Stochastic EPDiff Landmark Dynamics

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Abstract. We develop a variational method of deriving stochastic partial differential equations whose solutions follow the flow of a stochastic vector field. As an example in one spatial dimension we numerically simulate singular solutions (landmarks) of the stochastically perturbed EPDiff equation derived using this method. These numerical simulations show that singular solutions of the stochastically perturbed EPDiff equation persist, and some choices of stochastic perturbations allow landmarks to interpenetrate and exchange order on the real line in overtaking collisions, although this behaviour does not occur for singular solutions of the unperturbed deterministic EPDiff equation. This solution behaviour introduces the possibility of a topological change and may be of importance in registration of noisy images in computational anatomy.

Keywords: Geometric mechanics, cylindrical stochastic processes, stochastic soliton dynamics, symmetry reduced variational principles

1 Introduction

Trouvé and Vialard [14, 15] study the stochastic evolution of landmarks in LDDMM [12] as a stochastic perturbation of the canonical Hamiltonian system arising from the singular reduction to a finite dimensional system of Lagrangian particles of a solution of the EPDiff equation for the geodesics on the group of diffeomorphisms, which arises from the LDDMM variational principle [4]. From this viewpoint, the variational principle for shape analysis using LDDMM has a natural analogue in particle dynamics. In particular, papers [14, 15] suggest adding white noise to the Hamiltonian evolution equation for the landmark canonical "momentum", as though the noise were a random force acting on a system of particles. However, there exist many ways of introducing stochasticity into particle dynamics. Here, we will explore an alternative approach for including noise in Hamilton equations which is still consistent with the LDDMM variational principle for landmark evolution. For brevity, and to simplify matters, we will take the viewpoint of particle dynamics and defer its potential applications in landmark dynamics for LDDMM until later work.

Our approach is based on a generalisation in [7] of earlier work by Bismut [1], Lázaro-Camí and Ortega [11], and Bou-Rabee and Owhadi [2] for stochastic ordinary differential equations (SDE). The *parametric stochastic deformation*

(P-SD) approach of [7] unifies the Hamiltonian and Lagrangian approaches to temporal stochastic dynamics, and extends them to stochastic partial differential equations (SPDE) in the case of cylindrical noise in which the spatial dependence is *parametric*, while temporal dependence is stochastic.

Objectives. This paper has two main objectives. The first objective is the inclusion of parametric stochastic deformation (P-SD) in the variational principle for the EPDiff partial differential equation. The second objective is the numerical study of the statistical effects of parametric and canonically Hamiltonian stochastic deformations (CH-SD) on the soliton-like solutions of deterministic EPDiff in one spatial dimension, when the Lagrangian in Hamilton's principle is a Sobolev norm on the continuous vector fields. When the Lagrangian is the H^1 norm, the deterministic equation is the completely integrable CH equation [3] and the solutions are true solitons (peakons).

2 Stochastic variational perturbations in one spatial dimension

2.1 Singular peakon solutions of the EPDiff equations

Let $\mathrm{Diff}(\mathbb{R}^n)$ be the diffeomorphism group of $\mathbb{R}^n$, and $\mathfrak{X}(\mathbb{R}^n)$ its Lie algebra, i.e., the set of all smooth vector fields on $\mathbb{R}^n$. The EPDiff equation is obtained from the variational principle $\delta S = 0$ for the action functional $S = \int \ell(u)\, dt$ with the restricted variations $\delta u = \dot{v} - [u, v]$ (see [8]). The EPDiff(H^1) equation in the one-dimensional case when $\ell(u) = \frac{1}{2}\|u\|^2_{H^1} = \frac{1}{2}\int u^2 + \alpha^2 u_x^2 \,\mathrm{d}x$ is called the Camassa-Holm (CH) equation for $m = \delta\ell/\delta u = u - \alpha^2 u_{xx}$ with positive constant α^2; namely [3],

$$m_t + (um)_x + mu_x = 0 \quad \text{with} \quad m = u - \alpha^2 u_{xx}\,. \tag{1}$$

