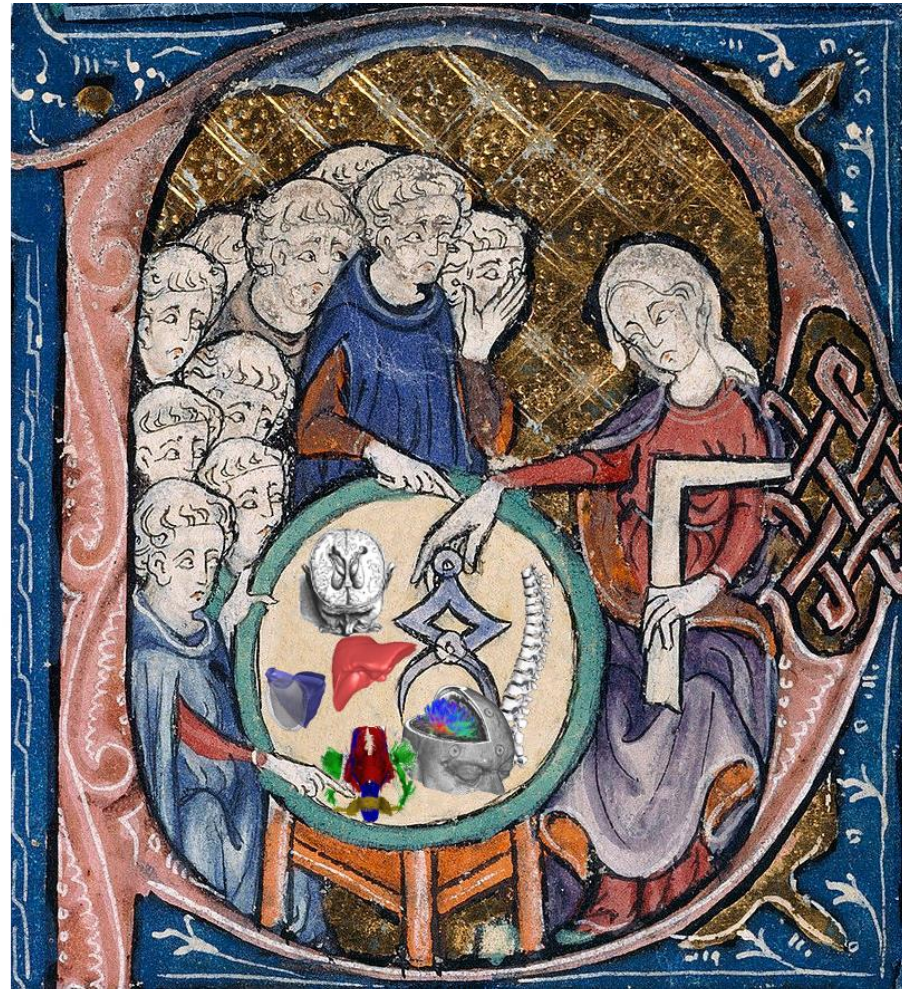
Xavier Pennec

Univ. Côte d'Azur and Inria, France

http://www-sop.inria.fr/asclepios/cours/Peyresq_2019/

Geometric Statistics

Mathematical foundations and applications in computational anatomy



Freely adapted from "Women teaching geometry", in Adelard of Bath translation of Euclid's elements, 1310.

3/ Metric and Affine Geometric Settings for Lie Groups

Ecole d'été de Peyresq, Jul 1-5 2019

Geometric Statistics

PhDs and Post-docs available

Epione e-patient / e-medicine

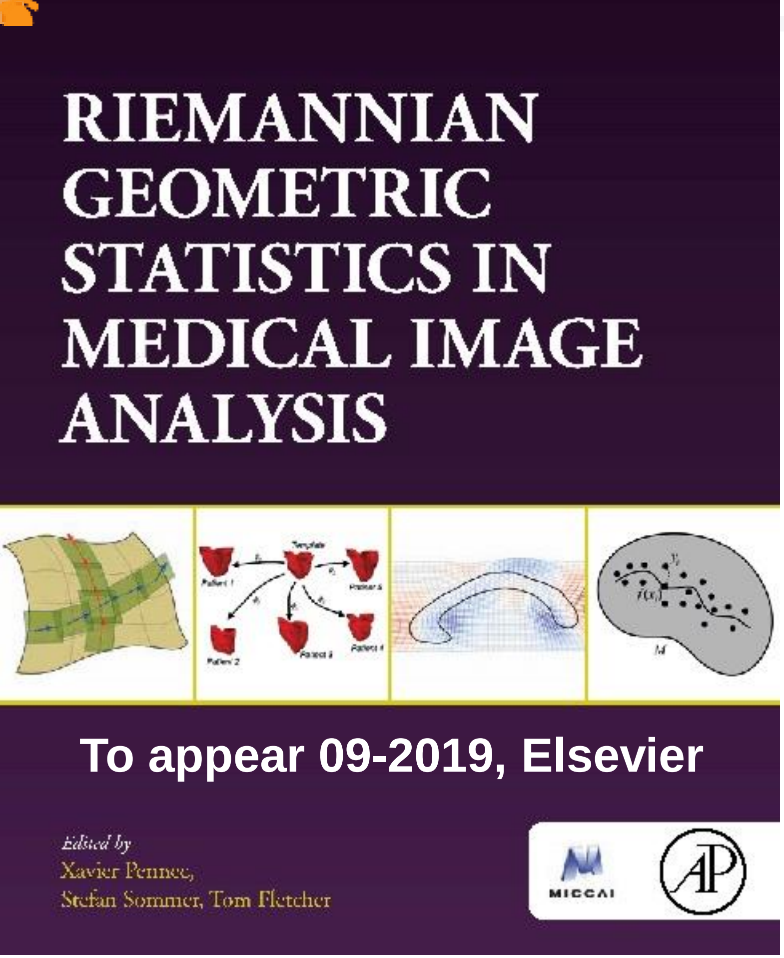
UNIVERSITÉ CÔTE D'AZUR

Inria

Supports for the course

http://www-sop.inria.fr/asclepios/cours/Peyresq_2019/

- 1/ Intrinsic Statistics on Riemannian Manifolds
  - Introduction to differential and Riemannian geometry. **Chapter 1**, RGSMIA. Elsevier, 2019.
  - Intrinsic Statistics on Riemannian Manifolds: Basic Tools for Geometric Measurements. JMIV 2006.
- 2/ SPD matrices and manifold-valued image processing
  - Manifold-valued image processing with SPD matrices. **Chapter 3** RGSMIA. Elsevier, 2019.
  - Historical reference: A Riemannian Framework for Tensor Computing. IJCV 2006.
- 3/ Metric and affine geometric settings for Lie groups
  - Beyond Riemannian Geometry The affine connection setting for transformation groups Chapter 5, RGSMIA. Elsevier, 2019.
- 4/ Parallel transport to analyze longitudinal deformations
  - Geodesics, Parallel Transport and One-parameter Subgroups for Diffeomorphic Image Registration. IJCV 105(2), November 2013.
  - Parallel Transport with Pole Ladder: a Third Order Scheme…[arXiv:1805.11436]
- 5/ Advanced statistics: central limit theorem and extension of PCA
  - Curvature effects on the empirical mean in Riemannian and affine Manifolds [arXiv:1906.07418]
  - Barycentric Subspace Analysis on Manifolds. Annals of Statistics. 46(6A):2711-2746, 2018. [arXiv:1607.02833]



Part 1: Foundations

- 1: Riemannian geometry [Sommer, Fetcher, Pennec]
- 2: Statistics on manifolds [Fletcher]
- 3: Manifold-valued image processing with SPD matrices [Pennec]
- 4: Riemannian Geometry on Shapes and Diffeomorphisms [Marsland, Sommer]
- **5: Beyond Riemannian: the affine connection setting for transformation groups [Pennec, Lorenzi]**

Part 2: Statistics on Manifolds and Shape Spaces

- 6: Object Shape Representation via Skeletal Models (s-reps) and Statistical Analysis [Pizer, Maron]
- 7: Inductive Fréchet Mean Computation on S(n) and SO(n) with Applications [Chakraborty, Vemuri]
- 8: Statistics in stratified spaces [Ferage, Nye]
- 9: Bias in quotient space and its correction [Miolane, Devilier,Pennec]
- 10: Probabilistic Approaches to Statistics on Manifolds: Stochastic Processes, Transition Distributions, and Fiber Bundle Geometry [Sommer]
- 11: Elastic Shape Analysis, Square-Root Representations and Their Inverses [Zhang, Klassen, Srivastava]

