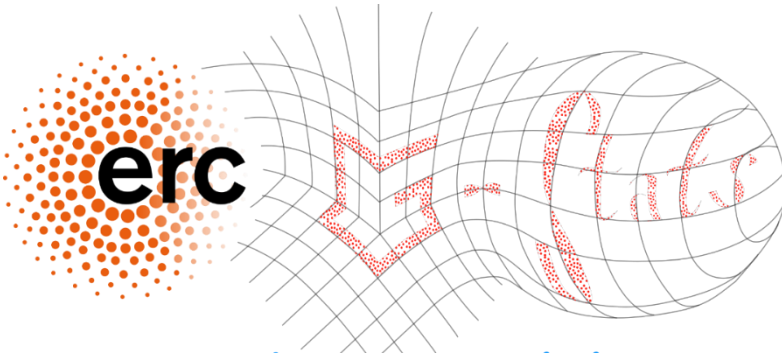


Xavier Pennec

Univ. Côte d'Azur and Inria, France



http://www-sop.inria.fr/asclepios/cours/Peyresq_2019/

Geometric Statistics

*Mathematical foundations
and applications in
computational anatomy*



Freely adapted from “Women teaching geometry”, in
Adelard of Bath translation of Euclid’s elements, 1310.

2/ Manifold-valued image processing

Ecole d’été de Peyresq, Jul 1-5 2019

Epione
e-patient / e-medicine

UNIVERSITÉ
CÔTE D’AZUR

Inria

Geometric Statistics: Mathematical foundations and applications in computational anatomy

Intrinsic Statistics on Riemannian Manifolds

Manifold-Valued Image Processing

Metric and Affine Geometric Settings for Lie Groups

Parallel Transport to Analyze Longitudinal Deformations

Advances Statistics: CLT & PCA

Geometric Statistics: Mathematical foundations and applications in computational anatomy

Intrinsic Statistics on Riemannian Manifolds

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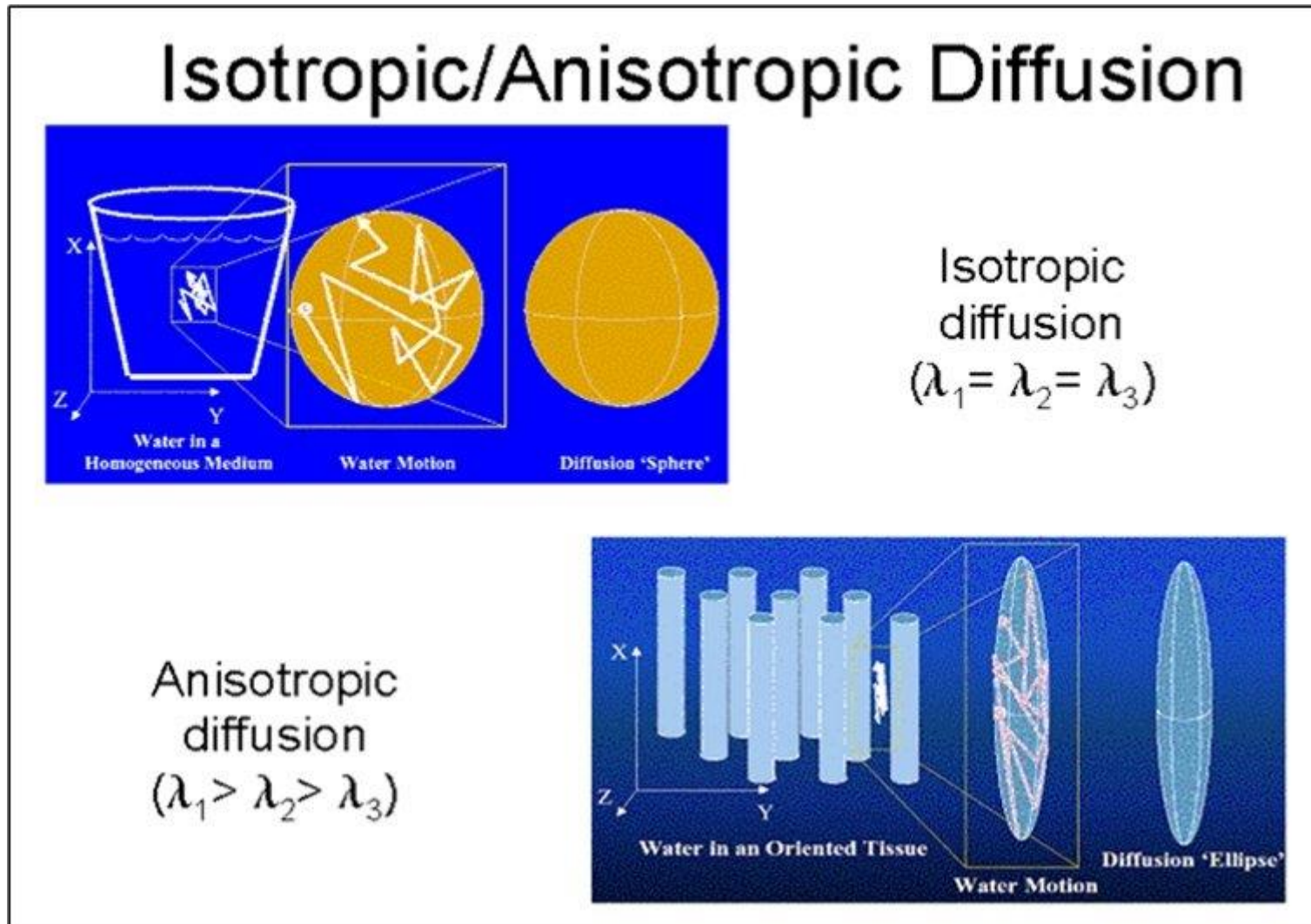
- Introduction to diffusion tensor imaging
- Interpolation, filtering, extrapolation of tensors
- Other metrics and applications

Metric and Affine Geometric Settings for Lie Groups

Parallel Transport to Analyze Longitudinal Deformations

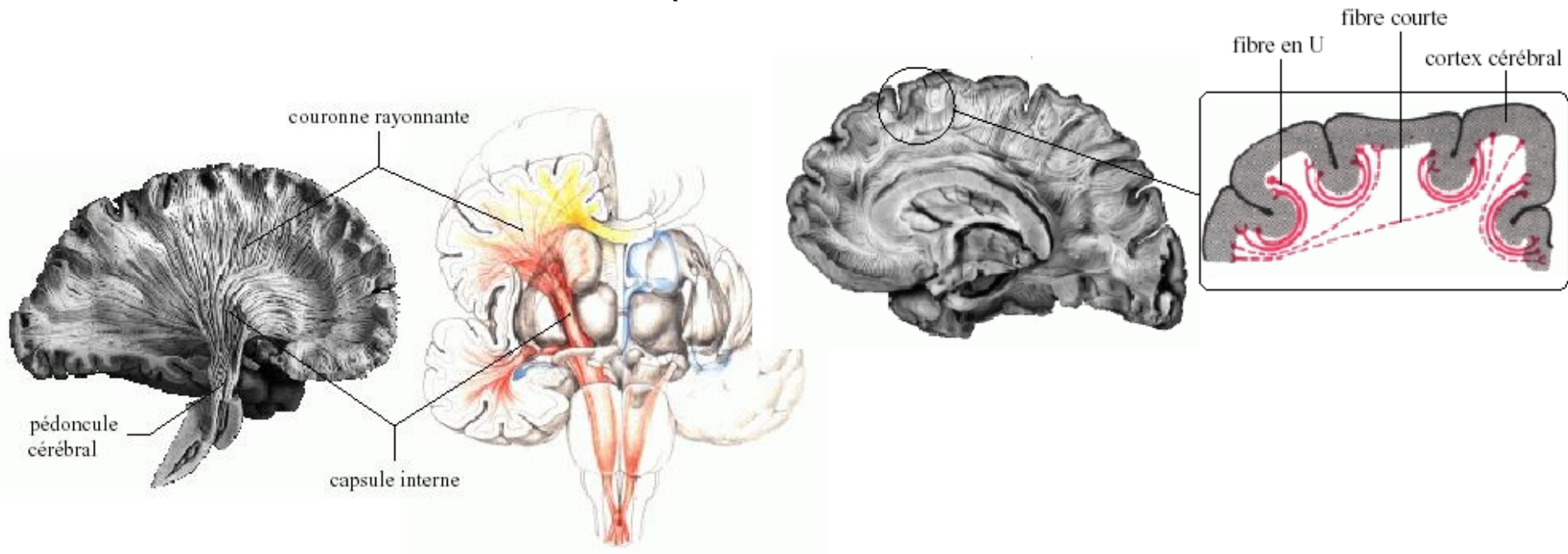
Advances Statistics: CLT & PCA

Introduction to diffusion tensor imaging (DTI)



Introduction to diffusion tensor imaging (DTI)

- Classical MRI:
 - white matter, grey matter, CSF
- Diffusion MRI:
 - MR technique born in the mid-80ies (Basser, LeBihan).
 - in vivo imaging of the white matter architecture
 - set of nervous fibers (axons): information highways of the brain!