This equation has singular peaked soliton (peakon) solutions, given by

$$m(x,t) = \sum_{a=1}^{N} p_a(t)\delta(x - q_a(t))\,, \quad \text{so that} \quad u(x,t) := \sum_{b=1}^{N} p_b(t)K(x - q_b(t)), \tag{2}$$

where $K(x-y) = \exp(-|x-y|/\alpha)$ is the Green's function for the Helmholtz operator $1 - \alpha^2\partial_x^2$. The peaked shape of the velocity profile of the soliton solution of the CH equation $u(x,t) := p(t)\exp(-|x-q(t)|/\alpha)$ provided the name, peakon.

Peakons are emergent singular solutions which dominate the initial value problem, since an initially confined smooth velocity distribution will decompose into peakon solutions and, in fact, *only* peakon solutions. The main point to notice is that the distance between any two peaks never passes through zero. That is, the peakons keep their order, even after any number of overtaking collisions (the taller peakons travel faster). Substituting the (weak) solution Ansatz (2) into the CH equation (1) and integrating against a smooth test function yields the following dynamical equations for the $2N$ solution parameters $q_a(t)$ and $p_a(t)$

$$\frac{dq_a}{dt} = u(q_a(t), t) \quad \text{and} \quad \frac{dp_a}{dt} = -\, p_a(t) \frac{\partial u(q_a(t), t)}{\partial q_a} \,. \tag{3}$$

The system of equations for the peakon parameters comprises a completely integrable canonical Hamiltonian system, whose solutions determine the positions $q_a(t)$ and amplitudes $p_a(t)$, for all N solitons, $a = 1, \dots, N$, and also describe the dynamics of their multi soliton interactions.

2.2 Singular momentum map version of the Stratonovich stochastic EPDiff equations

The objective of the remainder of the paper is to introduce stochasticity into the EPDiff equation and study its effects on the interactions of the peakon solutions of the CH equation. We consider the canonical Hamiltonian stochastic deformation (CH-SD), and also its special case, the parametric stochastic deformation (P-SD). To achieve our objective, we propose an action functional which contains a Stratonovich stochastic term, and treats q as an advected quantity, where the advection condition (the first equation in (3)) is enforced as a constraint with the help of the Lagrange multiplier p, and then prove the following theorem.

Theorem 1 (Canonical Hamiltonian Stochastic Deformation (CH-SD) of EPDiff).
The action $S(u, p, q)$ for the stochastic variational principle $\delta S = 0$ given by

$$S(u,p,q) = \underbrace{\int \left(\ell(u) + \sum_a \left\langle p_a \,,\, \frac{dq_a}{dt} - u(q_a, t) \right\rangle \right) dt}_{\text{Lebesgue integral}} - \underbrace{\int \sum_i h_i(q,p) \circ dW_i(t)}_{\text{Stratonovich integral}} \,, \tag{4}$$

leads to the following Stratonovich form of the *stochastic* EPDiff equation

$$\begin{aligned} dm &= -\pounds_u m \, dt + \sum_i \big\{ m \,,\, h_i(q,p) \big\} \circ dW_i(t) \,, \\ dq_a &= u(q_a, t) \, dt + \sum_i \big\{ q_a \,,\, h_i(q,p) \big\} \circ dW_i(t) \,, \\ dp_a &= -p_a(t) \frac{\partial u}{\partial x}(q_a, t) \, dt + \sum_i \big\{ p_a \,,\, h_i(q,p) \big\} \circ dW_i(t) \,, \end{aligned} \tag{5}$$

where the momentum density m and velocity u are given by

$$m(x,t) := \frac{\delta \ell}{\delta u} = \sum_{a=1}^{N} p_a \delta(x - q_a(t)) \,, \quad \text{and} \quad u(x,t) := \sum_{b=1}^{N} p_b K(x - q_b(t)). \tag{6}$$