Part 3: Deformations, Diffeomorphisms and their Applications

- 13: Geometric RKHS models for handling curves and surfaces in Computational Anatomy : currents, varifolds, f-shapes, normal cycles [Charlie, Charon, Glaunes, Gori, Roussillon]
- 14: A Discretize-Optimize Approach for LDDMM Registration [Polzin, Niethammer, Vialad, Modezitski]
- 15: Spatially varying metrics in the LDDMM framework [Vialard, Risser]
- 16: Low-dimensional Shape Analysis In the Space of Diffeomorphisms [Zhang, Fleche, Wells, Golland]
- 17: Diffeomorphic density matching, Bauer, Modin, Joshi]

Geometric Statistics: Mathematical foundations and applications in computational anatomy

Intrinsic Statistics on Riemannian Manifolds

Manifold-Valued Image Processing

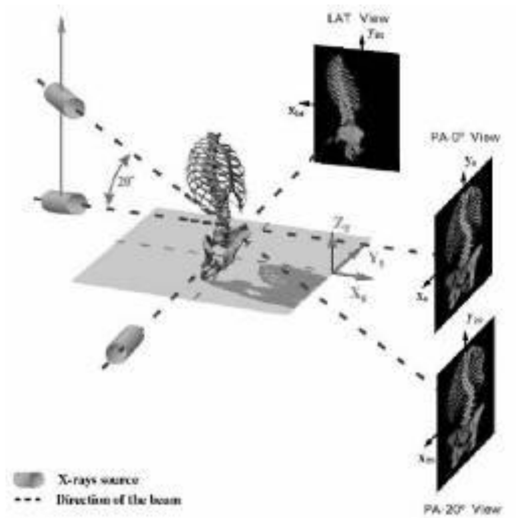
Metric and Affine Geometric Settings for Lie Groups

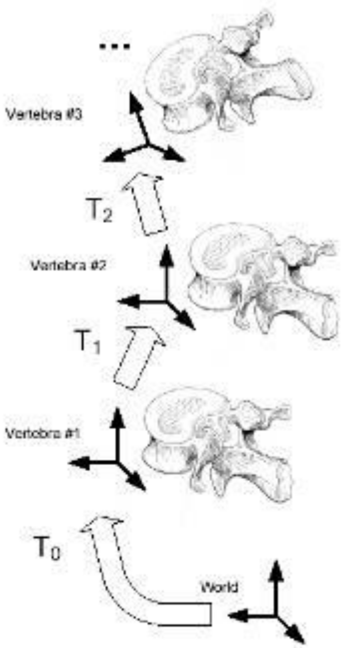
- Riemannian frameworks on Lie groups
- Lie groups as affine connection spaces
- Bi-invariant statistics with Canonical Cartan connection
- The SVF framework for diffeomorphisms

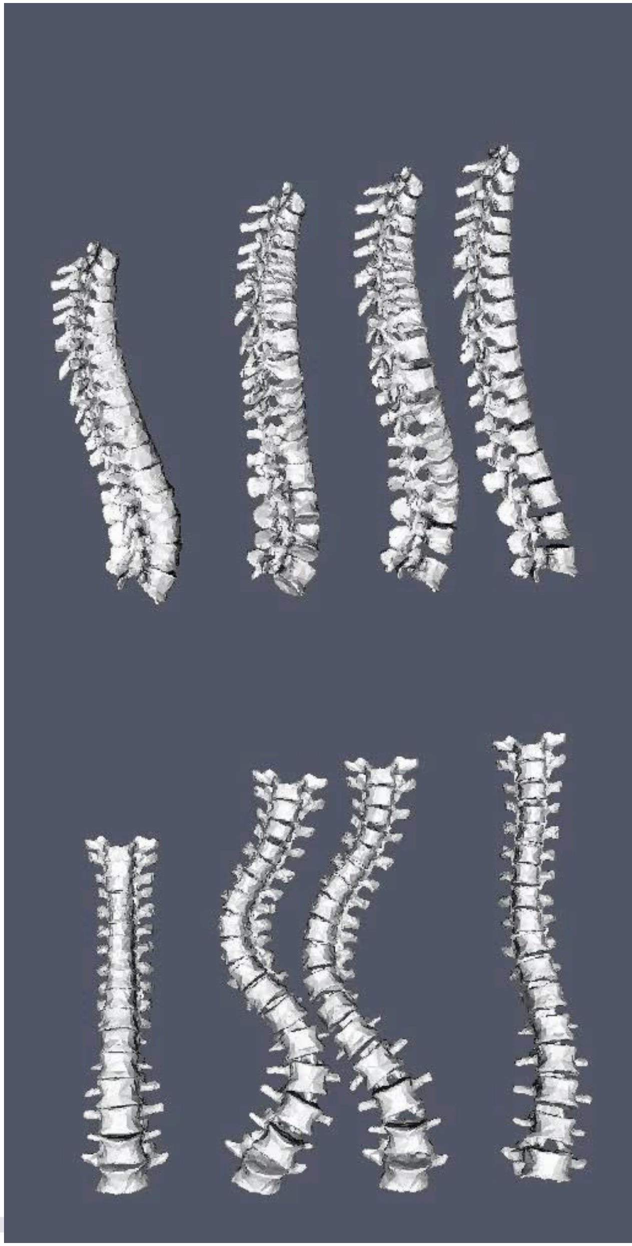
Parallel transport to analyze Longitudinal deformations

Advances Statistics: CLT & PCA

Statistical Analysis of the Scoliotic Spine







Data

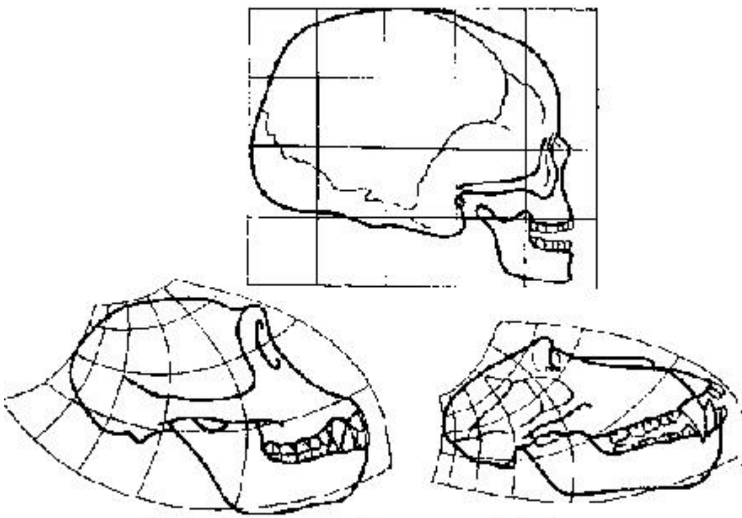
- 307 Scoliotic patients from the Montreal's St-Justine Hosp
- 3D Geometry from multi-planar X-rays
- Articulated model:17 relative pose of successive vertebras

Statistics

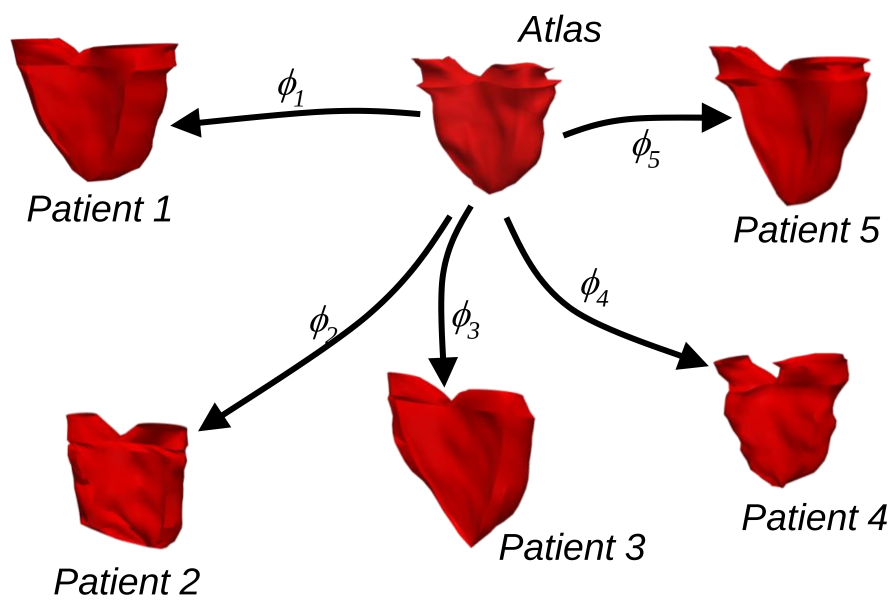
- Main translation variability is axial (growth?)
- Main rot. var. around anterior-posterior axis
- **4 first variation modes related to King's classes**