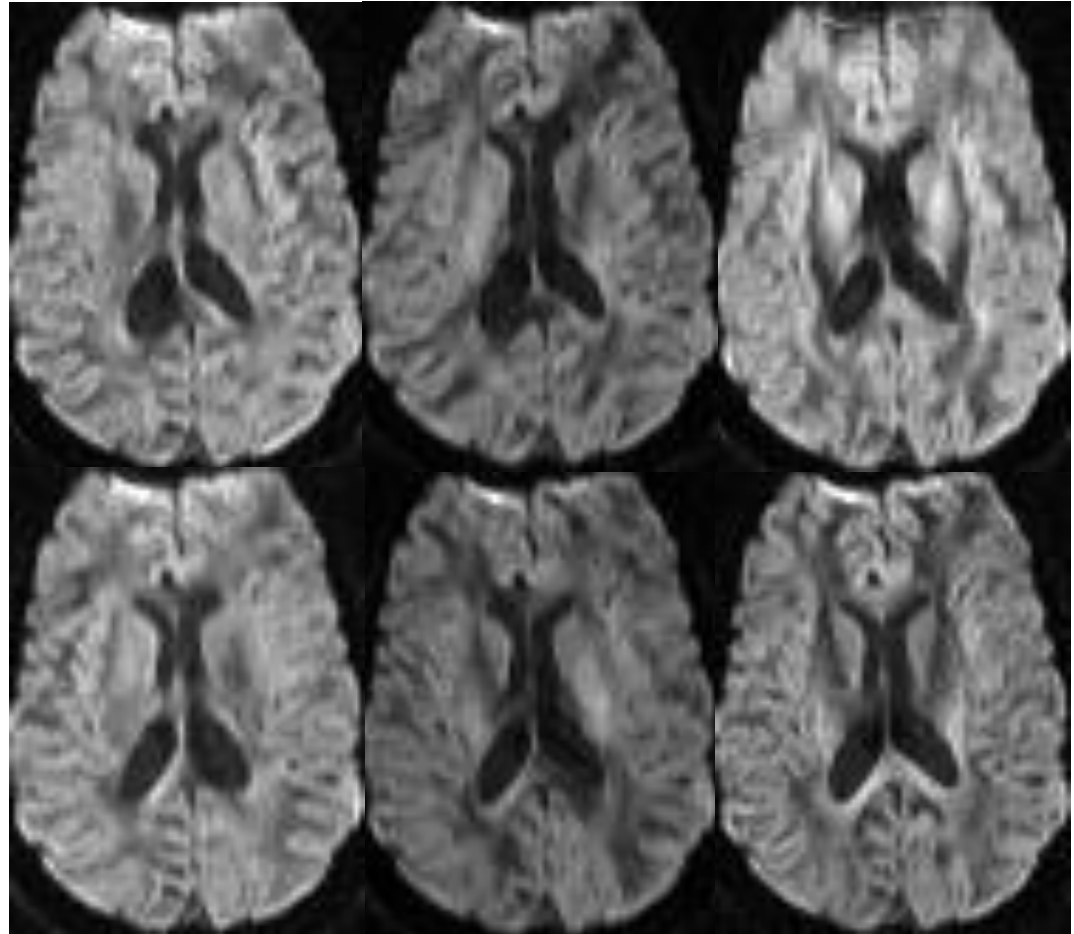


Anatomy of a diffusion MRI



T2 image

+



6 diffusion weighted images

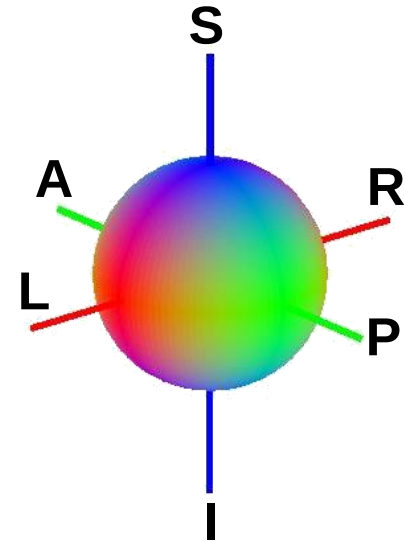
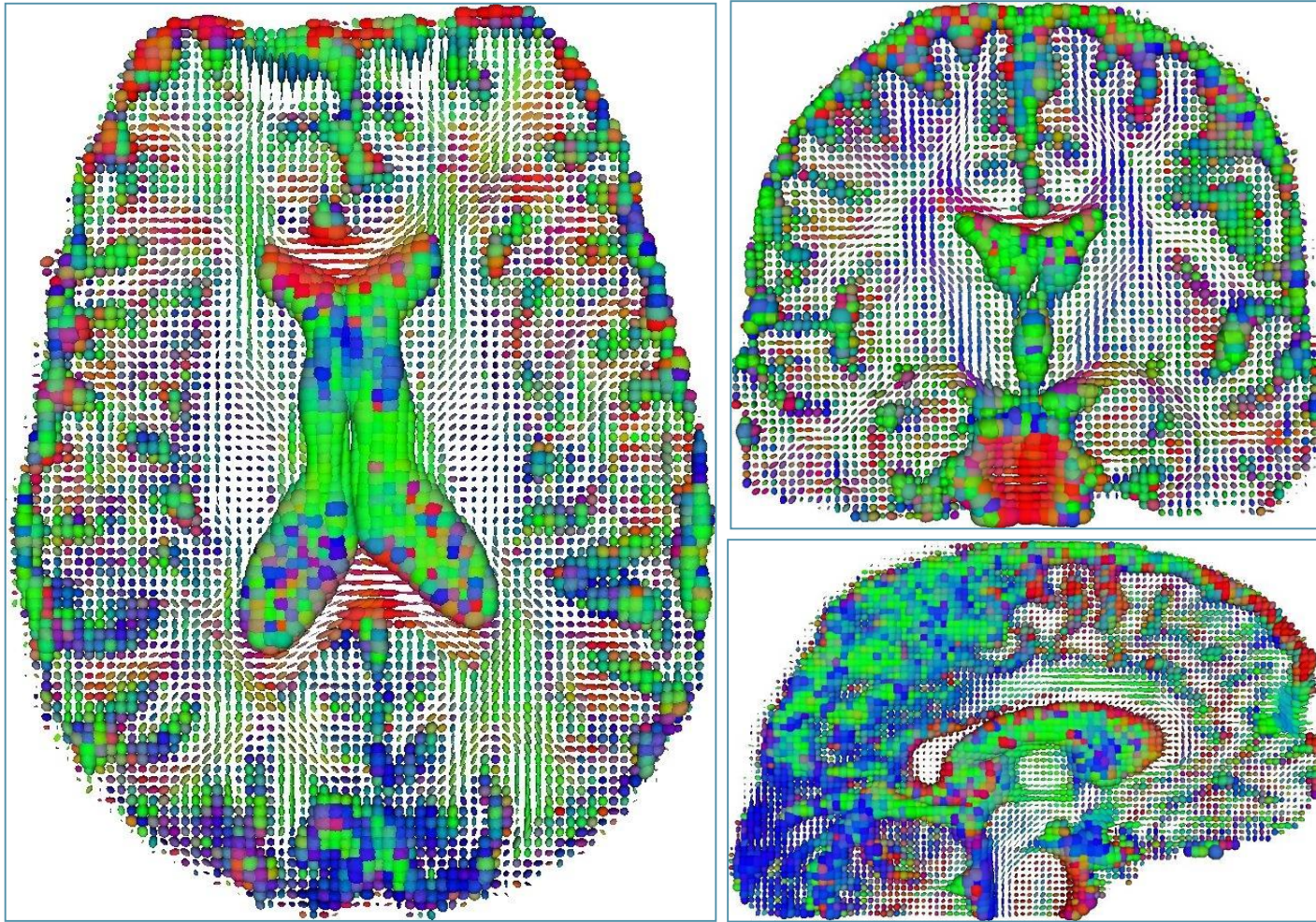
Stejskal & Tanner equation

Signal attenuation related to tensor using:

$$S_i = S_0 \exp(-b \cdot \overrightarrow{g_i}^t \cdot D \cdot \overrightarrow{g_i})$$

- S_i : diffusion weighted images;
- S_0 : T2 image;
- g_i : spatial direction of the diffusion gradient;
- b : b-value (related to physical parameters of the acquisition, including field strength and diffusion time);
- D : diffusion tensor;

Visualization using ellipsoids



Diffusion Tensor Imaging

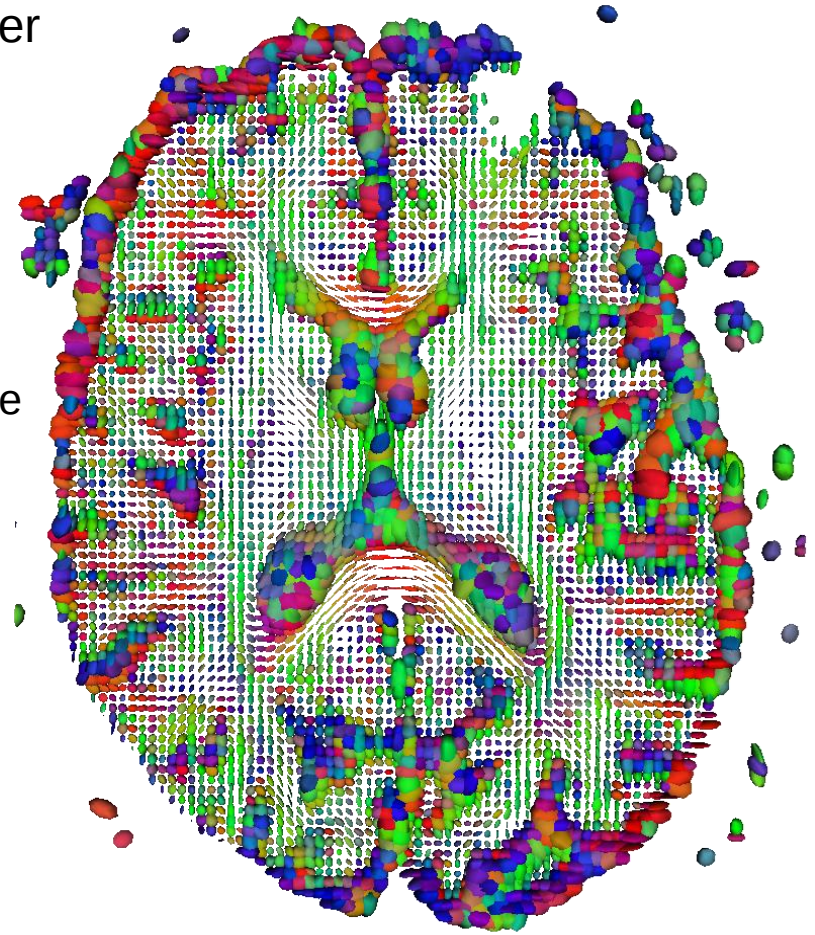
Covariance of the Brownian motion of water
-> Architecture of axonal fibers

Symmetric positive definite matrices

- Cone: Convex operations are stable
 - mean, interpolation
 - Null eigenvalues are reachable in a finite time : not physical!
- More complex operations are not
 - PDEs, gradient descent...

SPD-valued image processing

- Clinical images: Very noisy data
- Robust estimation
- Filtering, regularization
- Interpolation / extrapolation



Diffusion Tensor Field
(slice of a 3D volume)

Intrinsic computing on Manifold-valued images?