Proof. Take the variations of the action integral (4), to find

$$
\begin{aligned}
\delta u: &\quad \frac{\delta \ell}{\delta u} - \sum_{a=1}^{N} p_a \delta(x - q_a(t)) = 0\,, \\
\delta p: &\quad dq_a - u(q_a, t)\,dt - \sum_i \frac{\partial h_i}{\partial p_a}(q,p) \circ dW_i(t) = 0\,, \\
\delta q: &\quad -\,dp_a - p_a(t)\frac{\partial u}{\partial x}(q_a, t)\,dt - \sum_i \frac{\partial h_i}{\partial q_a}(q,p) \circ dW_i(t) = 0\,,
\end{aligned}
\tag{7}
$$

after integrations by parts with vanishing endpoint and boundary conditions. The first variational equation captures the relation (6), and latter two equations in (7) produce the corresponding equations in (5). Substituting the latter two equations in (7) into the time derivative of the first one yields the first equation in (5).

The particular choice of the functions $h_i(q,p) = \sum_{a=1}^{N} p_a \xi_i(q_a)$ lead to the parameterised stochastic deformation (P-SD) of the peakon solutions. We summarise this observation in the following Corollary.

Corollary 1. [P-SD is a special case of CH-SD for EPDiff] Given the set of diffusivities $\xi_i(x)$, $i = 1, \ldots, M$, let $h_i(q,p) = \sum_{a=1}^{N} p_a \xi_i(q_a)$. Then the momentum density $m(x,t)$ satisfies the equation

$$
dm + \pounds_{dx_t} m = 0\,, \tag{8}
$$

where the stochastic vector field $dx_t(x)$ is given by the P-SD formula,

$$
dx_t(x) = u(x,t)\,dt + \sum_i \xi_i(x) \circ dW_i(t)\,. \tag{9}
$$

Proof. Specialise to $h_i(q,p) = \sum_{a=1}^{N} p_a \xi_i(q_a)$ in the first line of equation (5) in Theorem 1.

Remark 1 (Outlook: Comparing results for P-SD and CH-SD). In Section 3 and Section 5 we will investigate the effects of choosing between two slightly different stochastic potentials on the interaction of two peakons, $N = 2$, corresponding to P-SD and CH-SD. The two options are $h_i^{(1)}(q,p) = \sum_{a=1}^{N} p_a \xi_i(q_a)$ and $h_i^{(2)}(q,p) = \sum_{a=1}^{N} p_a \varphi_{ia}(q)$, respectively. These are both linear in the peakon momenta and in the simplest case they have constant coefficients. Although these two choices are very similar, they will produce quite different solution behaviour in our numerical simulations of peakon-peakon overtaking collisions in Section 5.

Remark 2 (Stratonovich stochastic EPDiff equations in one dimension).

1. In one spatial dimension, equation (8) becomes

$$
\big(\partial_t m + u m_x + 2 m u_x\big)dt + m_x \sum_i \xi_i(x) \circ dW_i(t) + 2m \sum_i \xi_i'(x) \circ dW_i(t) = 0\,. \tag{10}
$$

Importantly, the multiplicative noise multiplies both the solution and its gradient. The latter is not a common form for stochastic PDEs. In addition, both the spatial correlations $\xi_i(x)$ and their derivatives $\xi_i'(x)$ are involved.

2. The equations for dq_a and dp_a in (5) are stochastic canonical Hamiltonian equations (SCHEs) in the sense of Bismut [1, 11]. These equations for dq_a and dp_a may be rewritten as

$$\begin{aligned} dq_a &= \frac{\partial H}{\partial p_a}(q,p)\,dt + \sum_i \frac{\partial h_i}{\partial p_a}(q,p)\circ dW_i(t)\,, \\ dp_a &= -\frac{\partial H}{\partial q_a}(q,p)\,dt - \sum_i \frac{\partial h_i}{\partial q_a}(q,p)\circ dW_i(t)\,, \end{aligned} \tag{11}$$

with the deterministic Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{a,b} p_a p_b K(q_a - q_b)\,. \tag{12}$$

