A mixed finite element method for deformed cubic meshes
Inria Project Lab C2S@Exa, annual meeting, Bordeaux

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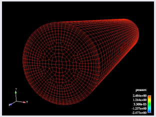
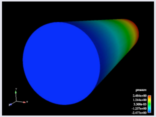
Goals

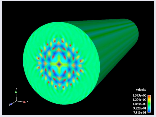
Numerical method

- Define a mixed finite element method for deformed cubes
- 1 pressure per cell
- 1 flux per face

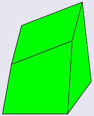
Mixed finite element methods







Deformed cube

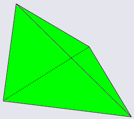


High performance computing

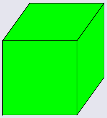
- Implementation in Traces (ANDRA)
- Parallelism and optimization

Mixed finite element methods

$\mathbf{RTN}_0$ for tetrahedra and cubes (Raviart-Thomas-Nédélec [1])

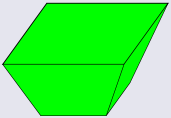


Tetrahedron

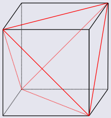


Cube

Composite element for hexahedron (Sboui-Jaffré-Roberts [2])



Hexahedron

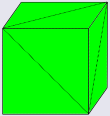


Hexahedron split into 5 tetrahedra

Composite mixed finite element method

Composite element with 5 tetrahedra

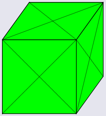
- Assume the faces are planar
- Choose the splitting



Cube split into 5 tetrahedra

Composite element with 24 tetrahedra

- Works with curved faces
- Single splitting
- Symmetry
- Conforming tetrahedral submesh



Cube split into 24 tetrahedra

Composite mixed finite element method

Incompressible Darcy flow

Find $\mathbf{u} \in \mathbf{H}(\mathrm{div}\,;\Omega)$ and $p \in \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} \mathbf{u} &= -\mathbf{K}\,\nabla\, p && \text{in } \Omega \\ \nabla\cdot\mathbf{u} &= f && \text{in } \Omega \\ p &= p_0 && \text{on } \partial\Omega \end{aligned}$$

Weak formulation

Find $\mathbf{u}_h \in \mathcal{W}_h$ and $p_h \in \mathcal{M}_h$ such that

$$\begin{aligned} \int_\Omega \mathbf{K}^{-1}\mathbf{u}_h\cdot\mathbf{v}_h - \int_\Omega p_h\,\nabla\cdot\mathbf{v}_h &= -\int_{\partial\Omega} p_0\mathbf{v}_h\cdot\mathbf{n} && \forall\mathbf{v}_h \in \mathcal{W}_h \\ -\int_\Omega q_h\,\nabla\cdot\mathbf{u}_h &= -\int_\Omega f q_h && \forall q_h \in \mathcal{M}_h \end{aligned}$$

Define the approximation spaces $\mathcal{W}_h$ and $\mathcal{M}_h$

Composite mixed finite element method

Conditions to meet for the approximation spaces $\mathcal{W}_h$ and $\mathcal{M}_h$

$$\mathbf{u}_h \in \mathcal{W}_h \text{ and } p_h \in \mathcal{M}_h$$

- p_h must be constant on each hexahedron E of the mesh $\mathcal{T}_h$
- $\nabla\cdot \mathbf{u}_h$ must be constant on E
- $\mathbf{u}_h$ must be in $\mathbf{RTN}_0$ inside the tetrahedral submesh $\mathcal{T}_E$ of E
- $\mathbf{u}_h$ must be uniquely defined by this value on each face F of the mesh

Definition of the approximation spaces $\mathcal{W}_h$ and $\mathcal{M}_h$

$$\mathcal{M}_h = \{q \in \mathbf{L}^2(E) : q|_E \text{ is constant on } E, \forall E \in \mathcal{T}_h\}$$