Geometric Statistics: Mathematical foundations and applications in computational anatomy

Intrinsic Statistics on Riemannian Manifolds

Manifold-Valued Image Processing

- Introduction to diffusion tensor imaging
- **Interpolation, filtering, extrapolation of tensors**
- Other metrics and applications

Metric and Affine Geometric Settings for Lie Groups
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GL(n)-Invariant Metrics on SPD matrices

Action of the Linear Group GL_n

$$A * \Sigma = A.\Sigma.A^T$$

Invariant metric $\langle W_1 | W_2 \rangle_{\Sigma} = \langle AW_1A^t | AW_2A^t \rangle_{A\Sigma A^t} \stackrel{def}{=} \langle \Sigma^{-1/2} * W_1, \Sigma^{-1/2} * W_2 \rangle_{Id}$

- Isotropy group at the identity: Rotations
- All rotationally invariant scalar products:

$$\langle W_1 | W_2 \rangle_{Id} \stackrel{def}{=} \text{Tr}(W_1^T W_2) + \beta \text{Tr}(W_1) \cdot \text{Tr}(W_2) \quad (\beta > -1/n)$$

- Geodesics at Id $\Gamma_{Id,W}(t) = \exp(tW)$

- Exponential map $Exp_{\Sigma}(\overrightarrow{\Sigma\Psi}) = \Sigma^{1/2} \exp(\Sigma^{-1/2} \cdot \overrightarrow{\Sigma\Psi} \cdot \Sigma^{-1/2}) \Sigma^{1/2}$

- Log map $\overrightarrow{\Sigma\Psi} = Log_{\Sigma}(\Psi) = \Sigma^{1/2} \log(\Sigma^{-1/2} \cdot \Psi \cdot \Sigma^{-1/2}) \Sigma^{1/2}$

- Distance

$$dist(\Sigma, \Psi)^2 = \langle \overrightarrow{\Sigma\Psi} | \overrightarrow{\Sigma\Psi} \rangle_{\Sigma} = \left\| \log(\Sigma^{-1/2} \cdot \Psi \cdot \Sigma^{-1/2}) \right\|_{Id}^2$$

GL(n)-Invariant Metrics on Tensors

$$\|W\|_{\Sigma}^2 = \text{Tr}(W \cdot \Sigma^{-1} W \Sigma^{-1}) + \beta \text{Tr}(W \Sigma^{-1})^2 \quad (\beta > -1/n)$$

Space of Gaussian distributions (Information geometry) ($\beta=0$)

- Fisher information metric [Burbea & Rao J. Multivar Anal 12 1982, Skovgaard Scand J. Stat 11 1984, Calvo & Oller Stat & Dec. 9 1991]
- Tensor segmentation [Lenglet RR04 & JMIV 25(3) 2006]

Affine-invariant metrics for DTI processing ($\beta=0$)

- [Pennec, Fillard, Ayache, IJCV 66(1), Jan 2006 / INRIA RR-5255, 2004]
- PGA on tensors [Fletcher & Joshi CVMIA04, SigPro 87(2) 2007]

Geometric means ($\beta=0$)

- Covariance matrices in computer vision [Forstner TechReport 1999]
- Math. properties [Moakher SIAM J. Matrix Anal App 26(3) 2004]
- Geodesic Anisotropy [Batchelor MRM 53 2005]

Homogeneous Embedding ($\beta=-1/(n+1)$)

- [Lovric & Min-Oo, J. Multivar Anal 74(1), 2000]

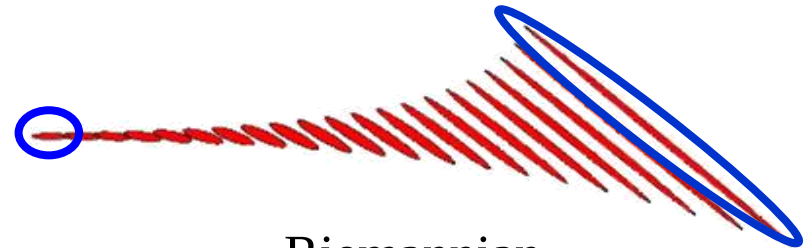
Tensor interpolation

Geodesic walking in 1D

$$\Sigma(t) = \exp_{\Sigma_1}(\overrightarrow{t\Sigma_1\Sigma_2})$$

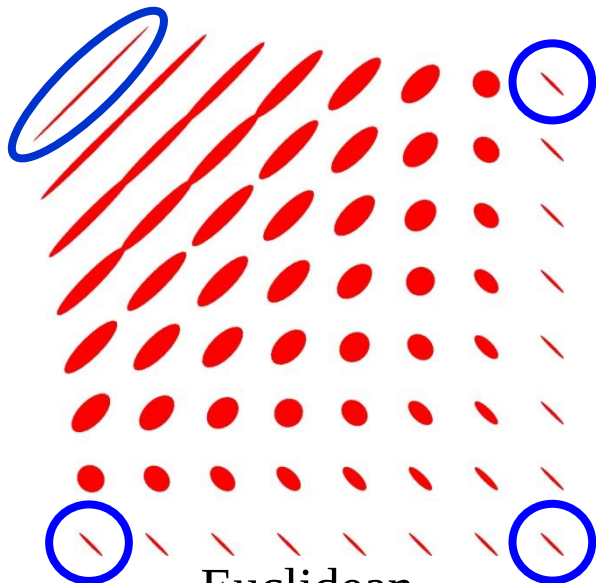


Euclidean

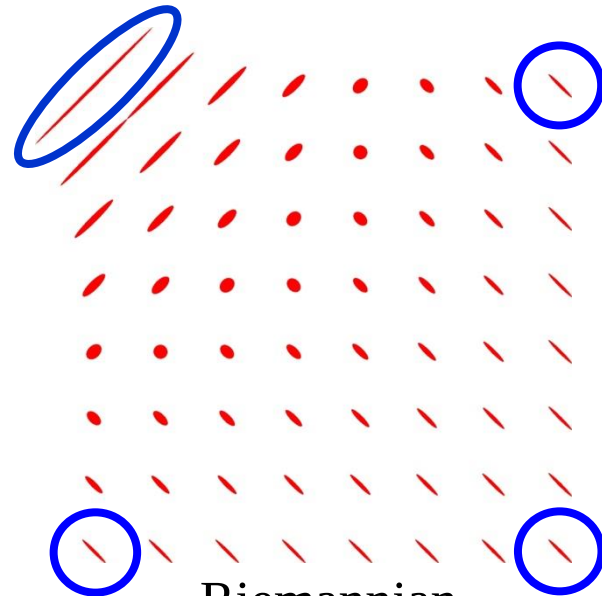


Riemannian

Representer theorem: $\Sigma(x) = \min_{\Sigma} \sum_i w_i(x) \text{dist}(\Sigma, \Sigma_i)^2$



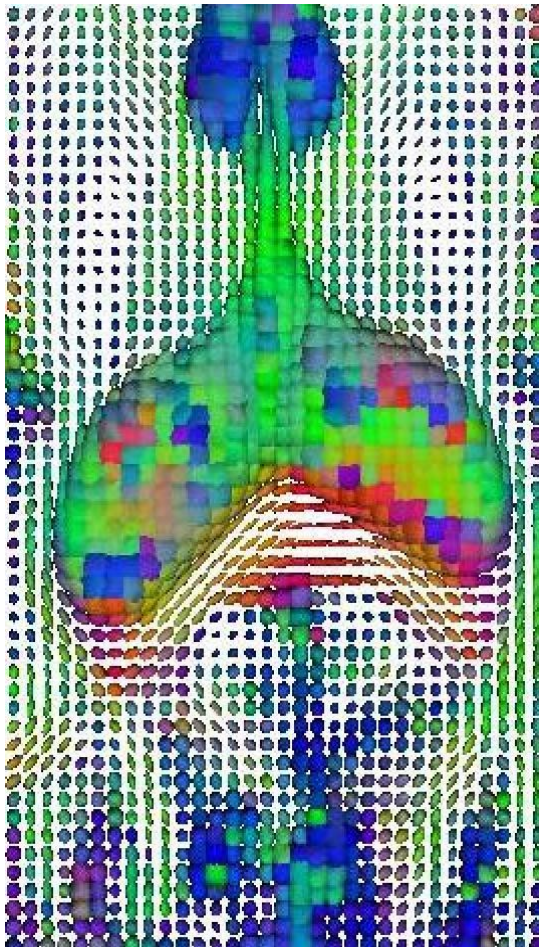
Euclidean



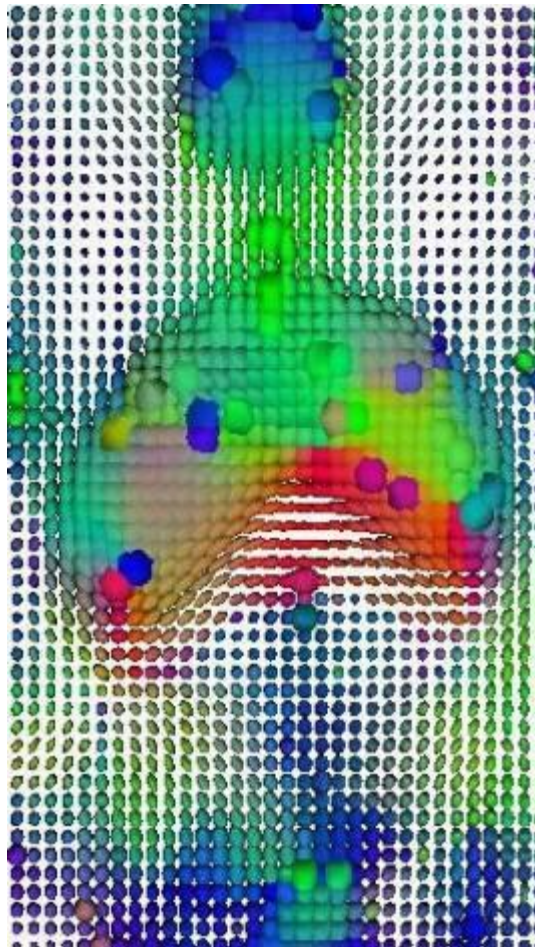
Riemannian

Gaussian filtering: Gaussian weighted mean

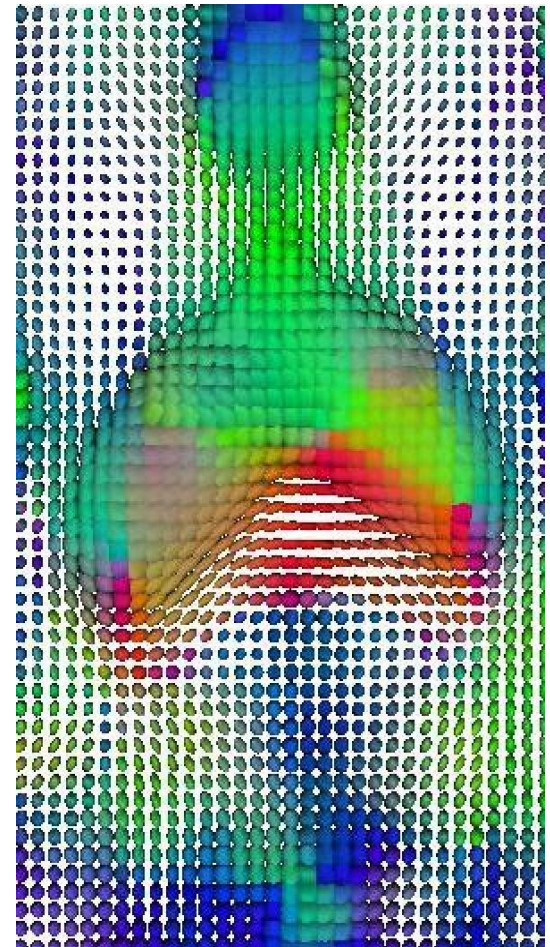
$$\Sigma(x) = \min \sum_{i=1}^n G_{\sigma}(x - x_i) \text{ dist}(\Sigma, \Sigma_i)^2$$



Raw



Coefficients $\sigma=2$



Riemann $\sigma=2$

PDE for filtering and diffusion

Harmonic regularization

$$C(\Sigma) = \int_{\Omega} \|\nabla \Sigma(x)\|_{\Sigma(x)}^2 dx$$

- Gradient = manifold Laplacian

$$\Delta \Sigma(x) = \sum_{i=x,y,z} \partial_i^2 \Sigma - \sum_{i=x,y,z} (\partial_i \Sigma) \Sigma^{(-1)} (\partial_i \Sigma) = \sum_u \frac{\overrightarrow{\Sigma(x) \Sigma(x+u)}}{\|u\|^2} + O(\|u\|^2)$$

- Trivial intrinsic numerical schemes with exponential maps!