The stochastic canonical Hamilton equations in (11) can also be obtained by extremising the *phase-space action functional*

$$S\big[q(t),p(t)\big] = \int_0^T \Big(\sum_{a=1}^N p_a \dot{q}_a - H(q,p)\Big)\,dt - \int_0^T \sum_{i=1}^M h_i(q,p)\circ dW_i(t)\,. \tag{13}$$

This is the restriction of (4) to the submanifold defined by the Ansatz (6).

3 The Fokker-Planck equation

The stochastic process in (11) for $(q(t),p(t))$ can be described with the help of a transition density function $\rho(t,q,p;\bar{q},\bar{p})$ which represents the probability density that the process, initially in the state $(\bar{q},\bar{p})$, will reach the state (q,p) at time t. The transition density function satisfies the Fokker-Planck equation corresponding to (11) (see [6], [9]). Let us examine the form of this equation in the case of $h_i(q,p) = \sum_{a=1}^N p_a\beta_{ia}$, where $\beta_{ia} = \text{const}$. In that case the noise in (11) is additive, and the Stratonovich and Itô calculus yield the same equations of motion.

3.1 Single-pulson dynamics

Consider a single pulson ($N = 1$) subject to one-dimensional (i.e., $M = 1$) Wiener process, with the stochastic potential $h(q,p) = \beta p$, where β is a nonnegative real parameter. The stochastic Hamiltonian equations (11) take the form $dq = p\,dt + \beta\circ dW(t)$, $dp = 0$. The corresponding Fokker-Planck equation takes the form

$$\frac{\partial\rho}{\partial t} + p\frac{\partial\rho}{\partial q} - \frac{1}{2}\beta^2\frac{\partial^2\rho}{\partial q^2} = 0 \tag{14}$$

with the initial condition $\rho(0,q,p;\bar{q},\bar{p}) = \delta(q-\bar{q})\delta(p-\bar{p})$. This advection-diffusion equation is easily solved with the help of the fundamental solution for the heat equation, and the solution yields

$$\rho_\beta(t,q,p;\bar{q},\bar{p}) = \frac{1}{\beta\sqrt{2\pi t}} e^{-\frac{(q-\bar{q}-pt)^2}{2\beta^2 t}} \delta(p-\bar{p}). \tag{15}$$

This solution means that the initial momentum $\bar{p}$ is preserved. The position has a Gaussian distribution which widens with time, and whose maximum is advected with velocity $\bar{p}$.

3.2 Two-pulson dynamics

The dynamics of two interacting pulsons has been thoroughly studied and possesses interesting features (see [5], [8]). It is therefore intriguing to see how this dynamics is affected by the presence of noise. Consider $N = 2$ pulsons subject to a two-dimensional (i.e., $M = 2$) Wiener process, with the stochastic potentials $h_1(q,p) = \beta_1 p_1$ and $h_2(q,p) = \beta_2 p_2$, where $q = (q_1,q_2)$, $p = (p_1,p_2)$. The corresponding Fokker-Planck equation takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q_1}\big[a_1\rho\big] + \frac{\partial}{\partial q_2}\big[a_2\rho\big] + \frac{\partial}{\partial p_1}\big[a_3\rho\big] + \frac{\partial}{\partial p_2}\big[a_4\rho\big] - \frac{1}{2}\beta_1^2\frac{\partial^2\rho}{\partial q_1^2} - \frac{1}{2}\beta_2^2\frac{\partial^2\rho}{\partial q_2^2} = 0 \tag{16}$$

with the initial condition $\rho\big(0,q,p;\bar{q},\bar{p}\big) = \delta(q_1-\bar{q}_1)\delta(p_1-\bar{p}_1)+\delta(q_2-\bar{q}_2)\delta(p_2-\bar{p}_2)$, where

$$\begin{aligned} a_1(q,p) &= p_1 + p_2 K(q_1-q_2), \qquad & a_3(q,p) &= -p_1p_2K'(q_1-q_2),\\ a_2(q,p) &= p_2 + p_1 K(q_1-q_2), \qquad & a_4(q,p) &= p_1p_2K'(q_1-q_2). \end{aligned} \tag{17}$$

