Domain Decomposition Methods developed in Alpines

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Outline

Outline

Motivation

Large Sparse discretized system with strongly heterogeneous coefficients (high contrast, nonlinear, multiscale)

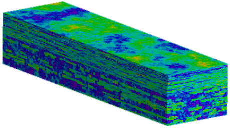
E.g. Darcy pressure equation, P^1-finite elements:

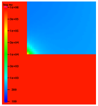
$$A\mathbf{U} = \mathbf{F}$$

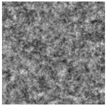
$$\mathrm{cond}(A) \sim \frac{\alpha_{\max}}{\alpha_{\min}} h^{-2}$$

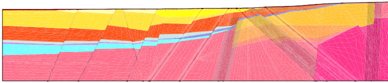
Goal:
iterative solvers
robust in size and heterogeneities

Applications:
flow in heterogeneous / stochastic / layered media
structural mechanics
time dependent waves
etc.









Black box solvers (solve(MAT,RHS,SOL))

	Direct Solvers	Iterative Solvers
Pros	Robustness	Naturally $\|\|$
Cons	Difficult to $\|\|$	Robustness

Domain Decomposition Methods (DDM): Hybrid solver $\rightarrow$ should be naturally parallel and robust

General form:

$$Au = f, \text{ solved with PCG for a preconditioner } M^{-1}.$$

What's Classical: Robustness with respect to problem size (scalability)

What's New here: Provable Robustness for the Schwarz method in the SPD case with respect to:

- Coefficients jumps
- System of PDEs

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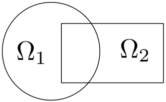
The original Schwarz Method (H.A. Schwarz, 1870)



$$
\begin{aligned}
-\Delta(u) &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Schwarz Method : $(u_1^n, u_2^n) \to (u_1^{n+1}, u_2^{n+1})$ with

$$
\begin{aligned}
-\Delta(u_1^{n+1}) &= f \quad \text{in } \Omega_1 \\
u_1^{n+1} &= 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \\
u_1^{n+1} &= u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}.
\end{aligned}
\qquad
\begin{aligned}
-\Delta(u_2^{n+1}) &= f \quad \text{in } \Omega_2 \\
u_2^{n+1} &= 0 \text{ on } \partial\Omega_2 \cap \partial\Omega \\
u_2^{n+1} &= u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.
\end{aligned}
$$

Parallel algorithm, converges but slowly, overlapping subdomains only. As a preconditioner in a Krylov method, convergence is acceptable.

Strong and Weak scalability

How to evaluate the efficiency of a domain decomposition?

Strong scalability (Amdahl)

"How the solution time varies with the number of processors for a fixed *total* problem size"

Weak scalability (Gustafson)

"How the solution time varies with the number of processors for a fixed problem size *per processor*."

Not achieved with the one level method

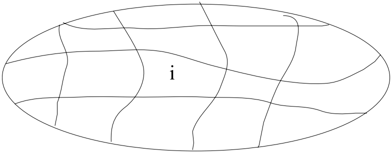
Number of subdomains	8	16	32	64
ASM	18	35	66	128

The iteration number increases linearly with the number of subdomains in one direction.

How to achieve scalability

Stagnation corresponds to a few very low eigenvalues in the spectrum of the preconditioned problem. They are due to the lack of a global exchange of information in the preconditioner.

$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$



The mean value of the solution in domain i depends on the value of f on all subdomains.

A classical remedy consists in the introduction of a coarse problem that couples all subdomains. This is closely related to deflation technique classical in linear algebra (see Y. Saad, J. Erhel, Nabben and Vuik) and multigrid techniques.

Adding a coarse space

We add a coarse space correction (*aka* second level)
Let V_H be the coarse space and Z be a basis, $V_H = \mathrm{span}\, Z$, writing $R_0 = Z^T$ we define the two level preconditioner as:

$$M^{-1}_{ASM,2} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i .$$

The Nicolaides approach (1987) is to use the kernel of the operator as a coarse space, this is the constant vectors, in local form this writes:

$$Z := (R_i^T D_i R_i \mathbf{1})_{1 \le i \le N}$$

where D_i are chosen so that we have a partition of unity:

$$\sum_{i=1}^{N} R_i^T D_i R_i = Id .$$

Theoretical convergence result

Theorem (Widlund, Dryija)

Let $M_{ASM,2}^{-1}$ be the two-level additive Schwarz method:

$$\kappa(M_{ASM,2}^{-1}A) \leq C\left(1+\frac{H}{\delta}\right)$$

where δ is the size of the overlap between the subdomains and H the subdomain size.