[J. Boisvert et al. ISBI'06, AMDO'06 and IEEE TMI 27(4), 2008]

Morphometry through Deformations



Skulls of a human, a chimpanzee and a baboon
and transformations between them



Measure of deformation [D'Arcy Thompson 1917, Grenander & Miller]

- Observation = "random" deformation of a reference template
- Deterministic template = anatomical invariants [Atlas ~ mean]
- Random deformations = geometrical variability [Covariance matrix]

Natural Riemannian Metrics on Transformations

Transformation are Lie groups: Smooth manifold G compatible with group structure

- Composition g o h and inversion g^{-1} are smooth
- Left and Right translation $L_g(f) = g \circ f \quad R_g(f) = f \circ g$
- Conjugation $Conj_g(f) = g \circ f \circ g^{-1}$
- Symmetry: $S_g(f) = g \circ f^{-1} \circ g$

Natural Riemannian metric choices

- Chose a metric at Id: $\langle x,y \rangle_{Id}$
- Propagate at each point g using left (or right) translation
  $\langle x,y \rangle_g = \langle DL_{g^{(-1)}}.x \, , DL_{g^{(-1)}}.y \rangle_{Id}$

Implementation

- Practical computations using left (or right) translations

$$\mathrm{Exp}_f(x) = f \circ \mathrm{Exp}_{Id}(DL_{f^{(-1)}}.x) \qquad \overrightarrow{fg} = \mathrm{Log}_f(g) = DL_f . \mathrm{Log}_{Id}(f^{(-1)} \circ g)$$

General Non-Compact and Non-Commutative case

No Bi-invariant Mean for 2D Rigid Body Transformations

- Metric at Identity: $dist(Id,(\theta;t_1;t_2))^2 = \theta^2 + t_1^2 + t_2^2$

- $T_1 = \left(\frac{\pi}{4}; -\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right) \quad T_2 = \left(0; \sqrt{2}; 0\right) \quad T_3 = \left(-\frac{\pi}{4}; -\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2}\right)$

- Left-invariant Fréchet mean: $(0;0;0)$
- Right-invariant Fréchet mean: $\left(0; \frac{\sqrt{2}}{3}; 0\right) \simeq (0; 0.4714; 0)$

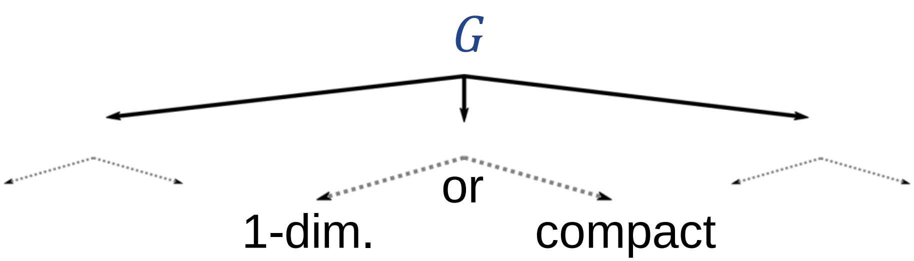
Questions for this talk:

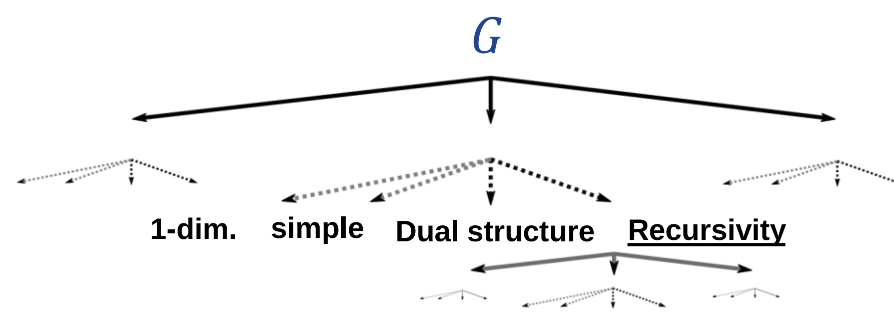
- **Can we design a mean compatible with the group operations?**
- **Is there a more convenient structure for statistics on Lie groups?**

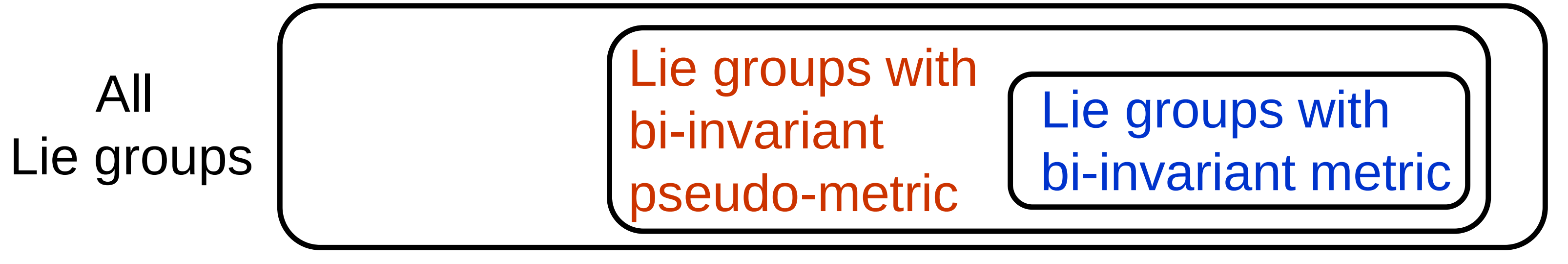
Existence of bi-invariant (pseudo) metrics

[Cartan 50's]:
Bi-invariant metric on G

[Medina, Revoy 80's]:
Bi-invariant pseudo-metric on G







[Miolane, Pennec, Computing Bi-Invariant Pseudo-Metrics on Lie Groups for Consistent Statistics. Entropy, 17(4):1850-1881, April 2015.]

- Algorithm: decompose the Lie algebra and find a bi-inv. pseudo-metric
- Test on rigid transformations SE(n): bi-inv. ps-metric for n=1 or 3 only

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Basics of Lie groups

Vector fields: algebra ΓG of derivations $\partial_X \emptyset$ of functions

- Lie bracket [X,Y]= $\partial_X \partial_Y - \partial_Y \partial_X$

Lie algebra of left invariant vector fields

- $DL\,\tilde{X} = \tilde{X} \circ L$ characterized by $\tilde{X}|_g = DL_g.x$ for $x \in T_e G$
- Lie bracket is also left-invariant
- Lie algebra: $\mathfrak{g} = (T_e G, +, \text{scal mult.}, [.,.])$
- Any VF can be decomposed on $\mathfrak{g}$ with scalar function components

Adjoint group

- Conjugation $\mathrm{Conj}_g(f) = g \circ f \circ g^{-1}$
- Adjoint representation $\mathrm{Ad}(g) = \mathrm{Dconj}_g: \mathfrak{g} \rightarrow GL(\mathfrak{g})$

  is compatible with composition: Ad(G) subgrp of $GL(\mathfrak{g})$

Basics of Lie groups (cont.)

Flow of a left invariant vector field $\tilde{X} = DL.x$ starting from e

- $\gamma_x(t)$ exists for all time
- One parameter subgroup: $\gamma_x(s+t) = \gamma_x(s).\gamma_x(t)$

Lie group exponential

- Definition: $x \in \mathfrak{g} \pi \rightarrow Exp(x) = \gamma_x(1) \epsilon G$
- Diffeomorphism from a a neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G (not true in general for inf. dim)
- Baker-Campbell Hausdorff (BCH) formula

$$BCH(x,y) = Log\big(Exp(x).Exp(y)\big) = x + y + \frac{1}{2}[x,y] + \ \ldots$$

3 curves at each point parameterized by the same tangent vector

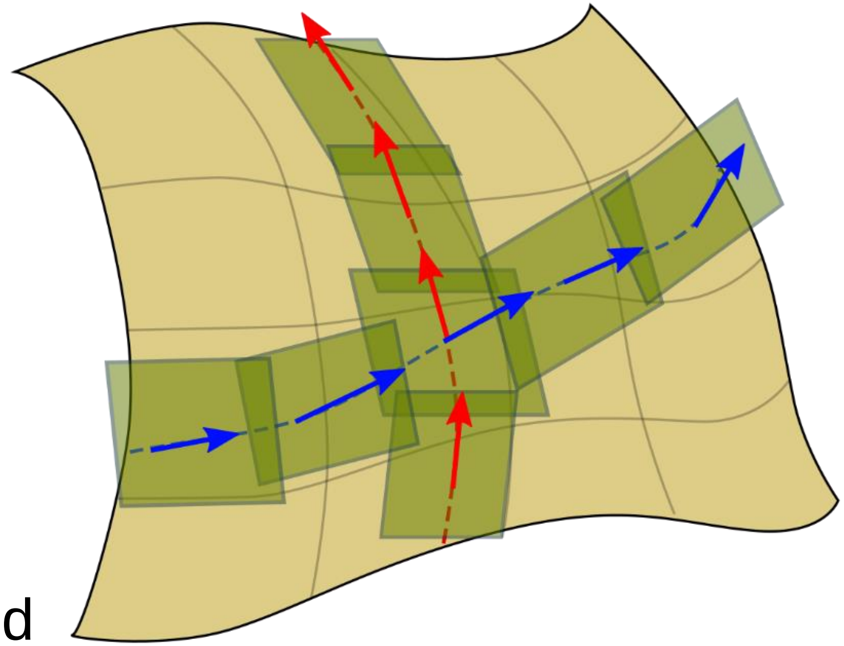
- Left / Right-invariant geodesics, one-parameter subgroups

Question: Can one-parameter subgroups be geodesics?

