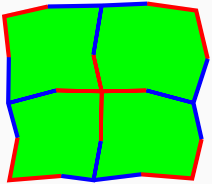
In matrix form $\mathbf{U}^+ = \mathbb{N}\bar{\mathbf{U}}$
Under some constraint on $\mathbb{N}$, we will construct a linear system with unknown $\mathbf{U}^+$ and $\bar{\mathbf{P}}$

Reduction to 1 flux per face

On the patch P :

Rectangular system (12×24)

$$\begin{pmatrix} \bar{\mathbb{A}}_P^r & \bar{\mathbb{A}}_P^b \\ \bar{\mathbb{B}}_P^r & \bar{\mathbb{B}}_P^b \end{pmatrix} \begin{pmatrix} \bar{\mathbf{U}}_P^r \\ \bar{\mathbf{U}}_P^b \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{F}}_P^{int} - \left(\bar{\mathbb{B}}_P\right)^t \bar{\mathbf{P}}_P \\ \bar{\mathbf{G}}_P \end{pmatrix}$$



A patch P

Add equations from the definition of $\mathbf{U}^+$

Square system (24×24), nonsingular when $n_{e,r} \neq n_{e,b}$ for all faces e and $(n_{e,r}, n_{e,b}) \neq (n_{e',r}, n_{e',b})$ for at least two faces e, e' in the patch :

$$\begin{pmatrix} \bar{\mathbb{A}}_P^r & \bar{\mathbb{A}}_P^b \\ \bar{\mathbb{B}}_P^r & \bar{\mathbb{B}}_P^b \\ \mathbb{N}_P^r & \mathbb{N}_P^b \end{pmatrix} \begin{pmatrix} \bar{\mathbf{U}}_P^r \\ \bar{\mathbf{U}}_P^b \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{F}}_P - \left(\bar{\mathbb{B}}_P\right)^t \bar{\mathbf{P}}_P \\ \bar{\mathbf{G}}_P \\ \bar{\mathbf{U}}_P^+ \end{pmatrix}$$

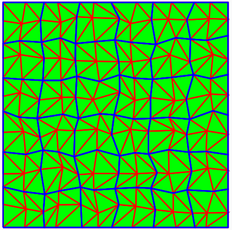
One can eliminate $\bar{\mathbf{U}}_P^r$ and $\bar{\mathbf{U}}_P^b$ in terms of $\bar{\mathbf{P}}_P$ and $\bar{\mathbf{U}}_P^+$ and proceed as in the first step.

Numerical experiment

Test case : 36 cells (288 triangles)

Initial values

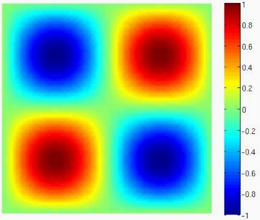
$$\mathbb{K} = \mathbb{I}$$
$$f = 0$$
$$g = -2(2\pi)^2 \sin(2\pi x)\sin(2\pi y)$$



The mesh $\mathcal{C}_h$ and the triangular sub-mesh $\mathcal{T}_h$

Exact solution

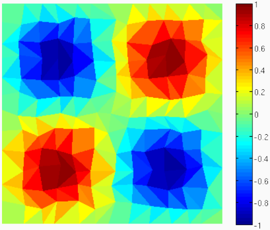
$$p = \sin(2\pi x)\sin(2\pi y)$$
$$\mathbf{u} = -2\pi \begin{pmatrix} \cos(2\pi x)\sin(2\pi y) \\ \sin(2\pi x)\cos(2\pi y) \end{pmatrix}$$



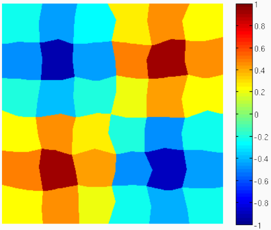
Exact pressure

Numerical experiment

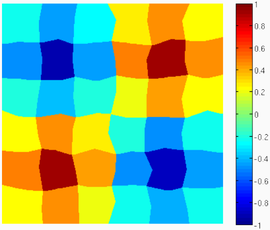
Test case : 36 cells (288 triangles)
Approximation of the pressure



MFE method on $\mathcal{T}_h$
$\|p - \mathbf{P}\|_{L^2} = 6.10^{-16}$



Intermediary method
$\left\|p - \bar{\mathbf{P}}\right\|_{L^2} = 7.10^{-16}$



Final method
$\left\|p - \bar{\mathbf{P}}\right\|_{L^2} = 7.10^{-14}$

Conditioning of the matrices

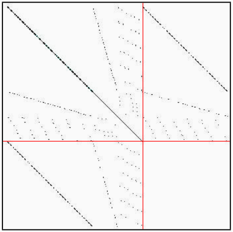
$\mathrm{cond} = 20$	$\mathrm{cond} = 55$	$\mathrm{cond} = 6.10^6$

Conditioning's mean of the local matrices

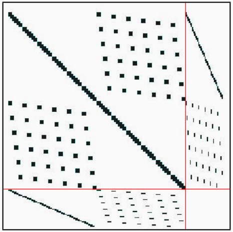
	$\mathrm{cond} = 49$	$\mathrm{cond} = 2.10^3$

Numerical experiment

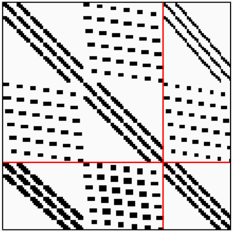
Test case : 36 cells (288 triangles)
Nonzero coefficients of the matrices



MFE method on $\mathcal{T}_h$
(744×744)



Intermediary method
(204×204)



Final method
(120×120)

Stencil of the matrices

$s = 7$ $\quad$ $s = 16$ $\quad$ $s = 29$

Conclusion and perspective

Conclusion

1. We succeeded at solving the problem with 1 pressure per cell and 1 flux per face on a very coarse mesh
2. Resulting matrix is lower dimension, much denser and badly conditioned

Perspective

Is it possible to improve dramatically the condition number by choosing a suitable $\mathbb{N}$?

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