This does indeed work very well

Number of subdomains	8	16	32	64
ASM	18	35	66	128
ASM + Nicolaides	20	27	28	27

$$
\begin{aligned}
-\nabla \cdot (\alpha(x,y)\nabla u) &= 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \\
u &= 0 \quad \text{on} \quad \partial\Omega_D, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega_N.
\end{aligned}
$$

Decomposition

$\alpha(x,y)$

Jump	1	10	10^2	10^3	10^4
ASM	39	45	60	72	73
ASM + Nicolaides	30	36	50	61	65

Our approach

Fix the problem by an optimal and proven choice of a coarse space Z.

Outline

Objectives

Strategy

Define an appropriate coarse space $V_{H2} = \mathrm{span}(Z_2)$ and use the framework previously introduced, writing $R_0 = Z_2^T$ the two level preconditioner is:

$$P_{ASM2}^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i .$$

The coarse space must be

- Local (calculated on each subdomain) $\rightarrow$ parallel
- Adaptive (calculated automatically)
- Easy and cheap to compute
- Robust (must lead to an algorithm whose convergence is proven not to depend on the partition nor the jumps in coefficients)

Problem setting – I

Given $f \in (V^h)^*$ find $u \in V^h$

$$
\begin{aligned}
a(u, v) &= \langle f, v \rangle \qquad \forall v \in V^h \\
\iff \qquad \mathbf{A}\,\mathbf{u} &= \mathbf{f}
\end{aligned}
$$

Assumption throughout: $\mathbf{A}$ *symmetric positive definite (SPD)*

Examples:

- Darcy $\quad a(u, v) = \int_\Omega \boldsymbol{\kappa} \nabla u \cdot \nabla v \, dx$
- Elasticity $\quad a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{C}\, \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \, dx$
- Eddy current $\quad a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\nu}\, \mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{v} + \boldsymbol{\sigma}\, \boldsymbol{u} \cdot \boldsymbol{v} \, dx$

Heterogeneities / high contrast in parameters

Problem setting – II

1. $V^h \dots$ FE space of functions in Ω based on mesh $\mathcal{T}^h = \{\tau\}$
2. $\{\phi_k\}_{k=1}^n$ (FE) basis of V^h
3. Technical assumptions fulfilled by standard FE and bilinear forms $a(\cdot, \cdot)$

Overlapping decomposition: $\Omega = \bigcup_{j=1}^{N} \Omega_j$ (Ω_j union of elements)

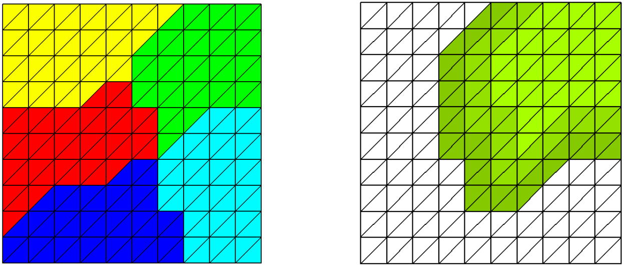
$$V_j := \mathrm{span}\{\phi_k : \mathrm{supp}(\phi_k) \subset \overline{\Omega}_j\}$$

such that every ϕ_k is contained in one of those spaces, i.e.

$$V^h = \sum_{j=1}^{N} V_j$$

Example: adding "layers" to non-overlapping partition

(partition and adding layers based on matrix information only!)