- $\mathcal{F}_h$ is the set of faces of the mesh $\mathcal{T}_h$
- $\mathcal{W}_h$ is defined as a vectorial space with the basis functions $\mathbf{w}_F$, $F \in \mathcal{F}_h$

Composite mixed finite element method

Definition of the approximation spaces $\mathcal{W}_h$ and $\mathcal{M}_h$

$$\mathcal{W}_h = \mathrm{Vect}\left\{\mathbf{w}_F, F \in \mathcal{F}_h : \mathbf{w}_F|_E \text{ is solution of } (\mathcal{P}_{E,F})\right\}$$

- A local problem $(\mathcal{P}_{E,F})$ is defined to meet the conditions for each $\mathbf{w}_F$
- The basis function $\mathbf{w}_F$ will solve the local problem $(\mathcal{P}_{E,F})$ inside E

The local approximation spaces $\widetilde{\mathcal{W}}_E$ and $\widetilde{\mathcal{M}}_E$

- $\mathcal{T}_E$ is the tetrahedral mesh of E
- $\widetilde{\mathcal{W}}_E$ and $\widetilde{\mathcal{M}}_E$ are the mixed finite element spaces

$$\widetilde{\mathcal{M}}_E = \{q \in \mathbf{L}^2(E) : q|_T \text{ is constant on } T, \forall\, T \in \mathcal{T}_E\}$$

$$\widetilde{\mathcal{W}}_E = \{\mathbf{v} \in \mathbf{H}(\mathrm{div}\,; E) : \mathbf{v}|_T \in \mathbf{RTN}_0(T), \forall\, T \in \mathcal{T}_E\}$$

Composite mixed finite element method

The local problem $(\mathcal{P}_{E,F})$ inside the composite element

Find $\mathbf{w}_F \in \widetilde{\mathcal{W}}_E$ and $\tilde{p}_F \in \widetilde{\mathcal{M}}_E$ such that

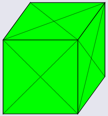
$$\begin{aligned} \int_E \mathbf{K}^{-1}\mathbf{w}_F \cdot \tilde{\mathbf{v}} - \int_E \tilde{p}_F \,\nabla\cdot \tilde{\mathbf{v}} &= 0 && \forall \tilde{\mathbf{v}} \in \widetilde{\mathcal{W}}_E \\ -\int_E \tilde{q}\,\nabla\cdot \mathbf{w}_F &= -\int_E \frac{1}{|E|}\tilde{q} && \forall \tilde{q} \in \widetilde{\mathcal{M}}_E \end{aligned} \qquad (\mathcal{P}_{E,F})$$

Explicit solution with 5 tetrahedra

Neumann boundary condition

$\mathbf{n}_F$ normal of the face F

$$\mathbf{w}_F \cdot \mathbf{n}_{F'} = \begin{cases} \frac{1}{|F|} & \text{if } F = F' \\ 0 & \text{else} \end{cases}$$



Cube split into 24 tetrahedra

Composite mixed finite element method with curved faces

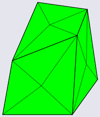
Problem with curved faces

Constant velocities are not inside the approximation space $\mathcal{W}_h$

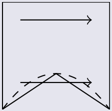
Proof (Nordbotten-Hægland [3])

- F is the union of 4 sub-faces F_i
- $\mathbf{n}_F = \mathbf{n}_{F_i}$ on the triangular sub-face F_i
- $\mathbf{u}$ is a constant velocity

$$\mathbf{u} \cdot \mathbf{n}_{F_i} \neq \mathbf{u} \cdot \mathbf{n}_{F_j}$$



Deformed cube

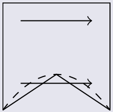


Constant velocity $\mathbf{u}$

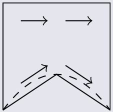
Composite mixed finite element method with curved faces

Neumann boundary condition with curved face

- $\mathbf{w}_F|_E$ solves a local problem inside E
- $\mathbf{w}_F \cdot \mathbf{n}_{F'} = 0$ if $F \neq F'$