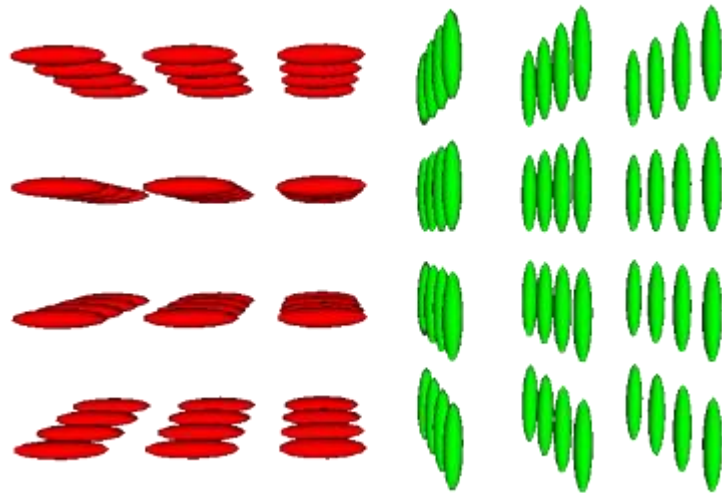
- Geodesic marching $\Sigma_{t+1}(x) = \exp_{\Sigma_t(x)}(-\varepsilon \nabla C(\Sigma)(x))$

Anisotropic diffusion

- Perona-Malik 90 / Gerig 92 $\Delta_w \Sigma(x) = \sum_u w(\|\partial_u \Sigma(x)\|_{\Sigma(x)}) \Delta_u \Sigma(x)$
- Robust functions $\text{Reg}(\Sigma) = \int \Phi(\|\nabla \Sigma(x)\|_{\Sigma(x)}^2) dx$

[Pennec, Fillard, Arsigny, IJCV 66(1), 2005, ISBI 2006]

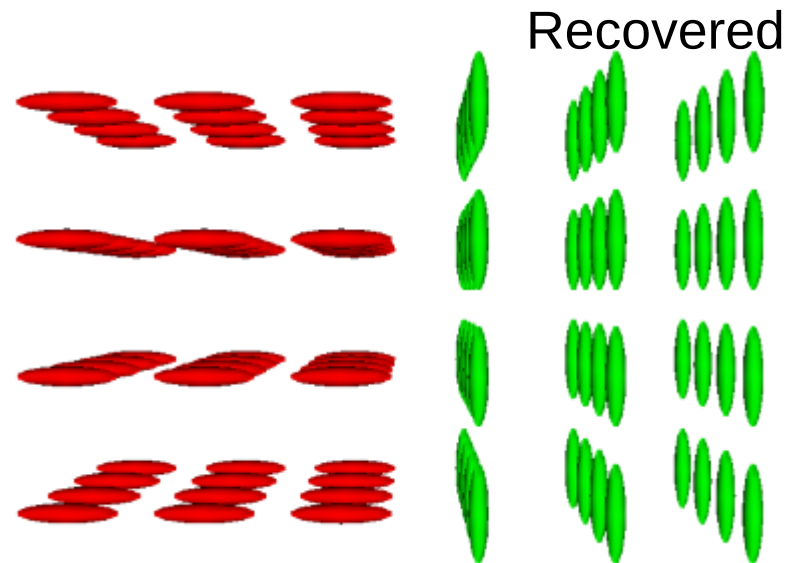
Anisotropic filtering



Initial



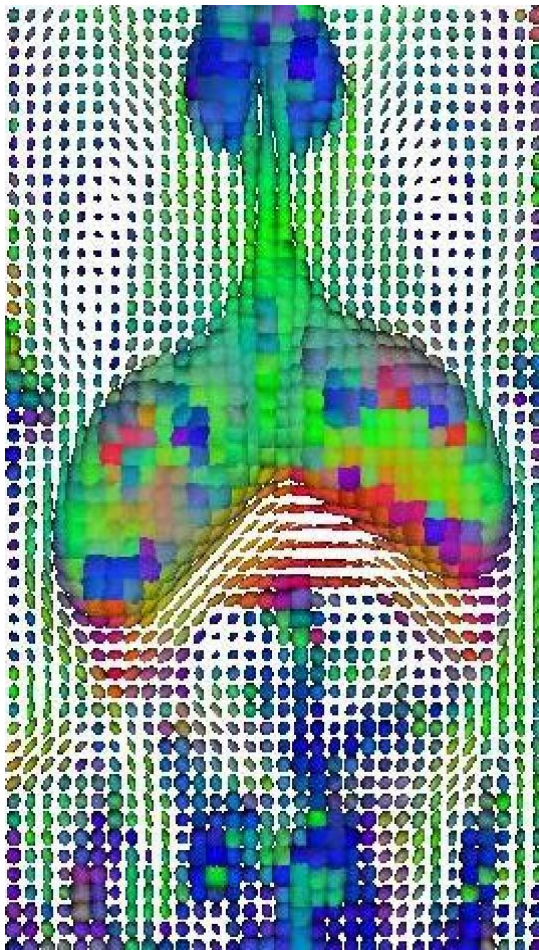
Noisy



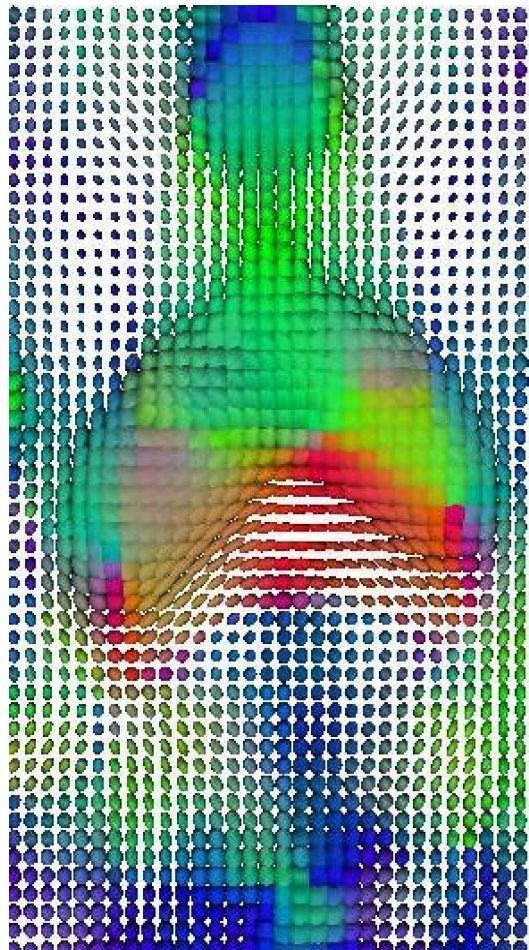
Recovered

Anisotropic filtering

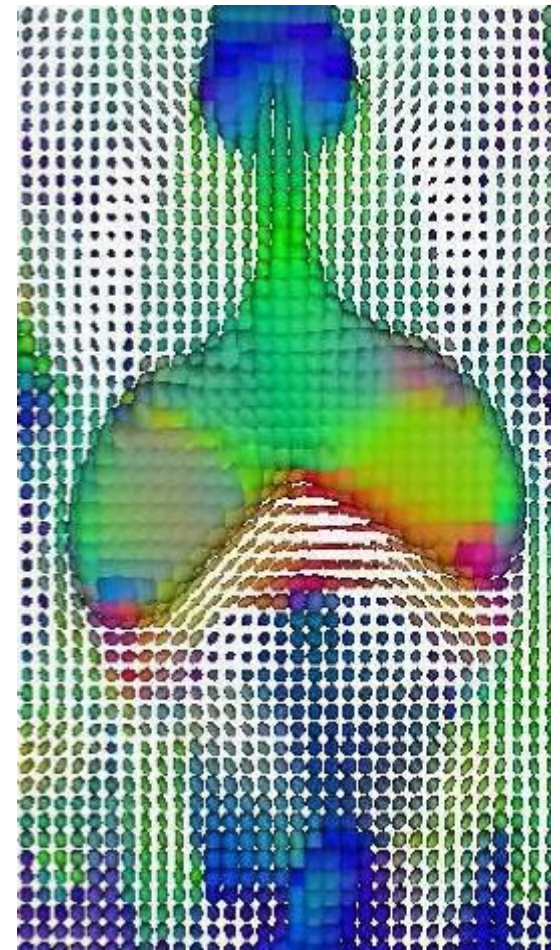
$$\hat{\Delta}\Sigma(x) = \sum_u w(\|\partial_u \Sigma(x)\|_{\Sigma(x)}) \Delta_u \Sigma(x) \quad \text{with} \quad w(t) = \exp(-t^2 / \kappa^2)$$
$$\Delta_u \Sigma(x) = \partial_u^2 \Sigma - (\partial_u \Sigma) \Sigma^{(-1)} (\partial_u \Sigma) \approx \overrightarrow{\Sigma(x) \Sigma(x+u)} / \|u\|^2$$