Local subspaces:

$$V_j \subset V^h \qquad j = 1, \ldots, N$$

Coarse space (defined later):

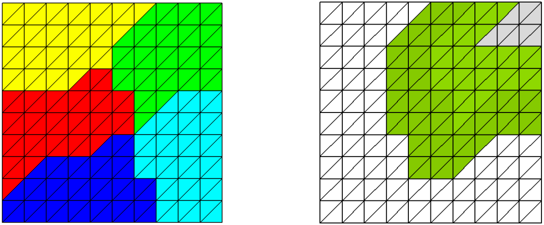
$$V_0 \ \subset V^h$$

Additive Schwarz preconditioner:

$$\mathbf{M}_{AS,2}^{-1} \ = \ \sum_{j=0}^{N} \mathbf{R}_j^\top \mathbf{A}_j^{-1} \mathbf{R}_j$$

where $\mathbf{A}_j = \mathbf{R}_j^\top \mathbf{A} \mathbf{R}_j$
and $\mathbf{R}_j^\top \leftrightarrow R_j^\top : V_j \to V^h$ natural embedding

Overlapping zone: $\Omega_j^\circ = \{x \in \Omega_j : \exists i \neq j : x \in \Omega_i\}$



Observation: partition of unity operator satisfies $\Xi_{j|\Omega_j \setminus \Omega_j^\circ} = \mathrm{id}$

Coarse space should be a sum of **local** contributions:

$$V_0 = \sum_{j=1}^{N} V_{0,j} \qquad \text{where } V_{0,j} \subset V_j$$

E.g. $V_{0,j} = \mathrm{span}\{\Xi_j p_{j,k}\}_{k=1}^{m_j}$

ASM theory needs **stable splitting**:

$$v \;=\; v_0 + \sum_{j=1}^{N} v_j$$

Suppose $v_0 = \sum_{j=1}^{N} \Xi_j \Pi_j v_{|\Omega_j}$ where $\Pi_j \ldots$ local projector

$$|\underbrace{\Xi_j(v - \Pi_j v)}_{v_j}|^2_{a,\Omega_j} \;=\; |\Xi_j(v - \Pi_j v)|^2_{a,\Omega_j^\circ} + |\Xi_j(v - \Pi_j v))|^2_{a,\Omega_j \setminus \Omega_j^\circ}$$

$$\overset{\text{HOW?}}{\leq} \quad C\,|v|^2_{a,\Omega_j}$$

(a,D denotes the restriction of a to D)

"Minimal" requirements:

- Π_j be a-orthogonal
- Stability estimate: $|\Xi_j(v - \Pi_j v)|^2_{a,\Omega_j^\circ} \;\leq\; c\,|v|^2_{a,\Omega_j}$

Choice of coarse space (continued)

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Fulfillment of 2

If there exist a non zero function w such that $|w|_{a,\Omega_j} = 0$, it is necessary to project on $Span(w)$.
The kernel of a Darcy equation is the constant function and that of elasticity is spanned by rigid body motions.
The corresponding coarse space will be refered to as ZEM (zero energy modes).

For highly heterogeneous problems, we take a larger coarse space deduced from the stability estimate.

Abstract eigenvalue problem

Geneo .EVP per subdomain:

Find $p_{j,k} \in V_{h|\Omega_j}$ and $\lambda_{j,k} \geq 0$:

$$a_{\Omega_j}(p_{j,k},\, v) \;=\; \lambda_{j,k}\, a_{\Omega_j^\circ}(\Xi_j p_{j,k},\, \Xi_j v) \qquad \forall v \in V_{h|\Omega_j}$$

$$A_j \mathbf{p}_{j,k} \;=\; \lambda_{j,k}\, \mathbf{X}_j A_j^\circ \mathbf{X}_j\, \mathbf{p}_{j,k} \qquad (\mathbf{X}_j \ldots \text{diagonal})$$

$a_D \ldots$ restriction of a to D

In the two-level ASM:

Choose first m_j eigenvectors per subdomain:

$$V_0 \;=\; \mathrm{span}\{\Xi_j p_{j,k}\}_{k=1,\ldots,m_j}^{j=1,\ldots,N}$$

This automatically includes Zero Energy Modes.

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Comparison with existing works

N. & al (SIAM 2011):

$$\int_{\Omega_j} \boldsymbol{\kappa} \, \nabla p_{j,k} \cdot \nabla v \, dx \; = \; \lambda_{j,k} \int_{\partial\Omega_j} \boldsymbol{\kappa} \, p_{j,k} \, v \, dx \qquad \forall v \in V_{h|\Omega_j}$$

Efendiev, Galvis, Lazarov & Willems (2011):

$$a_{\Omega_j}(p_{j,k}, \, v) \; = \; \lambda_{j,k} \sum_{i \in \text{neighb}(j)} a_{\Omega_j}(\xi_j \, \xi_i \, p_{j,k}, \, \xi_j \, \xi_i \, v) \qquad \forall v \in V_{|\Omega_j}$$

$\xi_j \ldots$ partition of unity, calculated adaptively (MS)

Our gen.EVP:

$$a_{\Omega_j}(p_{j,k}, \, v) \; = \; \lambda_{j,k} \, a_{\Omega_j^\circ}(\Xi_j p_{j,k}, \, \Xi_j v) \qquad \forall v \in V_{h|\Omega_j}$$

both matrices typically singular $\implies \lambda_{j,k} \in [0, \, \infty]$

Two technical assumptions.