Constant velocity $\mathbf{u}$



Approximated velocity $\mathbf{u}_h$

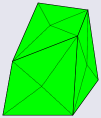
- The error between $\mathbf{u}$ and $\mathbf{u}_h$ depends on the mesh

Composite mixed finite element method with curved faces

Neumann boundary condition with curved face

$\mathbf{u} \cdot \mathbf{n}_F$ is not constant if F is a curved face

$$\mathbf{u} \cdot \mathbf{n}_{F_i} \neq \mathbf{u} \cdot \mathbf{n}_{F_j}$$



Deformed cube

Adapt the Neumann boundary condition

$$\mathbf{w}_F \cdot \mathbf{n}_F = \frac{1}{|F|}$$

$\bar{\mathbf{n}}_F$ is the mean of the normal $\mathbf{n}_{F_i}$

$$\bar{\mathbf{n}}_F = \frac{\sum_{i=1}^{4} |F_i| \, \mathbf{n}_{F_i}}{\left\| \sum_{i=1}^{4} |F_i| \, \mathbf{n}_{F_i} \right\|_{L^2}}$$

Neumann boundary condition

$$\mathbf{w}_F \cdot \mathbf{n}_F = \frac{\bar{\mathbf{n}}_F \cdot \mathbf{n}_F}{\int_F \bar{\mathbf{n}}_F \cdot \mathbf{n}_F}$$

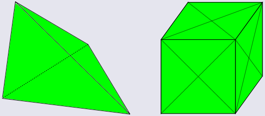
Numerical Experiment

Exact solution

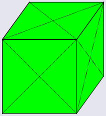
Convergence error inside the domain $\Omega = [0\,;1]^3$ with different meshes.

$$p = 2xz + \frac{y^2}{2} + z \qquad\qquad \mathbf{u} = -\begin{pmatrix} 2z \\ y \\ 2x+1 \end{pmatrix}$$

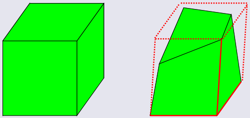
RTN_0 on tetrahedron



Composite RTN_0

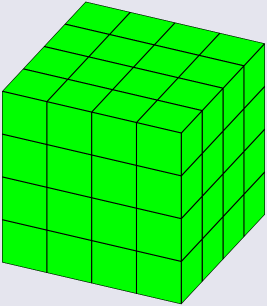


RTN_0 on hexahedron

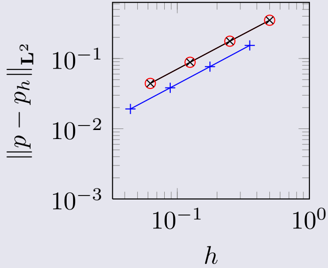


Numerical Experiment

Regular mesh



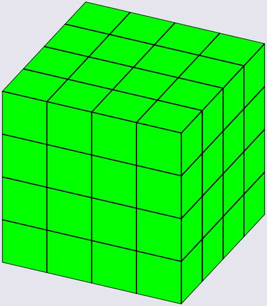
Convergence error on the regular mesh



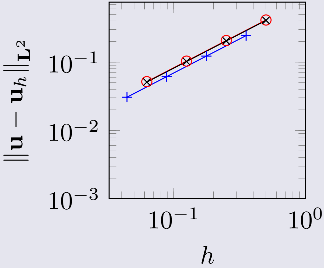
$\mathbf{RTN}_0$ on tetraedral mesh
Composite $\mathbf{RTN}_0$
$\mathbf{RTN}_0$ on hexaedral mesh

Numerical Experiment

Regular mesh



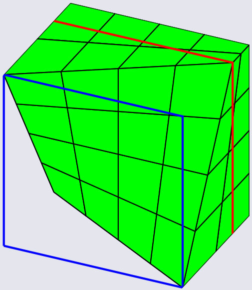
Convergence error on the regular mesh



$\mathbf{RTN}_0$ on tetraedral mesh
Composite $\mathbf{RTN}_0$
$\mathbf{RTN}_0$ on hexaedral mesh