Affine connection spaces:
Drop the metric, use connection to define geodesics

Affine Connection (infinitesimal parallel transport)

- Acceleration = derivative of the tangent vector along a curve
- Projection of a tangent space on a neighboring tangent space



Geodesics = straight lines

- Null acceleration: $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$
- 2nd order differential equation: Normal coordinate system
- Local exp and log maps, well defined in a convex neighborhood

[Lorenzi, Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. Int. J. of Computer Vision, 105(2):111-127, 2013.]

Canonical connections on Lie groups

Left invariant connections $\nabla_{DL.X} DL.Y = DL.\nabla_X Y$

- Characteric bilinear form on the Lie algebra $a(x, y) = \nabla_{\tilde{X}} \tilde{Y}|_e \in \mathfrak{g}$
- Symmetric part $\frac{1}{2}(a(x, y) + a(y, x))$ specifies geodesics
- Skew symmetric part $\frac{1}{2}(a(x, y) - a(y, x))$ specifies torsion along them

Bi-invariant connections

- $a(Ad(g).x, Ad(g).y) = Ad(g).a(x, y)$
- $a([z, x], y) + a(x, [z, y]) = [z, a(x, y)]$, x, y, z in $\mathfrak{g}$

Cartan Schouten connections (def. of Postnikov)

- Left-Inv connections for which one-parameter subgroups are geodesics
  - Matrices : M(t) = exp(t.V)
  - Diffeos : **translations of Stationary Velocity Fields (SVFs)**
- Uniquely determined by $a(x, x) = 0$ (skew symmetry)

Canonical Cartan connections on Lie groups

Bi-invariant Cartan Schouten connections

- Family $a(x,y) = \lambda[x,y]$ (-,0, + connections for λ=0,1/2,1)
  - Turner Laquer 1992: exhaust all of them for compact simple Lie groups except SU(n) (2-dimensional family)
- Same group geodesics ($a(x,y) + a(y,x) = 0$):
  one-parameter subgroups and their left and right translations
- Curvature: $\mathrm{R}(x,y) = \lambda(\lambda - 1)[[x,y],z]$
- Torsion: $T(x,y) = 2a(x,y) - [x,y]$

Left/Right Cartan-Schouten Connection (λ=0/λ=1)

- Flat space with torsion (absolute parallelism)
- Left (resp. Right)-invariant vector fields are covariantly constant
- Parallel transport is left (resp. right) translation

Unique symmetric bi-invariant Cartan connection (λ=1/2)

- $a(x,y) = \frac{1}{2}[x,y]$
- Curvature $R(x,y)z = -\frac{1}{4}\left[[x,y],z\right]$
- Parallel transport along geodesics: $\Pi_{\exp(y)} x = DL_{\exp(\frac{y}{2})} . DR_{\exp(\frac{y}{2})}$.x

Cartan Connections are generally not metric

Levi-Civita Connection of a left-invariant (pseudo) metric is left-invariant

- Metric dual of the bracket $< ad^* (x, y), z > = < [x, z], y >$
- $a(x, y) = \frac{1}{2}[x, y] - \frac{1}{2}\left(ad^*(x, y) + ad^*(y, x)\right)$

Bi-invariant (pseudo) metric => Symmetric Cartan connection

- A left-invariant (pseudo) metric is right-invariant if it is Ad-invariant
  $$< x, y > = < Ad_g(x), Ad_g(y) >$$
- Infinitesimaly: $< [x, z], y > + < x, [y, z] > = 0$ or $ad^*(x, y) + ad^*(y, x) = 0$

Existence of bi-invariant (pseudo) metrics

- A Lie group admits a bi-invariant metric iff Ad(G) is relatively compact
  $Ad(G) \subset O(\mathfrak{g}) \subset GL(\mathfrak{g})$
  - No bi-invariant metrics for rigid-body transformations
- Bi-invariant pseudo metric (Quadratic Lie groups):
  Medina decomposition **[Miolane MaxEnt 2014]**
  - Bi-inv. pseudo metric for SE(n) for n=1 or 3 only

Group Geodesics

Group geodesic convexity

- Take an NCN V at Id
- There exists a NCN $V_g = g \circ V \cap V \circ g$ at each g s.t.:
  - $g \circ Exp(x) = Exp(Ad(g).x) \circ g$
  - $Log\left(g \circ h \circ g^{(-1)}\right) = Ad(g).Log(h)$

Group geodesics in V_g

- $Exp_g\,(v) = \ g \circ Exp\left(DL_{g^{(-1)}}.v\right) = Exp\left(DR_{g^{(-1)}}.v\right) \circ g$
- $Log_g\,(h) = DL_g.Log\left(g^{(-1)} \circ h\right) = DR_g.Log\left(h \circ g^{(-1)}\right)$

Canonical Affine Connections on Lie Groups

A unique Cartan-Schouten connection

- Bi-invariant and symmetric (no torsion)
- Geodesics through Id are one-parameter subgroups (group exponential)
  - Matrices : M(t) = A exp(t.V)
  - Diffeos : **translations of Stationary Velocity Fields (SVFs)**

Levi-Civita connection of a bi-invariant metric (if it exists)

- Continues to exists in the absence of such a metric (e.g. for rigid or affine transformations)

Symmetric space with central symmetry $\boldsymbol{S_\psi(\phi) = \psi\phi^{-1}\psi}$

- Matrix geodesic symmetry: $S_A\big(M(t)\big) = A\exp(-tV)A^{-1}A = M(-t)$

[Lorenzi, Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. Int. J. of Computer Vision, 105(2):111-127, 2013.]

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Mean value on an affine connection space

Fréchet / Karcher means not usable (no distance) but:

$$\mathrm{E}[\mathbf{x}] = \underset{y \in \mathrm{M}}{\operatorname{argmin}} \left(\mathrm{E}\left[\operatorname{dist}(y, \mathbf{x})^2 \right] \right) \quad \Rightarrow \quad \mathrm{E}\left[\overrightarrow{\bar{\mathrm{x}}\mathbf{x}} \right] = \int_{\mathrm{M}} \overrightarrow{\bar{\mathrm{x}}\mathbf{x}} . p_{\mathbf{x}}(z) . d\mathrm{M}(z) = 0 \quad [P(C) = 0]$$

Exponential barycenters

- [Emery & Mokobodzki 91, Corcuera & Kendall 99]

$$\int Log_x(y)\ \mu(dy) = 0 \quad \text{or} \quad \sum_i Log_x(y_i) = 0$$

- Existence? Uniqueness?
- OK for convex affine manifolds with semi-local convex geometry [Arnaudon & Li, Ann. Prob. 33-4, 2005]
  - Use a separating function (convex function separating points) instead of a distance
- Algorithm to compute the mean: fixed point iteration (stability?)

Bi-invariant Mean on Lie Groups

Exponential barycenter of the symmetric Cartan connection

- Locus of points where $\sum Log\left(m^{-1}.g_i\right) = 0$ (whenever defined)
- Iterative algorithm: $m_{t+1} = m_t \circ Exp\left(\frac{1}{n}\sum Log(m_t^{-1}.g_i)\right)$
- First step corresponds to the Log-Euclidean mean
- Corresponds to the first definition of bi-invariant mean of [V. Arsigny, X. Pennec, and N. Ayache. Research Report RR-5885, INRIA, April 2006.]