Theorem (Spillane, Dolean, Hauret, N., Pechstein, Scheichl (Num. Math. 2013))

If for all j: $0 < \lambda_{j,m_{j+1}} < \infty$*:*

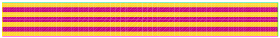
$$\kappa(M_{ASM,2}^{-1}A) \;\leq\; (1+k_0)\Big[2 + k_0\,(2k_0+1)\,\max_{j=1}^{N}\Big(1+\frac{1}{\lambda_{j,m_j+1}}\Big)\Big]$$

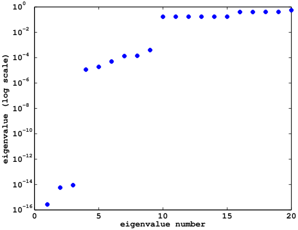
Possible criterion for picking m_j: (used in our Numerics)

$$\lambda_{j,m_j+1} \;<\; \frac{\delta_j}{H_j}$$

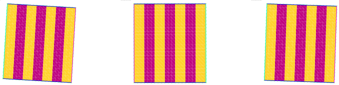
H_j ... subdomain diameter, δ_j ... overlap

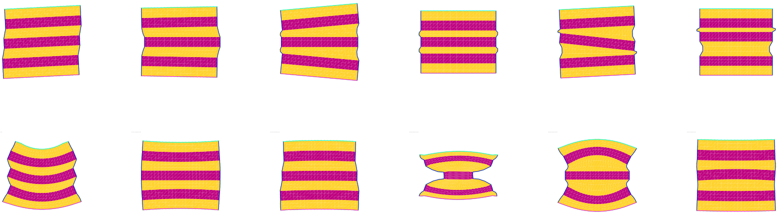
Eigenvalues and eigenvectors (Elasticity)