Despite its relatively simple structure, it does not appear to be possible to solve this equation analytically. It is nevertheless an elementary exercise to verify that the function

$$\rho(t,q_1,q_2,p_1,p_2;\bar{q}_1,\bar{q}_2,\bar{p}_1,\bar{p}_2) = \rho_{\beta_1}(t,q_1,p_1;\bar{q}_1,\bar{p}_1) + \rho_{\beta_2}(t,q_2,p_2;\bar{q}_2,\bar{p}_2), \tag{18}$$

where ρ_{β_i} is given by (15), satisfies (16) asymptotically as $q_1 - q_2 \longrightarrow \pm\infty$, assuming the Green's function and its derivative decay in that limit. This simple observation gives us an intuition that stochastic pulsons should behave like individual particles when they are far from each other, just like in the deterministic case. In order to study the stochastic dynamics of the collision of pulsons, we need to resort to Monte Carlo simulations.

In Section 4 we discuss our numerical algorithm, and in Section 5 we present the results of our numerical studies.

4 Stochastic variational integrator

Given the variational structure of the problem we have formulated in Theorem 1, it is natural to employ variational integrators for numerical simulations. For an extensive review of variational integrators we refer the reader to Marsden & West [13] and the references therein. Stochastic variational integrators were first introduced in Bou-Rabee & Owhadi [2]. These integrators were derived for Lagrangian systems using the Hamilton-Pontryagin variational principle. In our case, however, we find it more convenient to stay on the Hamiltonian side and use the discrete variational Hamiltonian mechanics introduced in Lall & West [10]. We combine the ideas of [2] and [10], and propose the following discretization of the phase-space action functional (13):

$$\begin{aligned} S_d = &\sum_{k=0}^{K-1} \bigg(\sum_{i=1}^{N} p_i^k \frac{q_i^{k+1}-q_i^k}{\Delta t} - H(q^{k+1},p^k) \bigg) \Delta t \\ &- \sum_{k=0}^{K-1} \sum_{m=1}^{M} \frac{h_m(q^k,p^k)+h_m(q^{k+1},p^{k+1})}{2} \Delta W_k^m, \end{aligned} \tag{19}$$

where $\Delta t = T/K$ is the time step, (q^k,p^k) denote the position and momentum at time $t_k = k\Delta t$, and $\Delta W_k^m \sim N(0,\Delta t)$ are independent normally distributed random variables for $m=1,\dots,M$ and $k=0,\dots,K-1$. Extremizing (19) with respect to q^k and p^k yields the following implicit stochastic variational integrator:

$$\begin{aligned} \frac{q_i^{k+1}-q_i^k}{\Delta t} &= \frac{\partial H}{\partial p_i}\big(q^{k+1},p^k\big) + \sum_{m=1}^{M} \frac{\partial h_m}{\partial p_i}\big(q^k,p^k\big) \frac{\Delta W_{k-1}^m + \Delta W_k^m}{2\Delta t}, \\ \frac{p_i^{k+1}-p_i^k}{\Delta t} &= -\frac{\partial H}{\partial q_i}\big(q^{k+1},p^k\big) - \sum_{m=1}^{M} \frac{\partial h_m}{\partial q_i}\big(q^{k+1},p^{k+1}\big) \frac{\Delta W_k^m + \Delta W_{k+1}^m}{2\Delta t}, \end{aligned} \tag{20}$$

for $i=1,\dots,N$. Knowing (q^k,p^k) at time t_k, the system above allows to solve for the position q^{k+1} and momentum p^{k+1} at the next time step. For increased computational efficiency, it is advisable to solve the first (nonlinear) equation for q^{k+1} first, and then the second equation for p^{k+1}. Assuming the stochastic potentials are of the form $h_i(q,p) = \sum_{a=1}^N p_a \varphi_{ia}(q)$, the second equation is a linear system for p^{k+1}, and in case $\varphi_{ia} = \text{const}$, it becomes an explicit update rule.