Mean is stable by left / right composition and inversion

- If m is a mean of $\{g_i\}$ and h is any group element, then
  - $h \circ m$ is a mean of $\{h \circ g_i\}$,
  - $m \circ h$ is a mean of the points $\{g_i \circ h\}$
  - and $m^{(-1)}$ is a mean of $\left\{g_i^{(-1)}\right\}$

[XP and V. Arsigny. Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups. In Matrix Information Geometry. 2012]

Bi-invariant Mean on Lie Groups

Fine existence

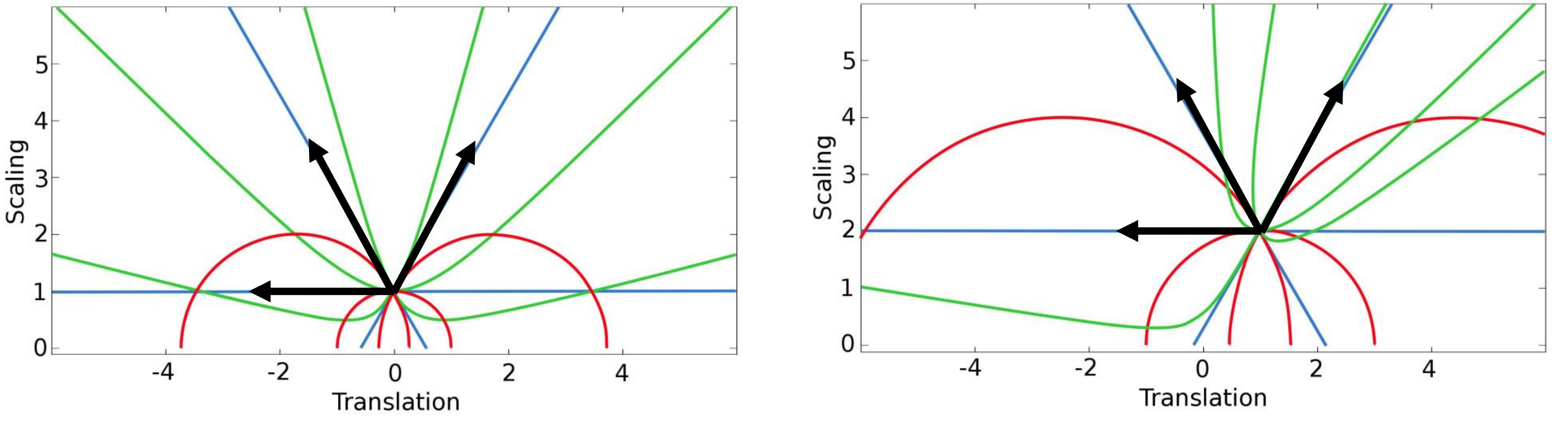
- If the data points belong to a sufficiently small normal convex neighborhood of some point, then there exists a unique solution in this NCN.
- Moreover, the iterated point strategy converges at least at a linear rate towards this unique solution, provided the initialization is close enough.
- Proof: using an auxiliary metric, the iteration is a contraction.

Closed-form for 2 points

- $m(t) = x \circ Exp\left(\frac{1}{2} Log\left(x^{(-1)} \circ y\right)\right)$

Scaling and translations ST(n)

- No bi-invariant metric
- Group geodesics defined globally, all points are reachable
- Existence and uniqueness of bi-invariant mean (closed form)



Group / left-invariant / right-invariant geodesics

Special matrix groups

Heisenberg Group (resp. Scaled Upper Unitriangular Matrix Group)

- No bi-invariant metric
- Group geodesics defined globally, all points are reachable
- Existence and uniqueness of bi-invariant mean (closed form resp. solvable)

Rigid-body transformations

- Logarithm well defined iff log of rotation part is well defined, i.e. if the 2D rotation have angles $|\theta_i| < \pi$
- Existence and uniqueness with same criterion as for rotation parts (same as Riemannian)

SU(n) and GL(n):

- log does not always exist (need 2 exp to cover)

Example mean of 2D rigid-body transformation

$$T_1 = \left(\frac{\pi}{4};\ -\frac{\sqrt{2}}{2};\frac{\sqrt{2}}{2}\right) \quad T_2 = (0;\ \sqrt{2};0\ ) \quad T_3 = \left(-\frac{\pi}{4};\ -\frac{\sqrt{2}}{2};-\frac{\sqrt{2}}{2}\right)$$

- Metric at Identity: $dist(Id,(\theta;t_1;t_2))^2 = \theta^2 + t_1^2 + t_2^2$
- Left-invariant Fréchet mean: $(0;0;0\ )$
- Log-Euclidean mean: $\left(0;\frac{\sqrt{2}-\pi/4}{3};0\right) \simeq (0;0.2096;0)$
- Bi-invariant mean: $\left(0;\frac{\sqrt{2}-\pi/4}{1+\pi/4(\sqrt{2}+1)};0\right) \simeq (0;0.2171;0)$
- Right-invariant Fréchet mean: $\left(0;\frac{\sqrt{2}}{3};0\right) \simeq (0;0.4714;0)$

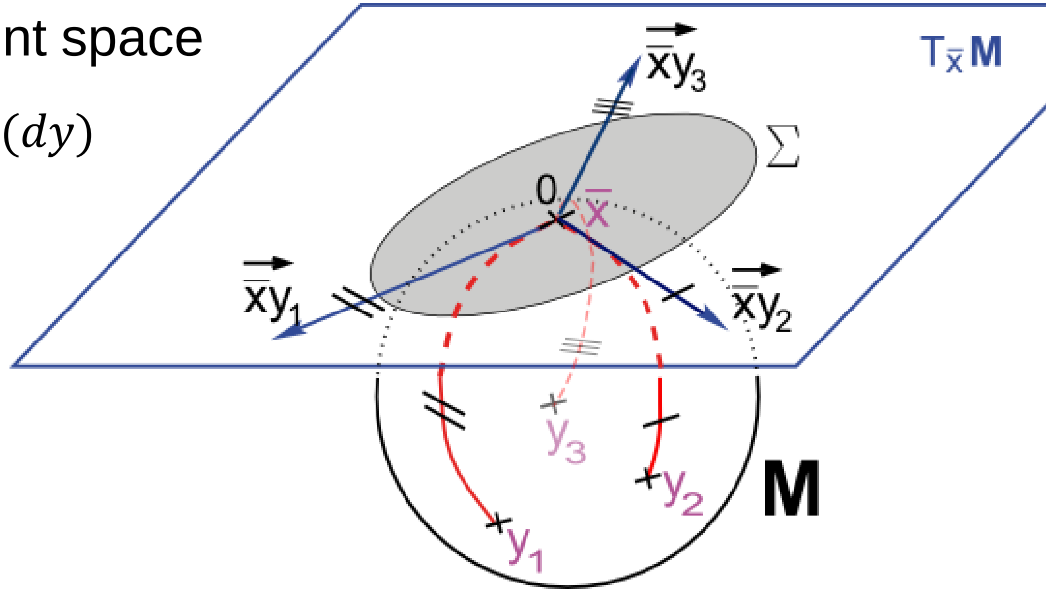
Generalization of the Statistical Framework

Covariance matrix & higher order moments

- Defined as tensors in tangent space

$$\Sigma = \int Log_x(y) \otimes Log_x(y)\, \mu(dy)$$

- Matrix expression changes according to the basis



Other statistical tools

- Mahalanobis distance well defined and bi-invariant

$$\mu_{(m,\Sigma)}(g) = \int [Log_m(g)]^i \Sigma_{ij}^{(-1)} [Log_m(g)]^j \mu(dy)$$

- ~~Tangent Principal Component Analysis (t-PCA)~~
- Principal Geodesic Analysis (PGA), provided a data likelihood
- Independent Component Analysis (ICA)

Cartan Connections vs Riemannian

What is similar

- Standard differentiable geometric structure [curved space without torsion]
- Normal coordinate system with Exp_x et Log_x [finite dimension]

Limitations of the affine framework

- No metric (but no choice of metric to justify)
- The exponential does always not cover the full group
  - Pathological examples close to identity in finite dimension
  - In practice, similar limitations for the discrete Riemannian framework

What we gain

- A globally invariant structure invariant by composition & inversion
- Simple geodesics, efficient computations (stationarity, group exponential)
- The simplest linearization of transformations for statistics?