Raw



Riemann Gaussian

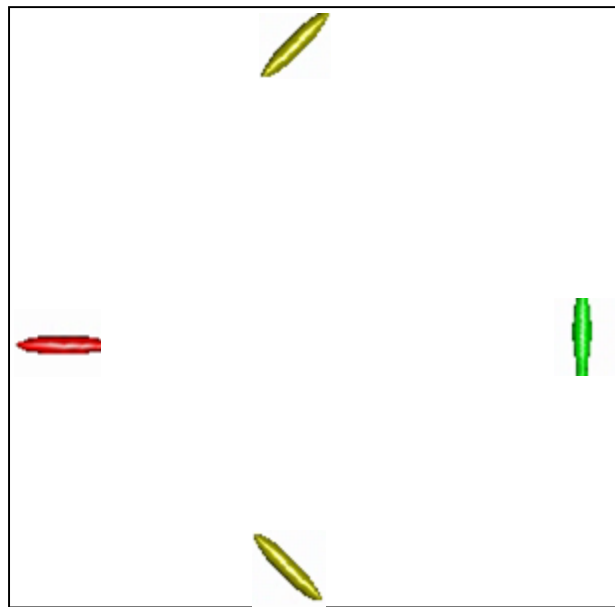


Riemann anisotropic

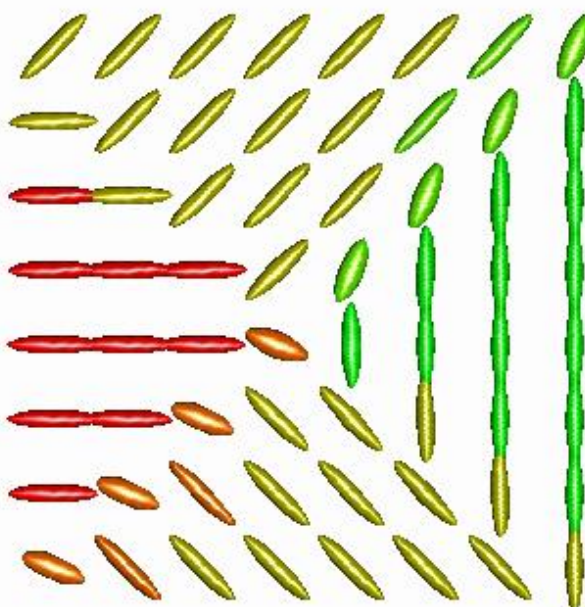
Extrapolation by Diffusion

$$C(\Sigma) = \frac{1}{2} \int_{\Omega} \sum_{i=1}^n G_{\sigma}(x - x_i) \text{dist}(\Sigma(x), \Sigma_i)^2 dx + \frac{\lambda}{2} \int_{\Omega} \|\nabla \Sigma(x)\|_{\Sigma(x)}^2$$

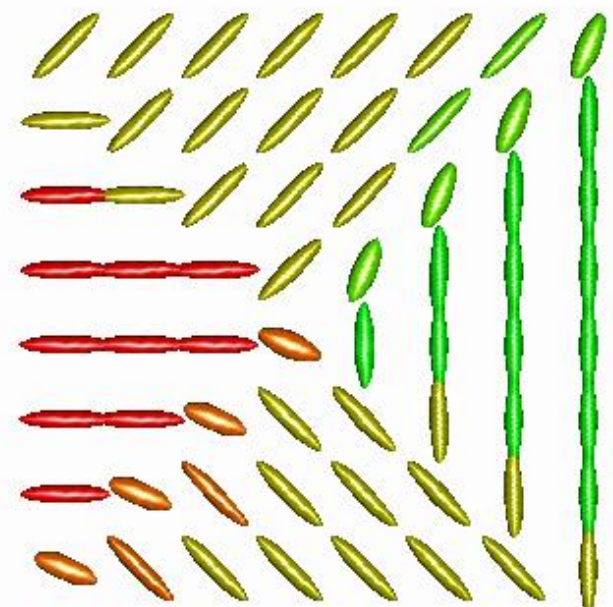
$$\nabla C(\Sigma)(x) = - \sum_{i=1}^n G_{\sigma}(x - x_i) \overrightarrow{\Sigma(x) \Sigma_i} - \lambda (\Delta \Sigma)(x)$$



Original tensors



Diffusion $\lambda=0.01$



Diffusion $\lambda=\infty$

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Advances Statistics: CLT & PCA

Log Euclidean Metric on Tensors

Exp/Log: global diffeomorphism Tensors/sym. matrices

- Vector space structure carried from the tangent space to the manifold

- Log. product
- Log scalar product
- Bi-invariant metric

$$\Sigma_1 \otimes \Sigma_2 \equiv \exp(\log(\Sigma_1) + \log(\Sigma_2))$$

$$\alpha \bullet \Sigma \equiv \exp(\alpha \log(\Sigma)) = \Sigma^\alpha$$

$$\text{dist}(\Sigma_1, \Sigma_2)^2 \equiv \|\log(\Sigma_1) - \log(\Sigma_2)\|^2$$

Properties

- Invariance by the action of similarity transformations only
- Very simple algorithmic framework

[Arsigny, MICCAI 2005 & MRM 56(2), 2006]

Riemannian Frameworks on tensors

Affine-invariant Metric (Curved space – Hadamard)

- Dot product $\langle V | W \rangle_{\Sigma} = \langle AVA^T | AWA^T \rangle_{A\Sigma A^T} = \langle \Sigma^{-1/2} V \Sigma^{-1/2} | \Sigma^{-1/2} W \Sigma^{-1/2} \rangle_{Id}$
- Geodesics $Exp_{\Sigma}(t, \overrightarrow{\Sigma\Psi}) = \Sigma^{1/2} \exp(t \cdot \Sigma^{-1/2} \cdot \overrightarrow{\Sigma\Psi} \cdot \Sigma^{-1/2}) \Sigma^{1/2}$
- Distance $\text{dist}(\Sigma, \Psi)^2 = \langle \overrightarrow{\Sigma\Psi} | \overrightarrow{\Sigma\Psi} \rangle_{\Sigma} = \left\| \log(\Sigma^{-1/2} \cdot \Psi \cdot \Sigma^{-1/2}) \right\|_{L_2}^2$

[Pennec, Fillard, Ayache, IJCV 66(1), 2006, Lenglet JMIV'06, etc]

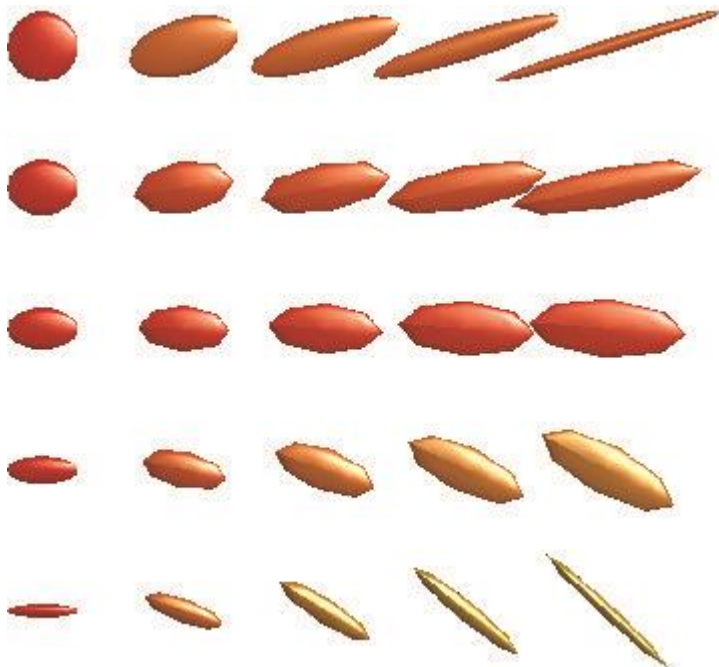
Log-Euclidean similarity invariant metric (vector space)

- Transport Euclidean structure through matrix exponential
- Dot product $\langle V | W \rangle_{\Sigma} = \langle \partial_V \log(\Sigma) | \partial_W \log(\Sigma) \rangle_{Id}$
- Geodesics $Exp_{\Sigma}(t, \overrightarrow{\Sigma\Psi}) = \exp(\log(\Sigma) + t \cdot \partial_{\overrightarrow{\Sigma\Psi}} \log(\Sigma))$
- Distance $\text{dist}(\Sigma_1, \Sigma_2)^2 \equiv \left\| \log(\Sigma_1) - \log(\Sigma_2) \right\|^2$

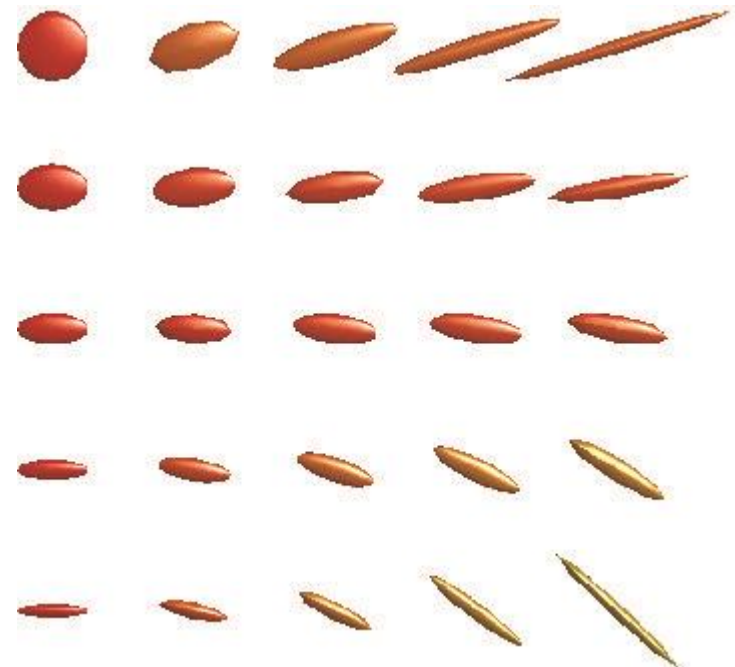
[Arsigny, Pennec, Fillard, Ayache, SIAM'06, MRM'06]

Log Euclidean vs Affine invariant

- Means are geometric (vs arithmetic for Euclidean)
- Log Euclidean slightly more anisotropic
- Speedup ratio: 7 (aniso. filtering) to >50 (interp.)