The integrator (20) is symplectic, and preserves momentum maps corresponding to (discrete) symmetries of the discrete Hamiltonian—for instance, if $H(q,p)$ and all $h_i(q,p)$ are translationally invariant, as in our simulations in Section 5, then the total momentum $\sum_{i=1}^N p_i$ is numerically preserved. The proof of these facts trivially follows from [2], keeping in mind that the momenta p_i and velocities $\dot{q}_i$ are related via the Legendre transform. By a straightforward application of the Stratonovich-Taylor expansion (see [9]), one can show that the integrator (20) has strong order of convergence 0.5, and weak order of convergence 1.

5 Numerical experiments

We performed numerical simulations of the rear-end collision of two pulsons for two different Green's functions, namely $K(q_1-q_2)=e^{-(q_1-q_2)^2}$ and $K(q_1-q_2)=e^{-2|q_1-q_2|}$. In the latter case, the corresponding pulsons are commonly called 'peakons'. We investigated the initial conditions $\bar{q}_1=0$, $\bar{q}_2=10$, $\bar{p}_2=1$ together with the following four initial values: $\bar{p}_1=8$, $\bar{p}_1=4$, $\bar{p}_1=2$, $\bar{p}_1=1$.

That is, we varied the initial momentum of the faster pulson. We perturbed the slower pulson by introducing a one-dimensional Wiener process with the stochastic potential $h(q,p)=\beta p_2$ (this corresponds to $\beta_1=0$, $\beta_2=\beta$ in Section 3.2). The pulsons were initially well-separated, so their initial evolution was described by (18). The parameter β was varied in the range $[0,6.5]$. We used the time step $\Delta t=0.02$, and for each choice of the parameters 50000 sample solutions were computed until the time $T=100$.

5.1 Sample paths and mean solutions

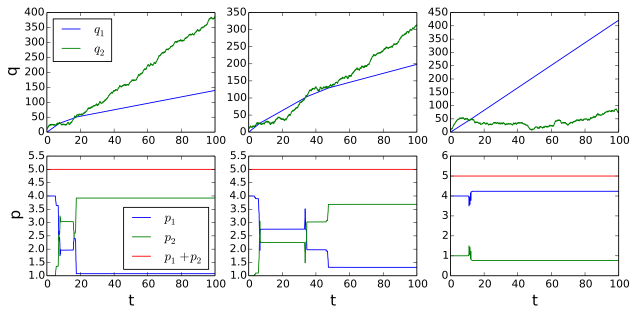

Fig. 1. Example numerical sample paths for Gaussian pulsons for the simulations with $\bar{p}_1=4$ and $\beta=4$. The positions are depicted in the plots in the upper row, and the corresponding momenta are shown in the plots in the lower row.

Figure 1 shows a few sample paths from the simulations of the interaction of Gaussian pulsons for the case with $\bar{p}_1=4$ and $\beta=4$. The simulations for $\bar{p}_1=8$ and $\bar{p}_1=2$, as well as the simulations for peakons, gave qualitatively similar results. The most striking feature is that the faster pulson/peakon may in fact cross the slower one. In the deterministic case one can show that the faster pulson can never pass the slower one—they just exchange their momenta. The proof relies on the fact that both the Hamiltonian and total momentum are preserved (see [5], [8]). In our case, however, the Hamiltonian (12) is not preserved due to the presence of the time-dependent noise, which allows much richer dynamics of the interactions. This may find interesting applications in landmark matching—see the discussion in Section 6.