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Riemannian Metrics on diffeomorphisms

Space of deformations

- Transformation y=ϕ(x)
- Curves in transformation spaces: ϕ(x,t)
- Tangent vector = speed vector field

$$v_t(x) = \frac{d\phi(x,t)}{dt}$$

Right invariant metric

$$\|v_t\|_{\phi_t} = \|v_t \circ \phi_t^{-1}\|_{Id}$$

- Eulerian scheme
- Sobolev Norm H_k or H_∞ (RKHS) in LDDMM → diffeomorphisms [Miller, Trouve, Younes, Holm, Dupuis, Beg… 1998 – 2009]

Geodesics determined by optimization of a time-varying vector field

- Distance

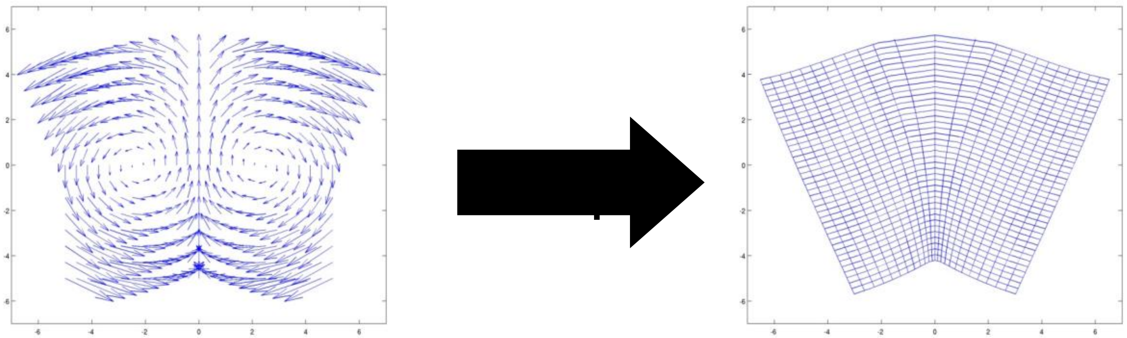
$$d^2(\phi_0, \phi_1) = \arg\min_{v_t} \left(\int_0^1 \|v_t\|_{\phi_t}^2 \, dt \right)$$

- Geodesics characterized by initial velocity / momentum
- Optimization for images is quite tricky (and lenghty)

The SVF framework for Diffeomorphisms

Idea: **[Arsigny MICCAI 2006, Bossa MICCAI 2007, Ashburner Neuroimage 2007]**

- Exponential of a smooth vector field is a diffeomorphism
- Parameterize deformation by ~~time-varying~~ Stationary Velocity Fields



Stationary velocity field — Diffeomorphism

Direct generalization of numerical matrix algorithms

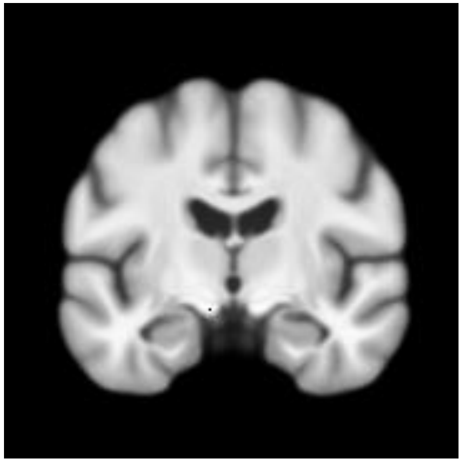
- Computing the deformation: **Scaling and squaring** **[Arsigny MICCAI 2006]**
  recursive use of exp(***v***)=exp(***v***/2) o exp(***v***/2)
- Computing the Jacobian: Dexp(v) = Dexp(v/2) o exp(v/2) . Dexp(v/2)
- Updating the deformation parameters: **BCH formula** **[Bossa MICCAI 2007]**

  $\exp(\boldsymbol{v}) \circ \exp(\varepsilon \boldsymbol{u}) = \exp(\boldsymbol{v} + \varepsilon \boldsymbol{u} + [\boldsymbol{v},\varepsilon \boldsymbol{u}]/2 + [\boldsymbol{v},[\boldsymbol{v},\varepsilon \boldsymbol{u}]]/12 + \dots)$

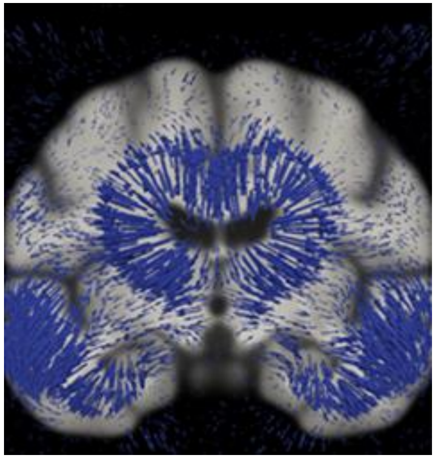
  - Lie bracket $[\boldsymbol{v},\boldsymbol{u}](p) = \mathrm{Jac}(\boldsymbol{v})(p).\boldsymbol{u}(p) - \mathrm{Jac}(\boldsymbol{u})(p).\boldsymbol{v}(p)$

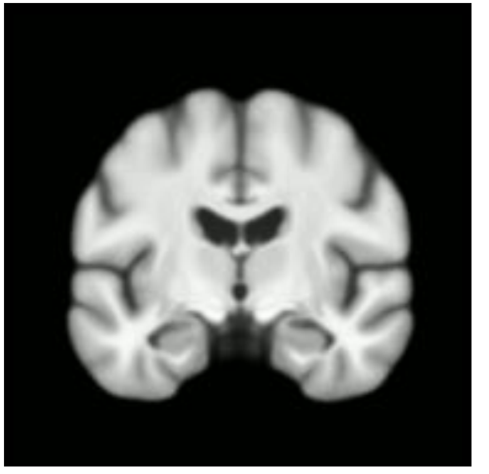
Measuring Temporal Evolution with deformations

Optimize LCC with deformation parameterized by SVF



$$\varphi_t(x) = exp(t.v(x))$$





https://team.inria.fr/asclepios/software/lcclogdemons/

[Lorenzi, Ayache, Frisoni, Pennec, Neuroimage 81, 1 (2013) 470-483]

The Stationnary Velocity Fields (SVF) framework for diffeomorphisms

- SVF framework for diffeomorphisms is algorithmically simple
- Compatible with "inverse-consistency"
- Vector statistics directly generalized to diffeomorphisms.

Registration algorithms using log-demons:

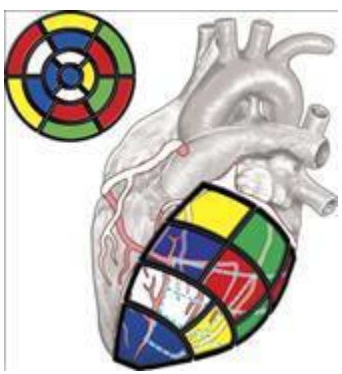
- Log-demons: Open-source ITK implementation (Vercauteren MICCAI 2008)
  http://hdl.handle.net/10380/3060
  [MICCAI Young Scientist Impact award 2013]
- Tensor (DTI) Log-demons (Sweet WBIR 2010):
  https://gforge.inria.fr/projects/ttk
- LCC log-demons for AD (Lorenzi, Neuroimage. 2013)
  https://team.inria.fr/asclepios/software/lcclogdemons/
- 3D myocardium strain / incompressible deformations (Mansi MICCAI'10)
- Hierarchichal multiscale polyaffine log-demons (Seiler, Media 2012)
  http://www.stanford.edu/~cseiler/software.html
  [MICCAI 2011 Young Scientist award]

A powerful framework for statistics

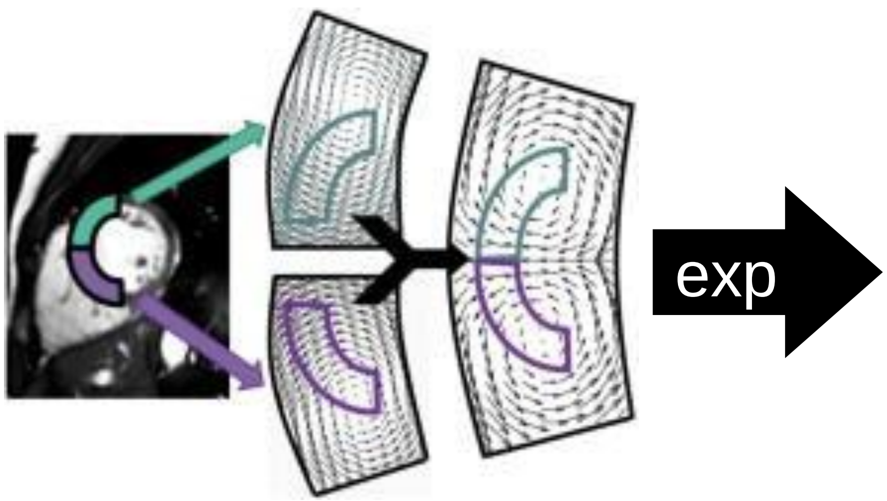
Parametric diffeomorphisms [Arsigny et al., MICCAI 06, JMIV 09]