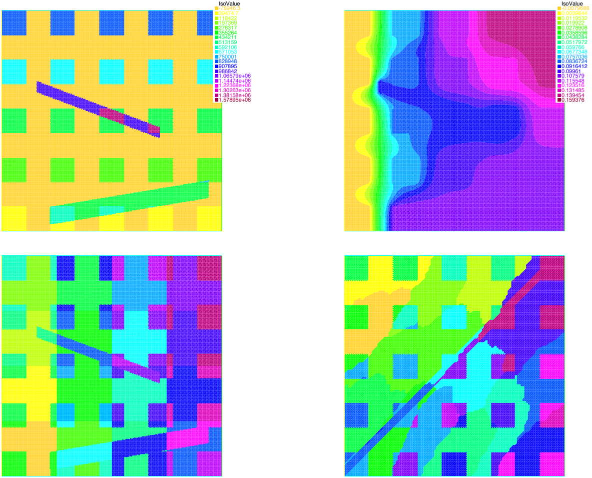




Logarithmic scale

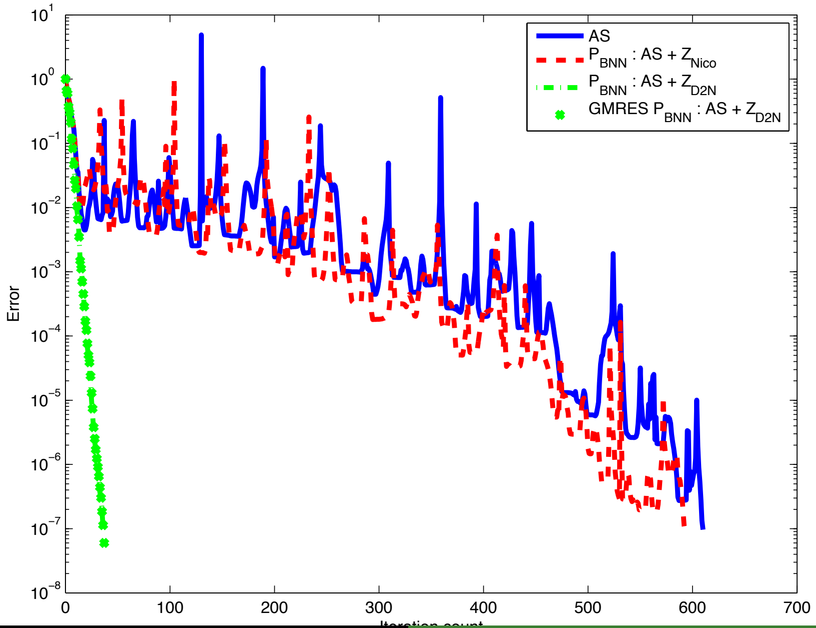






Channels and inclusions: $1 \leq \alpha \leq 1.5 \times 10^6$, the solution and partitionings (Metis or not)

Convergence



Numerical results – Optimality

m_i is given automatically by the strategy.

$\#Z$ per subd.	ASM	ASM+Z_{Nico}	ASM+Z_{Geneo}
$\max(m_i - 1, 1)$			273
m_i	614	543	36
$m_i + 1$			32

- Taking one fewer eigenvalue has a huge influence on the iteration count
- Taking one more has only a small influence

Numerical results via a Domain Specific Language

FreeFem++ (http://www.freefem.org/ff++), with:

- Metis Karypis and Kumar 1998
- SCOTCH Chevalier and Pellegrini 2008
- UMFPACK Davis 2004
- ARPACK Lehoucq et al. 1998
- MPI Snir et al. 1995
- Intel MKL
- PARDISO Schenk et al. 2004
- MUMPS Amestoy et al. 1998
- PaStiX Hénon et al. 2005
- Slepc via PETSC

Runs on PC (Linux, OSX, Windows) and HPC (Babel@CNRS, HPC1@LJLL, Titane@CEA via GENCI PRACE)

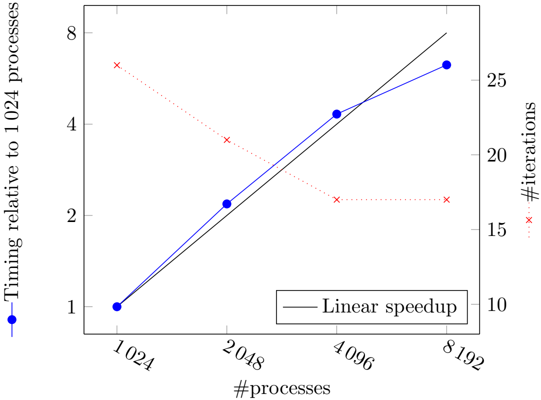
Why use a DS(E)L instead of `C/C++/Fortran/..` ?

- performances close to low-level language implementation,
- hard to beat something as simple as:

$$\mathbf{varf}\ \mathrm{a(u,\ v)} = \mathbf{int3d}(\mathbf{mesh})([\mathbf{dx}(\mathrm{u}),\ \mathbf{dy}(\mathrm{u}),\ \mathbf{dz}(\mathrm{u})]'\ *\ [\mathbf{dx}(\mathrm{v}),\ \mathbf{dy}(\mathrm{v}),\ \mathbf{dz}(\mathrm{v})]) \\ + \mathbf{int3d}(\mathbf{mesh})(\mathrm{f}\ *\ \mathrm{v}) + \mathbf{on}(\mathrm{boundary\_mesh})(\mathrm{u} = 0)$$

Strong scalability in two dimensions heterogeneous elasticity (P. Jolivet with Frefeem ++)

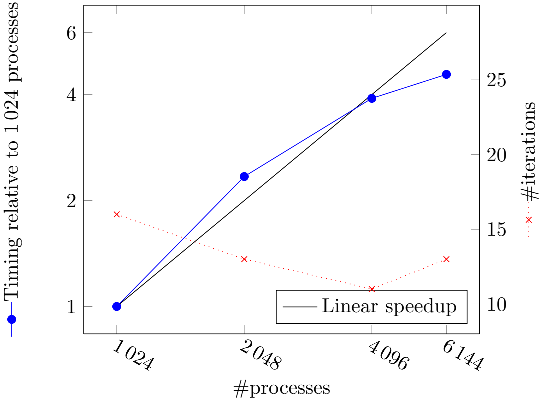
Elasticity problem with heterogeneous coefficients with automatic mesh partition



Speed-up for a 1.2 billion unknowns 2D problem. Direct solvers in the subdomains. Peak performance wall-clock time: 26s.