Euclidean

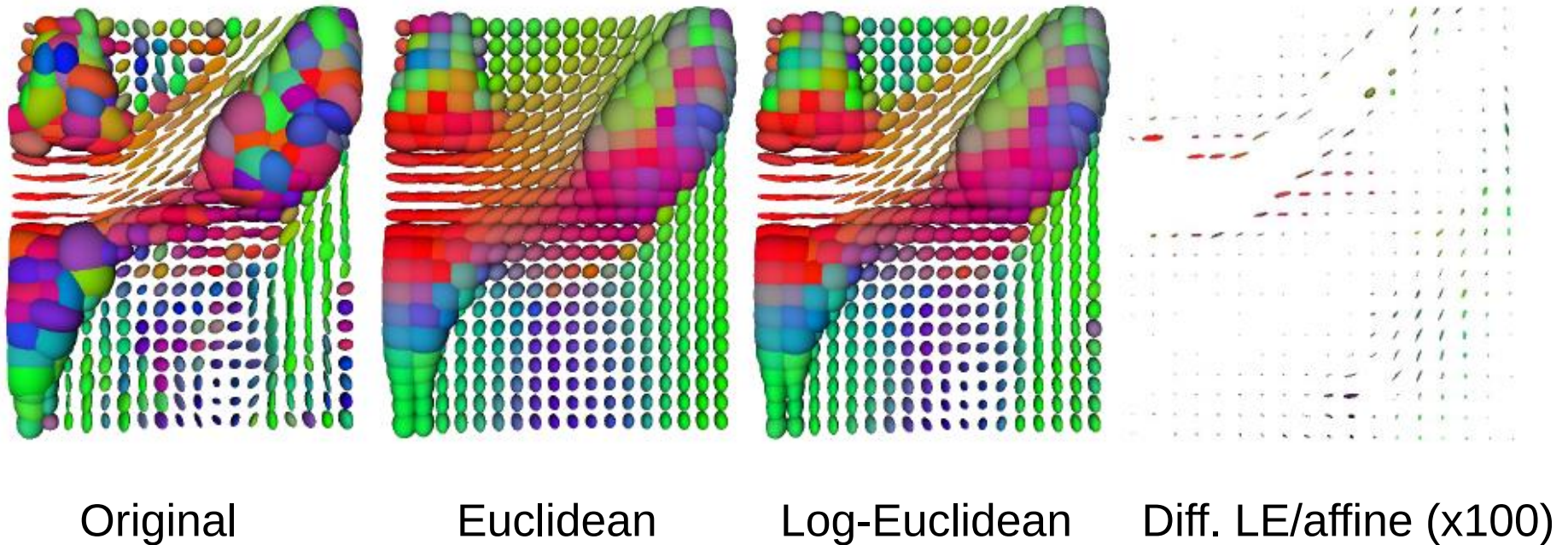


Affine Invariant Log Euclidean

Log Euclidean vs Affine invariant

Real DTI images: anisotropic filtering

- Difference is not significant
- Speedup of a factor 7 for Log-Euclidean



Some other metrics on tensors

Square root metrics

- Cholesky [Wang Vemuri et al, IPMI'03, TMI 23(8) 2004.]
- Size and shape space [Dryden, Koloydenko & Zhou, 2008]
- Power Euclidean [Dryden & Pennec, arXiv 2011]

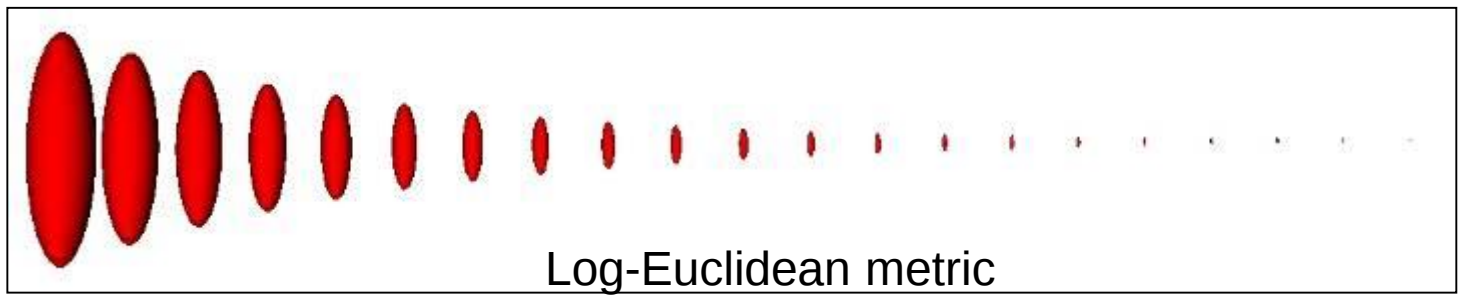
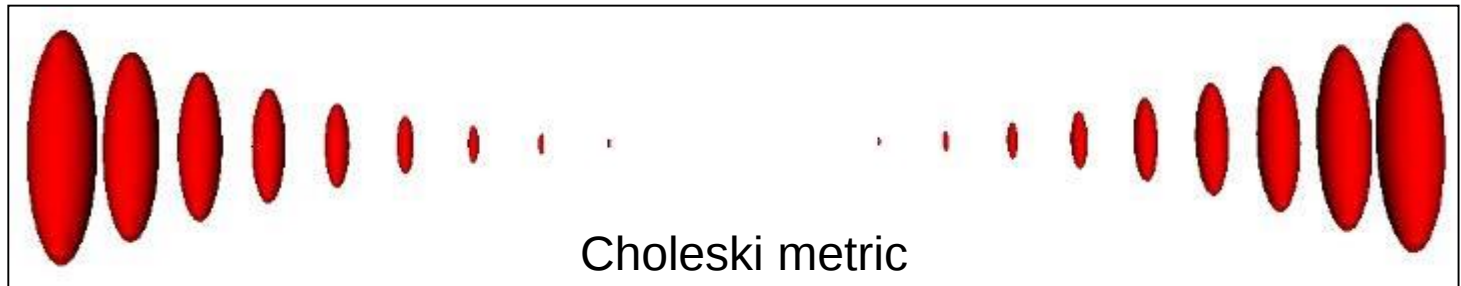
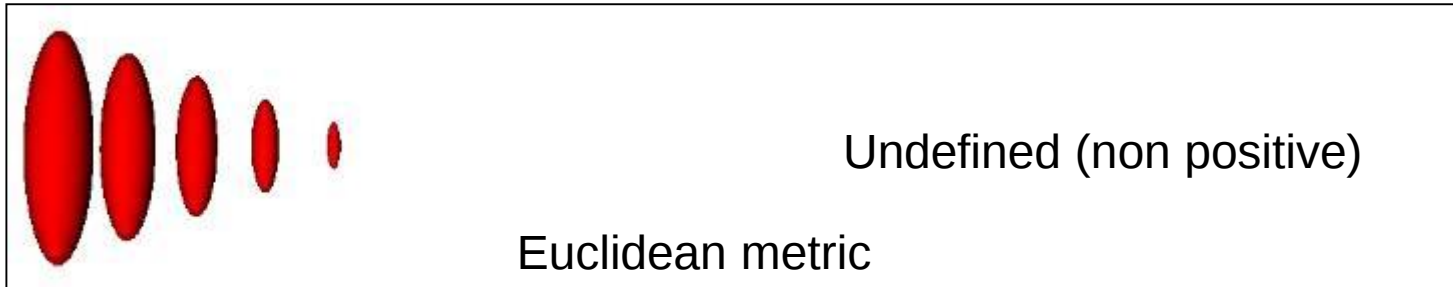
Non Riemannian distances

- J-Divergence [Wang & Vemuri, TMI 24(10), 2005]
- Geodesic Loxodromes [Kindlmann et al. MICCAI 2007]

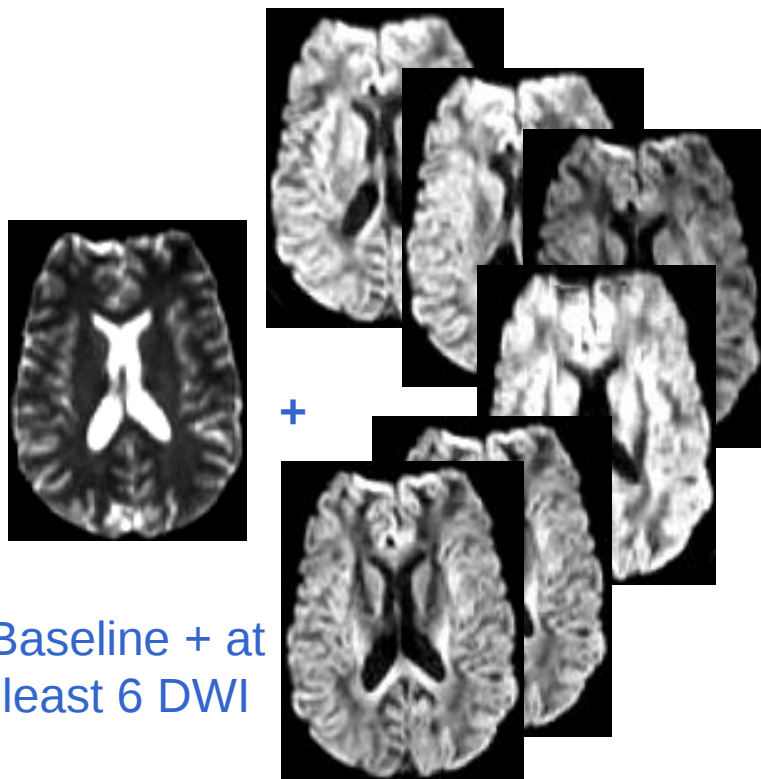
4th order tensors

- [Gosh, Descoteau & Deriche MICCAI'08]

Geodesic shooting in tensors spaces



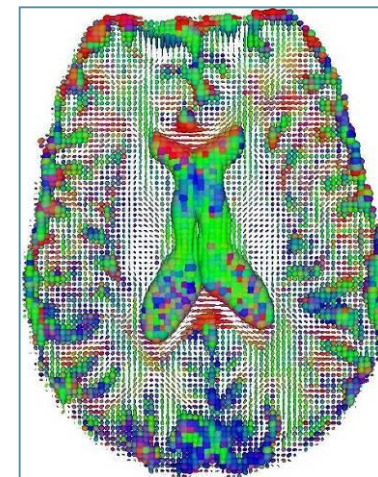
DTI Estimation from DWI



Baseline + at
least 6 DWI

Stejskal & Tanner
diffusion equation

$$S_i = S_0 \exp(-bg_i^T Dg_i)$$



Diffusion Tensor Field

Estimation / Regularization on complex DWI:

- Anisotropic diffusion on Choleski factors [Wang & Vemuri, TMI'04]

Estimation with a Rician noise

- Smoothing DWI before estimation [Basu & Fletcher, MICCAI 2006]
- ML (MMSE) [Aja-Fernández et al, TMI 2008]
- **MAP with log-Euclidean prior** [Fillard et al., ISBI 2006, TMI 2007]