- One affine transformation per region (polyaffines transformations)
- Cardiac motion tracking for **each subject** [McLeod, Miccai 2013]

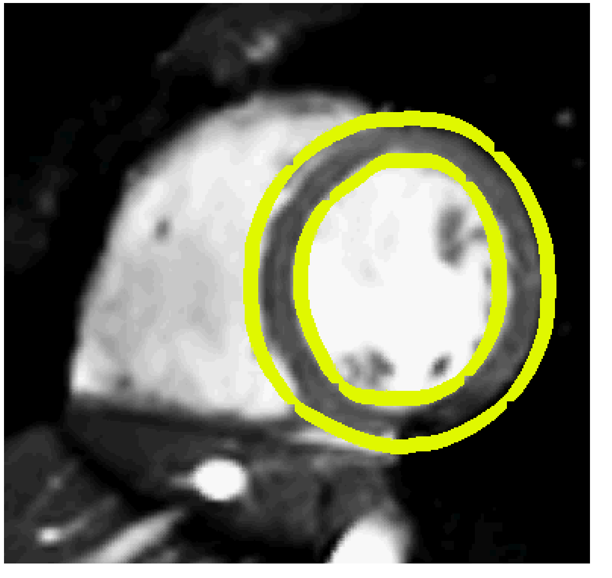
Log demons projected but with 204 parameters instead of a few millions



AHA regions



Stationary velocity fields



Diffeomorphism

A powerful framework for statistics

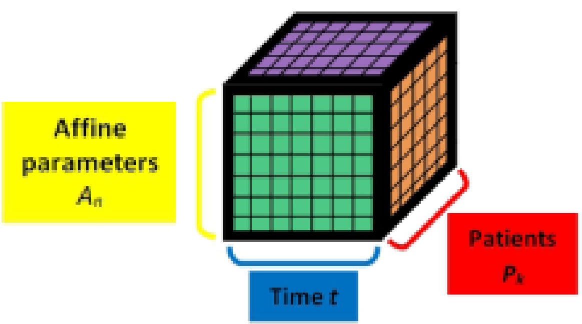
Parametric diffeomorphisms [Arsigny et al., MICCAI 06, JMIV 09]

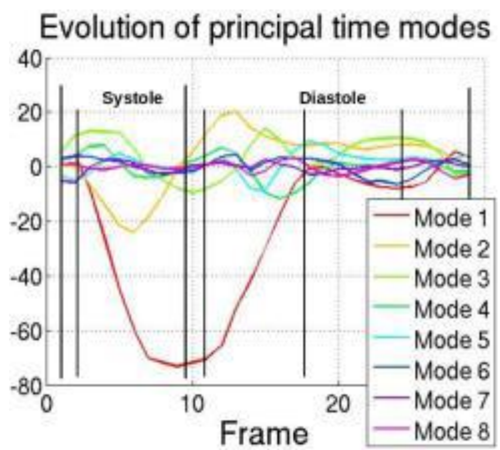
- One affine transformation per region (polyaffines transformations)
- Cardiac motion tracking for **each subject** [McLeod, Miccai 2013]

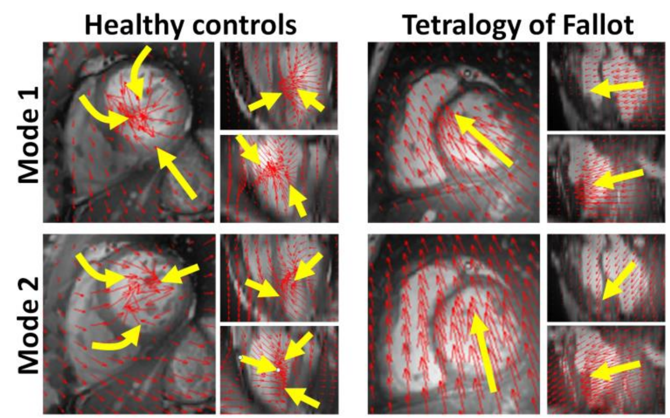
Log demons projected but with 204 parameters instead of a few millions

- **Group analysis** using tensor reduction : reduced model

8 temporal modes x 3 spatial modes = 24 parameters (instead of 204)



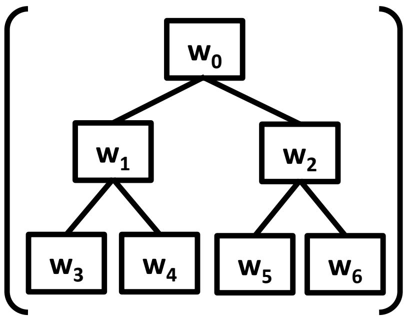


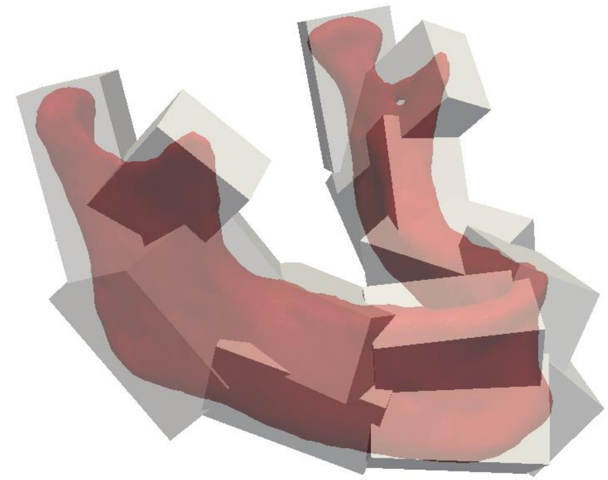


Hierarchical Deformation model

Population level:

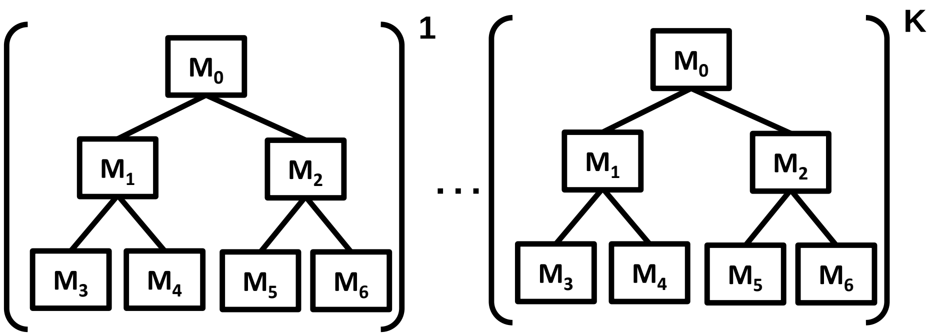
Spatial structure of the anatomy common to all subjects



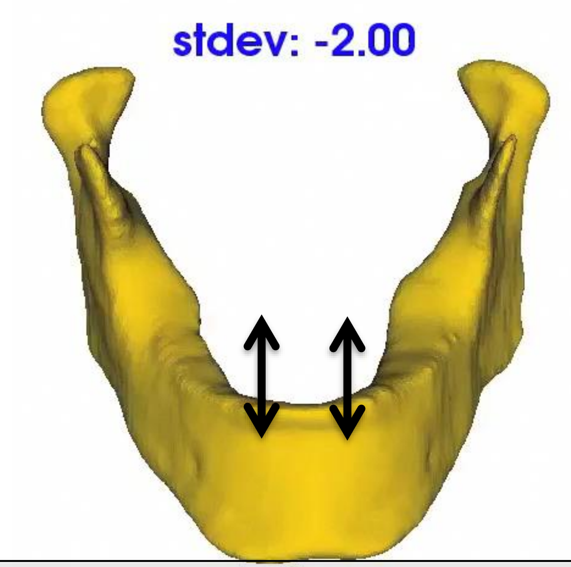


Subject level:

Varying deformation atoms for each subject



Aff(3) valued trees



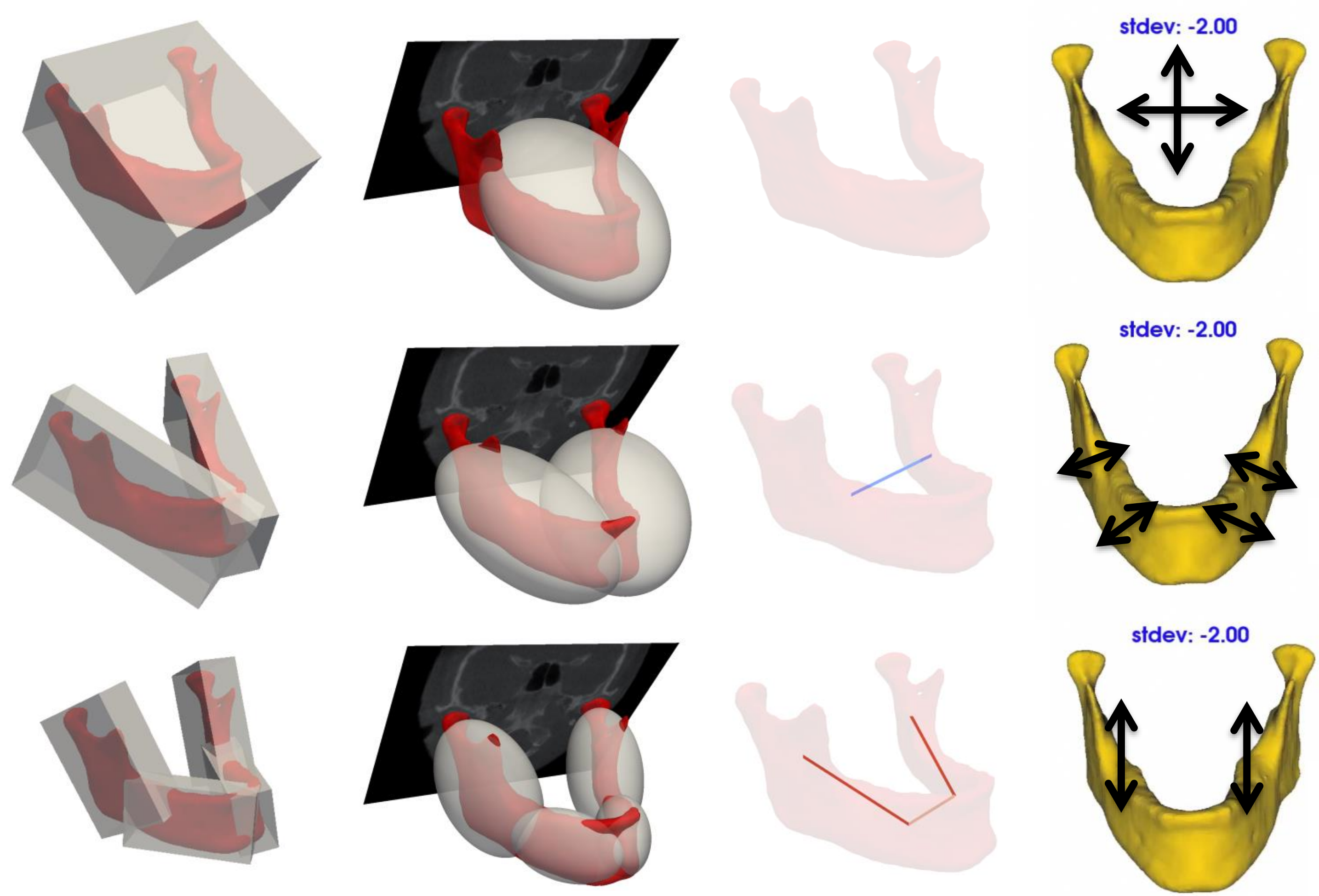
Hierarchical Estimation of the Variability

Oriented bounding boxes | Weights | Structure | First mode of variation

Level 0

Level 1

Level 2



Global scaling

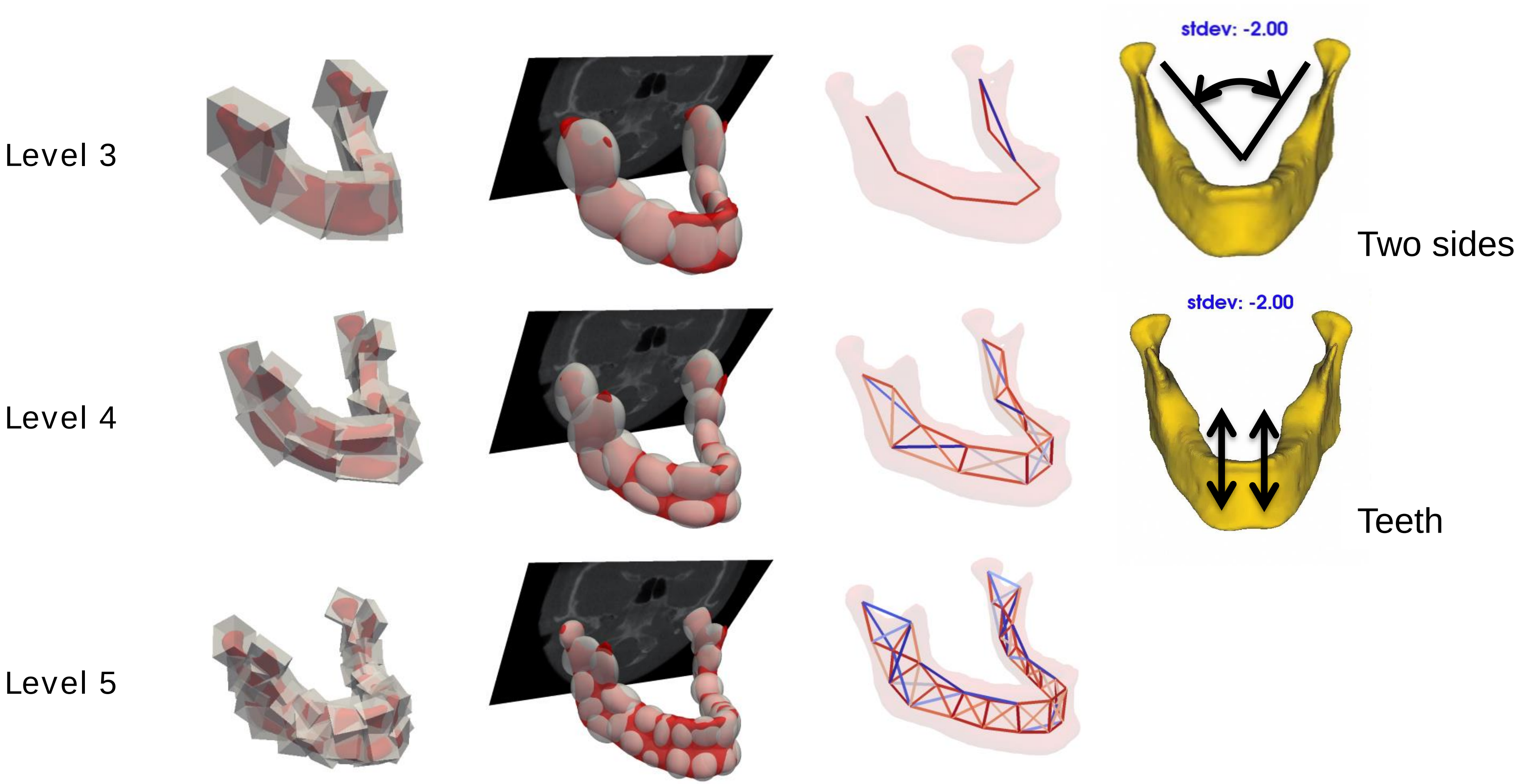
Thickness

Angle and ramus

47 subjects

[Seiler, Pennec, Reyes, Medical Image Analysis 16(7):1371-1384, 2012]

Hierarchical Estimation of the Variability



47 subjects **[Seiler, Pennec, Reyes, Medical Image Analysis 16(7):1371-1384, 2012]**

References for Statistics on Manifolds and Lie Groups

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- Xavier Pennec. Intrinsic Statistics on Riemannian Manifolds: Basic Tools for Geometric Measurements. Journal of Mathematical Imaging and Vision, 25(1):127-154, July 2006. http://www.inria.fr/sophia/asclepios/Publications/Xavier.Pennec/Pennec.JMIV06.pdf

Invariant metric on SPD matrices and of Frechet mean to define manifold-valued image processing algorithms

- Xavier Pennec, Pierre Fillard, and Nicholas Ayache. A Riemannian Framework for Tensor Computing. International Journal of Computer Vision, 66(1):41-66, Jan. 2006. http://www.inria.fr/sophia/asclepios/Publications/Xavier.Pennec/Pennec.IJCV05.pdf

Bi-invariant means with Cartan connections on Lie groups

- Xavier Pennec and Vincent Arsigny. Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups. In Frederic Barbaresco, Amit Mishra, and Frank Nielsen, editors, Matrix Information Geometry, pages 123-166. Springer, May 2012. http://hal.inria.fr/hal-00699361/PDF/Bi-Invar-Means.pdf

Cartan connexion for diffeomorphisms:

- Marco Lorenzi and Xavier Pennec. Geodesics, Parallel Transport & One-parameter Subgroups for Diffeomorphic Image Registration. International Journal of Computer Vision, 105(2), November 2013 https://hal.inria.fr/hal-00813835/document