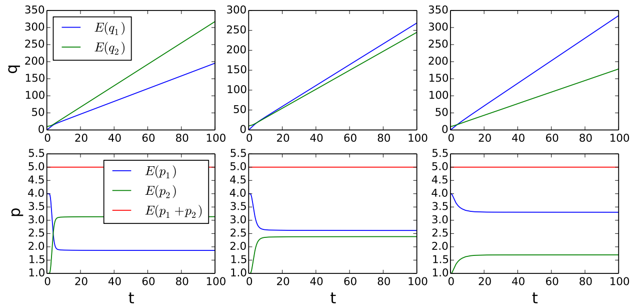

Fig. 2. Numerical mean paths for Gaussian pulsons for the simulations with $\bar{p}_1 = 4$. Results for three example choices of the parameter β are presented: $\beta = 1.5$ (*left*), $\beta = 2.5$ (*middle*), and $\beta = 4.5$ (*right*). The positions are depicted in the plots in the upper row, and the corresponding momenta are shown in the plots in the lower row.

Looking at Figure 1 we also note that our variational integrator exactly preserves the total momentum, as expected. Figure 2 depicts the mean solution for Gaussian pulsons with the initial condition $\bar{p}_1 = 4$ for different values of the noise intensity β. We see that for small noise the mean solution resembles the deterministic one, but as the parameter β is increased, the mean solution represents two pulsons passing through each other with increasingly less interaction. We study the probability of crossing in more detail in Section 5.2.

We observed that pulsons may cross even when they have the same initial momentum ($\bar{p}_1 = 1$). In the deterministic case they would just propagate in the same direction, retaining their relative distance.

5.2 Probability of crossing

We studied in more detail the distance between the pulsons $\Delta q(t) = q_2(t) - q_1(t)$ at the end of the simulation, that is, at time $t = 100$. The probability of crossing as a function of the noise intensity β is depicted in Figure 3. We see that this probability seems to approach unity for the simulations with $\bar{p}_1 > 1$, and 0.5 for $\bar{p}_1 = 1$.

5.3 Noise screening

In the numerical experiments described above we observed that the presence of noise causes pulsons to cross with a non-zero probability. The functions $q_1(t)$, $p_1(t)$, $q_2(t)$ and $p_2(t)$ define a transformation of the real line through (6). In the deterministic case this transformation is a diffeomorphism, but not when noise is added, since the crossing of pulsons introduces topological changes in the image of the real line under this transformation. This may be of interest in image matching, as in [14], when one would like to construct a deformation between two images which are not exactly diffeomorphic. However, with that application

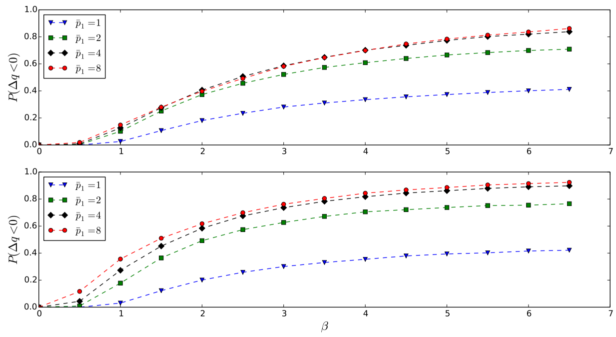

Fig. 3. The probability of crossing, that is, the probability that $q_2(t) < q_1(t)$ at time $t = 100$, as a function of the parameter β for Gaussian pulsons (*top*) and peakons (*bottom*).