Noise is Gaussian in complex DW signal

- Rician = amplitude of complex Gaussian
- LSQ on Rician noise = bias for low SNRs [Sijbers, TMI 1998]

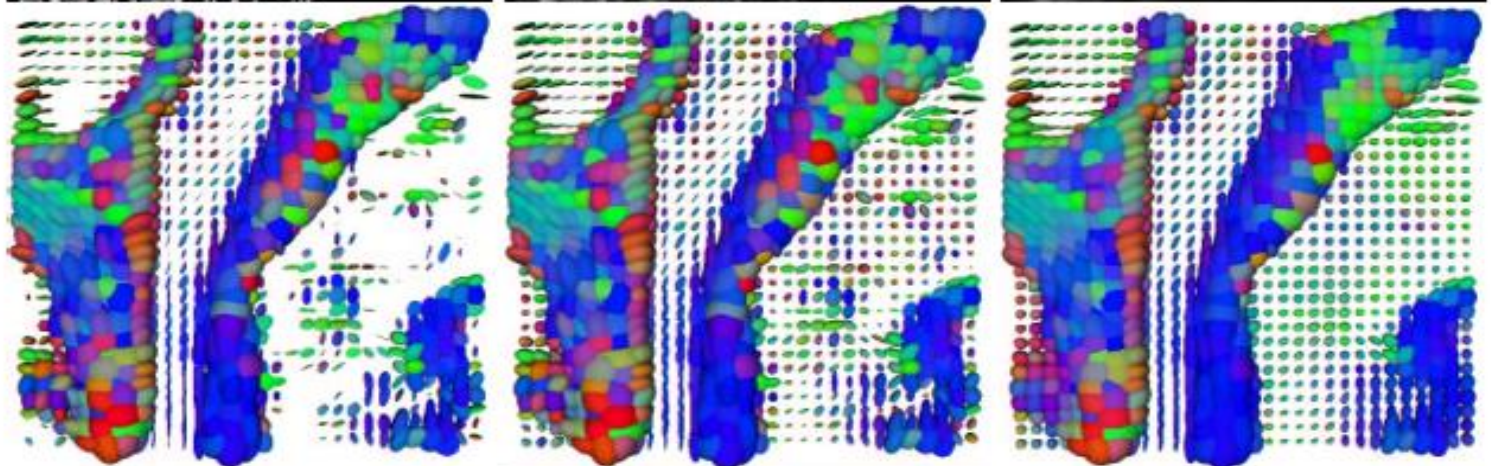
Rician MAP estimation with Riemannian spatial prior

$$MAP(\Sigma) = -\sum_{i=1}^N \int \log \left(\frac{\hat{S}_i}{\sigma^2} \exp \left(-\frac{\hat{S}_i^2 + S_i(\Sigma)^2}{2\sigma^2} \right) I_0 \left(\frac{S_i(\Sigma)\hat{S}_i}{\sigma^2} \right) \right) dx + \int \Phi \left(\|\nabla \Sigma(x)\|_{\Sigma(x)}^2 \right) dx$$

FA



Estimated
tensors



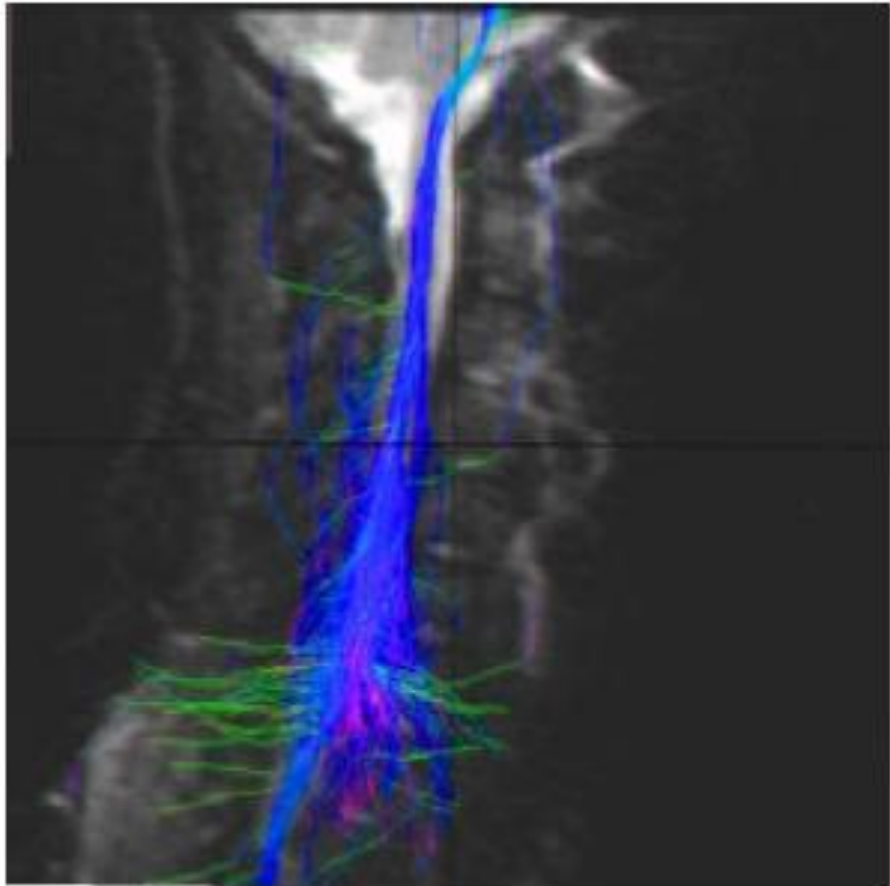
Standard

ML Rician

MAP Rician

[Fillard, Arsigny, Pennec, Ayache ISBI'06, TMI 26(11) 2007]

Clinical DTI of the spinal cord: fiber tracking

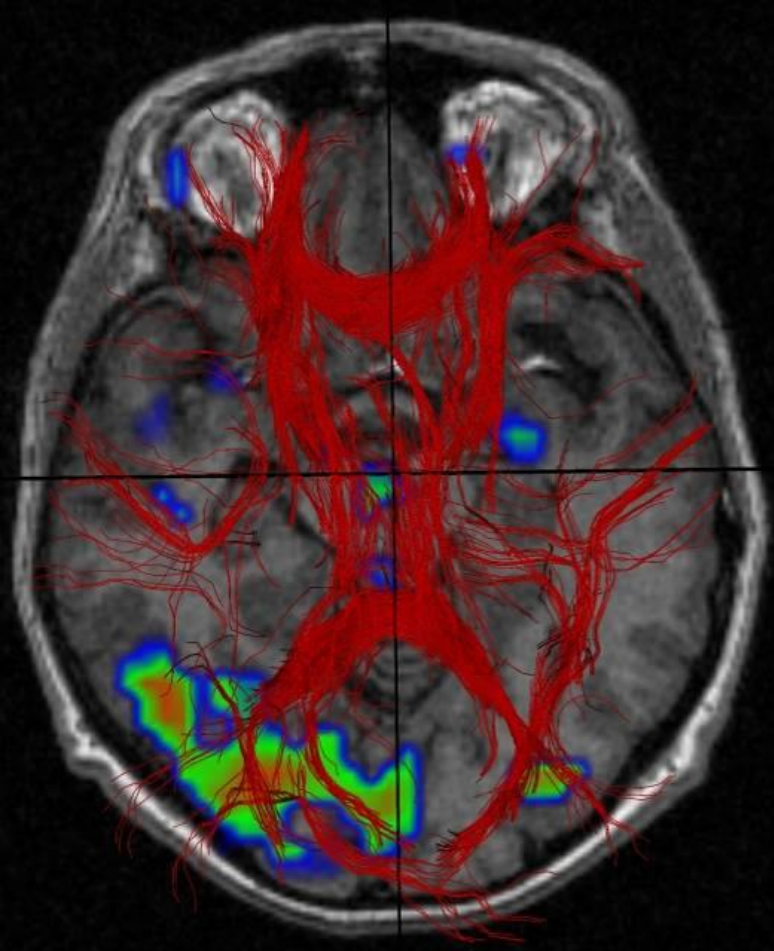


Standard

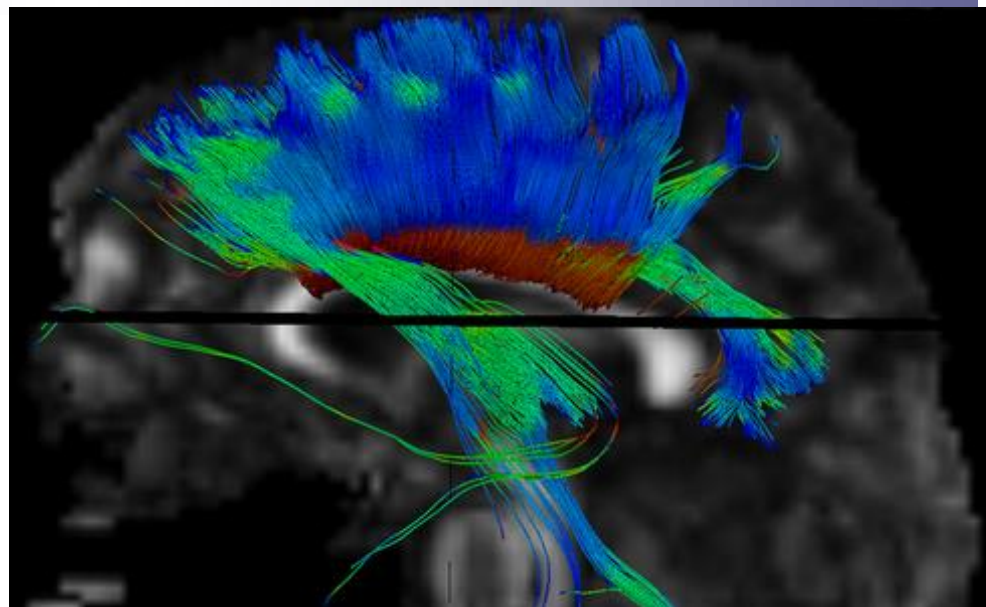


MAP Rician

[Fillard, Arsigny, Pennec, Ayache ISBI'06, TMI 26(11) 2007]