Strong scalability in three dimensions heterogeneous elasticity

Elasticity problem with heterogeneous coefficients with automatic mesh partition



Speed-up for a 160 million unknowns 3D problem. Direct solvers in subdomains. Peak performance wall-clock time: 36s.

Darcy pressure equation

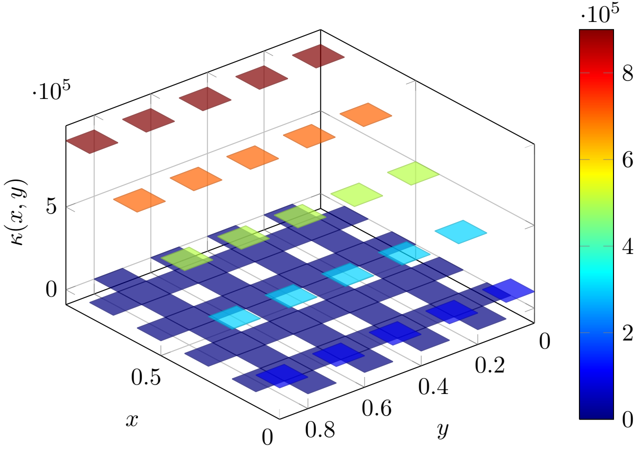
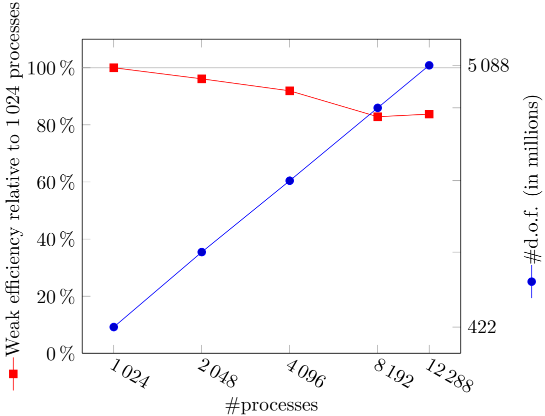


Figure : Two dimensional diffusivity κ

Weak scalability in two dimensions

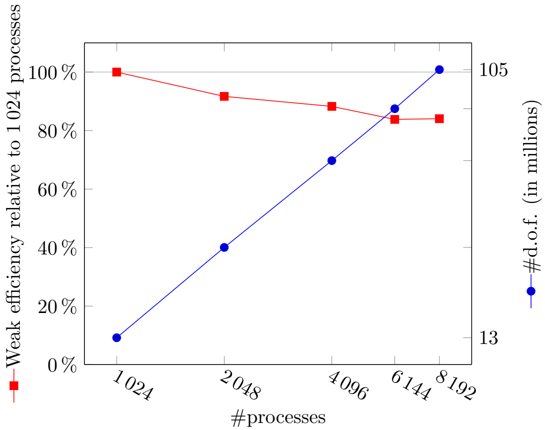
Darcy problems with heterogeneous coefficients with automatic mesh partition



Efficiency for a 2D problem. Direct solvers in the subdomains.
Final size: 22 billion unknowns. Wall-clock time: $\simeq$ 200s.

Weak scalability in three dimensions

Darcy problems with heterogeneous coefficients with automatic mesh partition



Efficiency for a 3D problem. Direct solvers in the subdomains.
Final size: 2 billion unknowns. Wall-clock time: $\simeq$ 200s.

Outline

Conclusion

Summary

- Using generalized eigenvalue problems and projection preconditioning we are able to achieve a targeted convergence rate.
- Works for BNN and FETI methods as well (not shown here)
- Implementation requires only element stiffness matrices

Future work

- Build the coarse space on the fly.
- Nonlinear time dependent problem (Reuse of the coarse space)
- Multigrid like three (or more) level methods
- Non symmetric, undefinite problems
- Purely algebraic setting

Bibliography Coarse Space for Schwarz

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THANK YOU FOR YOUR ATTENTION!

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