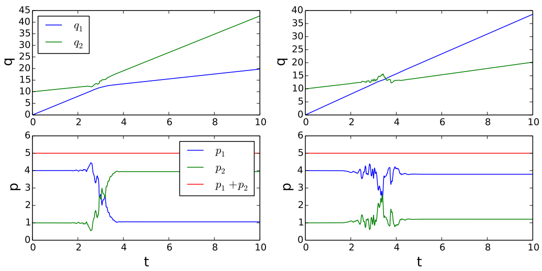

Fig. 4. Example numerical sample paths for Gaussian pulsons for the simulations with the initial conditions $\bar{q}_1 = 0$, $\bar{p}_1 = 4$, $\bar{q}_2 = 10$, and $\bar{p}_2 = 1$, and the stochastic potential $h(q,p) = \beta p_2 \exp(-(q_2 - q_1)^2)/\gamma$, with the parameters $\beta = 4$ and $\gamma = 4$. The positions are depicted in the plots in the upper row, and the corresponding momenta are shown in the plots in the lower row.

in mind, one may want to restrict the stochastic effects only to the situation when two pulsons get close to each other. This can be obtained by applying the stochastic potential $h(q,p) = \beta p_2 \exp(-(q_2 - q_1)^2)/\gamma$. The parameter β adjusts the noise intensity, just as before, while the parameter γ controls the range over which the stochastic effects are non-negligible. We performed a few simulations with this stochastic potential. A few sample paths are depicted in Figure 4. Note that this stochastic potential is translation-invariant, so the total momentum is preserved. As expected, our variational integrator preserves the total momentum exactly.

5.4 Restriction to parametric noise and additive noise in the momentum equation

Interestingly, crossing of pulsons does not seem to occur for the case of parametric stochastic deformation with the restriction $\varphi_{ai}(q) = \xi_i(q_a)$ as in Lemma 1. We ran numerical experiments for the potential $h(q,p) = \beta(p_1 + p_2)$, which has the form as in Lemma 1 with $\xi(x) = \beta$, but observed no interpenetration. We also did not observe crossing when the stochastic potential is independent of p. For instance, we performed simulations with the potential $h(q,p) = \beta q_2$. Such a potential results in additive noise in the momentum equation in (11) only, as in [14]. In many cases the pulsons would asymptotically approach each other, but never pass. We observed similar behavior for the (translationally invariant) potential $h(q,p) = \beta \exp(-(q_1 - q_2)^2/\gamma)$.

6 Summary and prospects

We have seen in Section 2 that the finite-dimensional peakon solutions for the EPDiff partial differential equation in one spatial dimension persist under both parametric stochastic deformation (P-SD) and canonical Hamiltonian stochastic deformations (CH-SD) of the EPDiff variational principle. We took advantage of the flexibility of CH-SD to study stochastic peakon-peakon collisions in which noise was introduced into *only one* of the peakon position equations (rather than symmetrically into both of the canonical position equations, as occurs with P-SD), while at the same time not introducing any noise into either of the corresponding canonical momentum equations. Our numerical experiments in Section 5 revealed that this type of noise allows the soliton-like singular peakon and pulson solutions of EPDiff to interpenetrate and change order on the real line, although this is not possible for the diffeomorphic flow represented by the solutions of the unperturbed deterministic EPDiff equation. This crossing of peakon paths was observed and its statistics were studied in detail. In contrast, crossing of peakon paths was *not* observed for the corresponding P-SD simulations in which the noise enters symmetrically in both position equations. Crossing of peakon paths was also not observed when stochasticity was added only in the canonical momentum equations, as studied in Trouvé and Vialard [14], where the authors considered the equations

$$dq_a = u(q_a, t)\, dt \quad \text{and} \quad dp_a = -p_a(t) \frac{\partial u}{\partial x}(q_a, t)\, dt + \sigma dW(t) \tag{21}$$

for stochastic landmark matching in computational anatomy. This perturbation corresponds to (11) with $h_1(q,p) = \sigma \sum_a q_a$, and enforces a stochastic Brownian force on the particles, rather than making particle paths stochastic. Trouvé and Vialard showed that this simple additive noise in the momentum equation is in general enough to account for correlations between points on the curve during landmark evolution under stochastic forcing. Our results in Section 5 demonstrated that noise in the position equations may additionally allow the landmarks to change their order on the line, thus allowing matching of two images which are not diffeomorphic.

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