Numerical Experiment

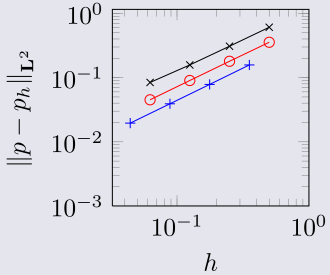
Deformed mesh



$y = 0.2$

$y = 0.8$

Convergence error on the deformed mesh



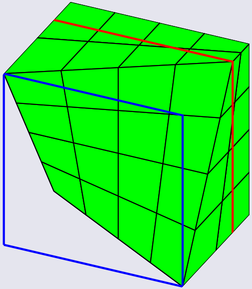
$\mathbf{RTN}_0$ on tetraedral mesh

Composite $\mathbf{RTN}_0$

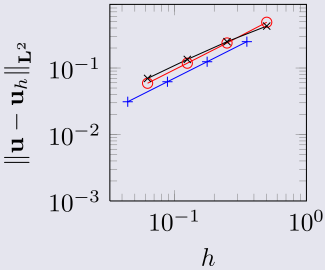
$\mathbf{RTN}_0$ on hexaedral mesh

Numerical Experiment

Deformed mesh



Convergence error on the deformed mesh

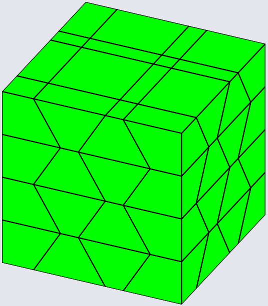


$y = 0.2$
$y = 0.8$

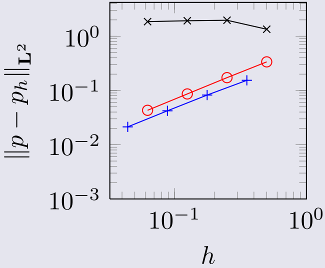
$\mathbf{RTN}_0$ on tetraedral mesh
Composite $\mathbf{RTN}_0$
$\mathbf{RTN}_0$ on hexaedral mesh

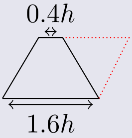
Numerical Experiment (Fixed aspect ratio)

Hexahedral mesh



Convergence error on the hexahedral mesh

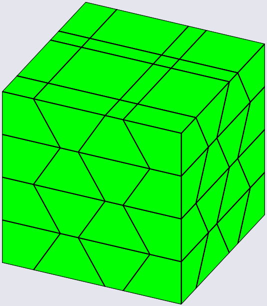




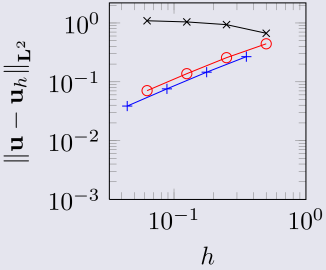
- $\mathbf{RTN}_0$ on tetraedral mesh
- Composite $\mathbf{RTN}_0$
- $\mathbf{RTN}_0$ on hexaedral mesh

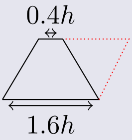
Numerical Experiment (Fixed aspect ratio)

Hexahedral mesh



Convergence error on the hexahedral mesh

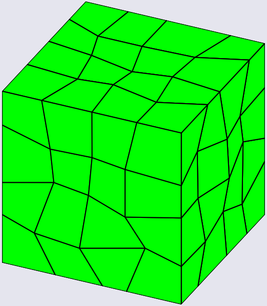




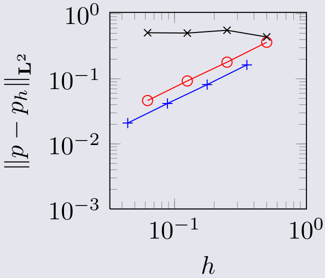
- $\mathbf{RTN}_0$ on tetraedral mesh
- Composite $\mathbf{RTN}_0$
- $\mathbf{RTN}_0$ on hexaedral mesh

Numerical Experiment (Fixed aspect ratio)

Random mesh



Convergence error on the random mesh

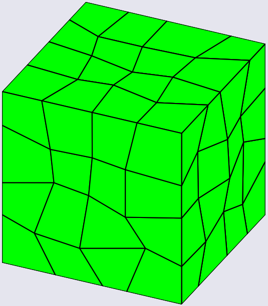


Shift randomly
the vertices
$\pm 0.3h$

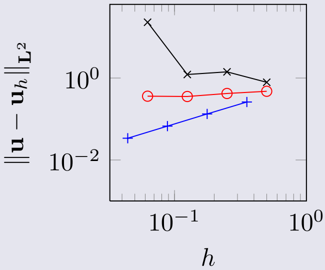
$\mathbf{RTN}_0$ on tetraedral mesh
Composite $\mathbf{RTN}_0$
$\mathbf{RTN}_0$ on hexaedral mesh

Numerical Experiment (Fixed aspect ratio)

Random mesh



Convergence error on the random mesh

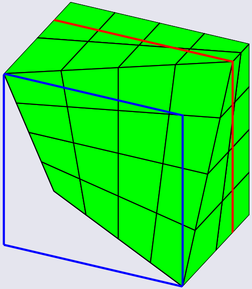


Shift randomly the vertices $\pm 0.3h$

- $\mathbf{RTN}_0$ on tetraedral mesh
- Composite $\mathbf{RTN}_0$
- $\mathbf{RTN}_0$ on hexaedral mesh

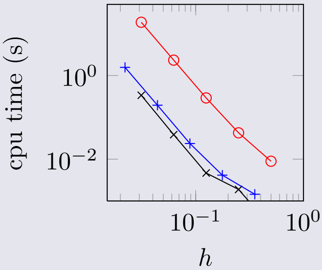
Numerical Experiment (Hybrid form - pcg - Traces)

Deformed mesh



32768 hexahedra
786432 tetrahedra

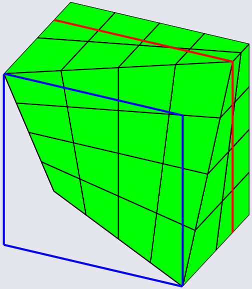
CPU time to build the linear equation



$\mathbf{RTN}_0$ on tetraedral mesh
Composite $\mathbf{RTN}_0$
$\mathbf{RTN}_0$ on hexaedral mesh

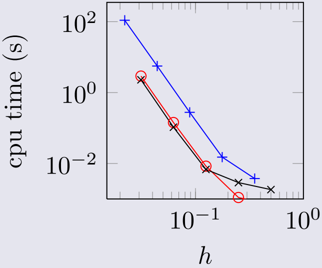
Numerical Experiment (Hybrid form - pcg - Traces)

Deformed mesh



32768 hexahedra
786432 tetrahedra

CPU time to solve the linear equation



- $\mathbf{RTN}_0$ on tetraedral mesh
- Composite $\mathbf{RTN}_0$
- $\mathbf{RTN}_0$ on hexaedral mesh

Conclusion

Convergence error

- The convergence is optimal for planar faces
- If the curved faces are fixed, the velocity converges
- The composite error is between the two $\mathbf{RTN}_0$ errors

Prisms and pyramids

- The same methodology can be apply for prisms and pyramids
- Local and conforming refinement

Projection of the solution into the $\mathbf{RTN}_0$ tetrahedral space

- With $\widetilde{p}_F$ in $(\mathcal{P}_{E,F})$ the pressure is defined on the tetrahedral submesh
- A posteriori error estimation (Vohralík [4])

Traces

Implementation in Traces

- Put the composite method in Traces (hexahedron, prism, pyramid)
- Check the matrix
- Check the method with basic test cases

Perspective

- Build a specific test case
- Study the parallelism
- Try another solver
- Do a posteriori error estimation

References I

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