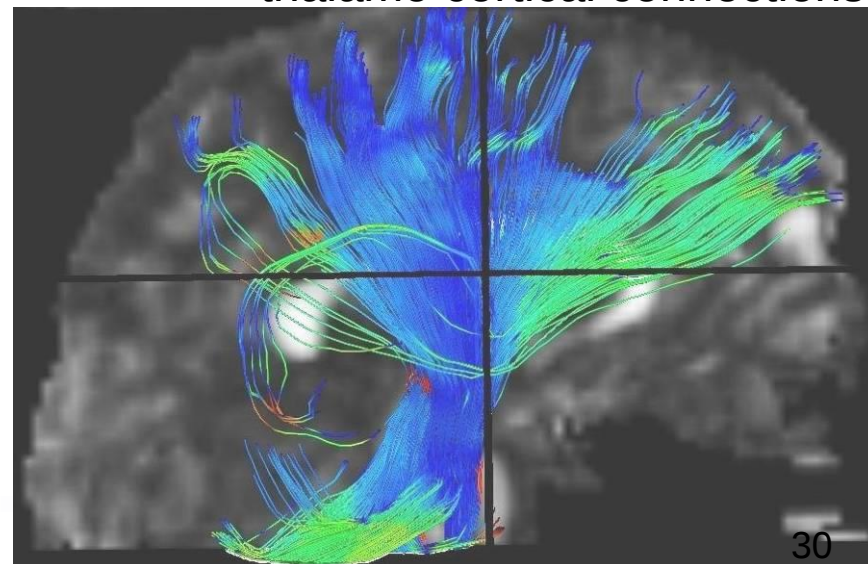
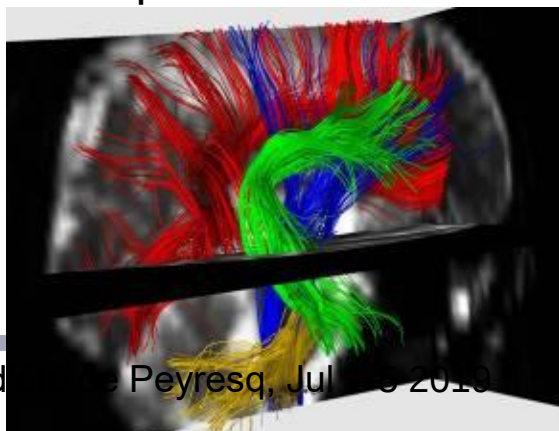
T1 + Activation map + fibers

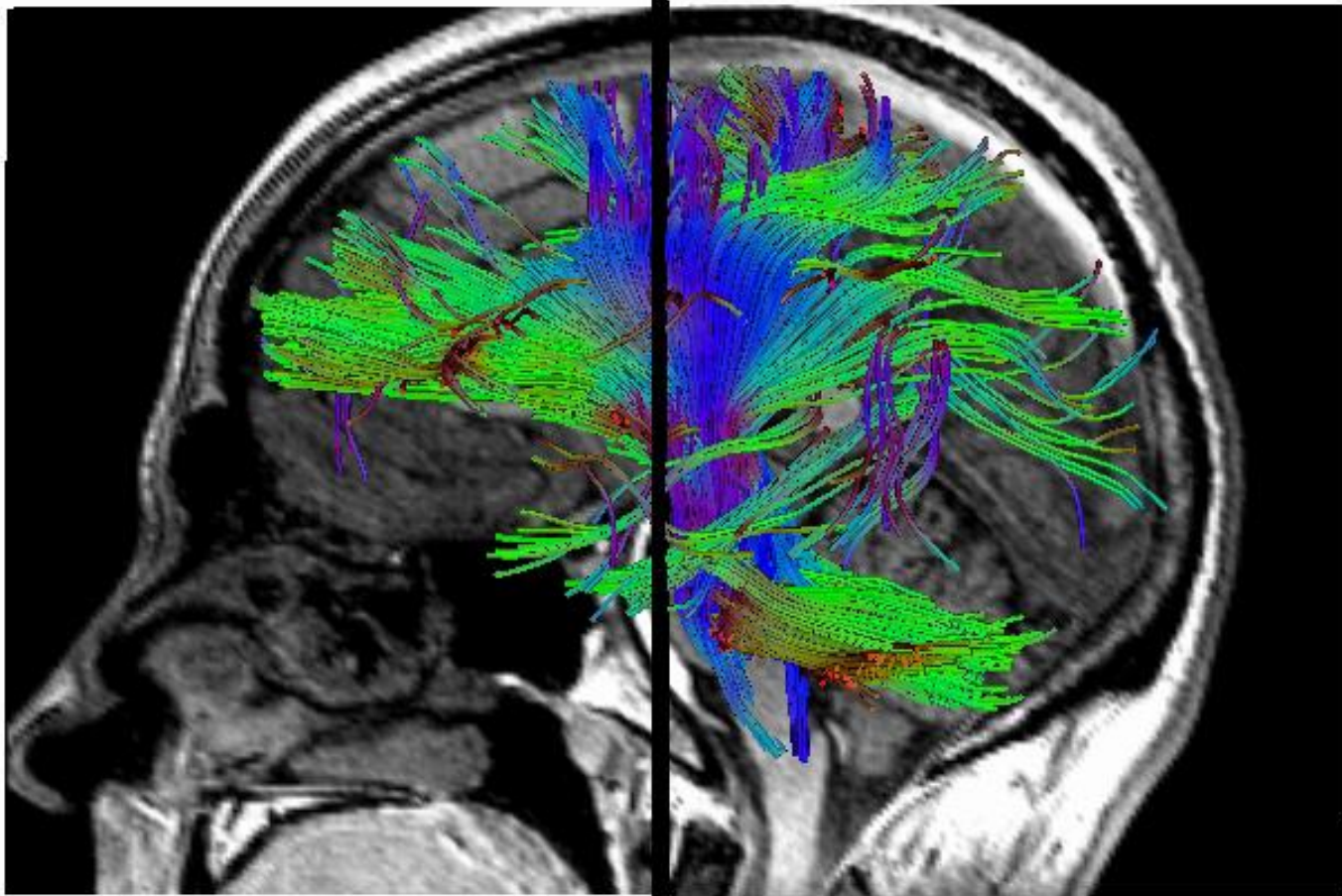


Corpus callosum + cingulum

*Courtesy of P. Fillard
using MedINRIA*

Corticospinal tract and
thalamo cortical connections





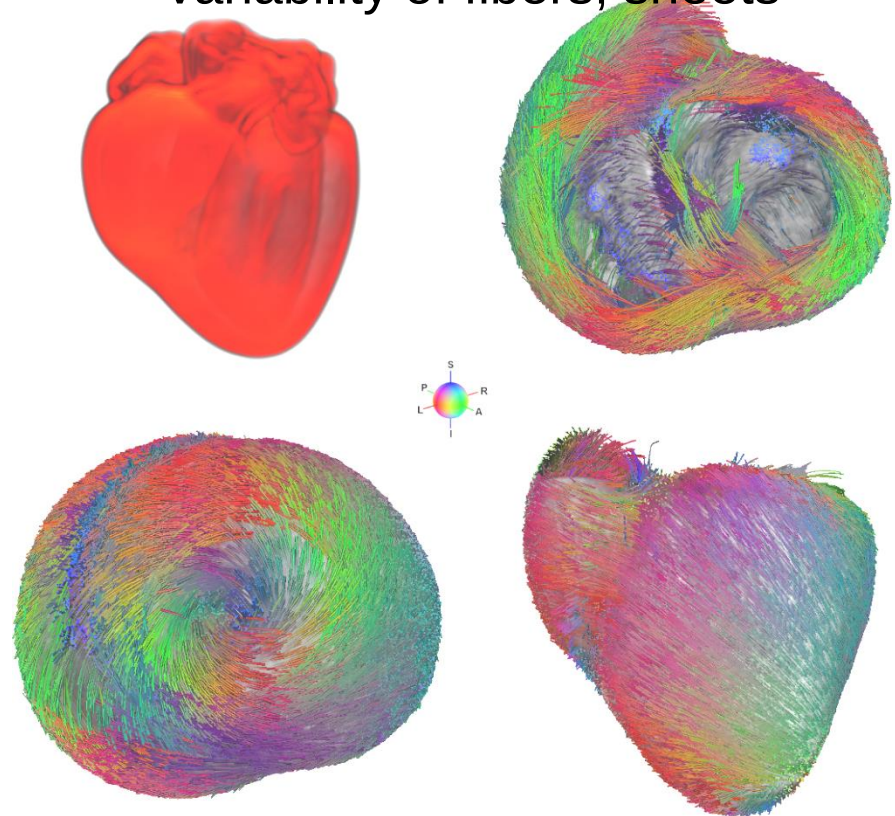
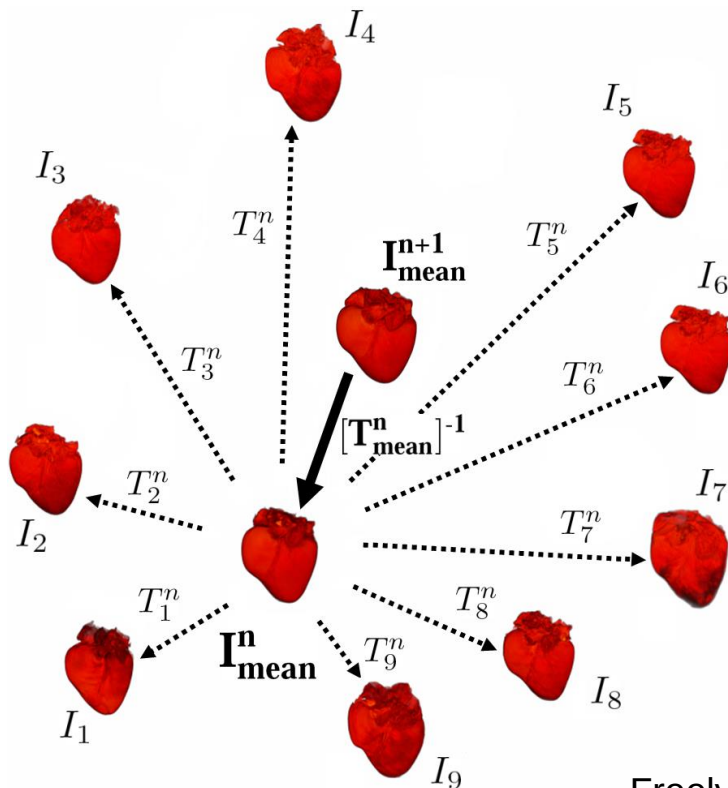
A Statistical Atlas of the Cardiac Fiber Structure

[J.M. Peyrat, et al., MICCAI'06, TMI 26(11), 2007]

Database

- 7 canine hearts from JHU
- Anatomical MRI and DTI

- Average cardiac structure
- Variability of fibers, sheets



Freely available at <http://www-sop.inria.fr/asclepios/data/heart>

A Statistical Atlas of the Cardiac Fiber Structure

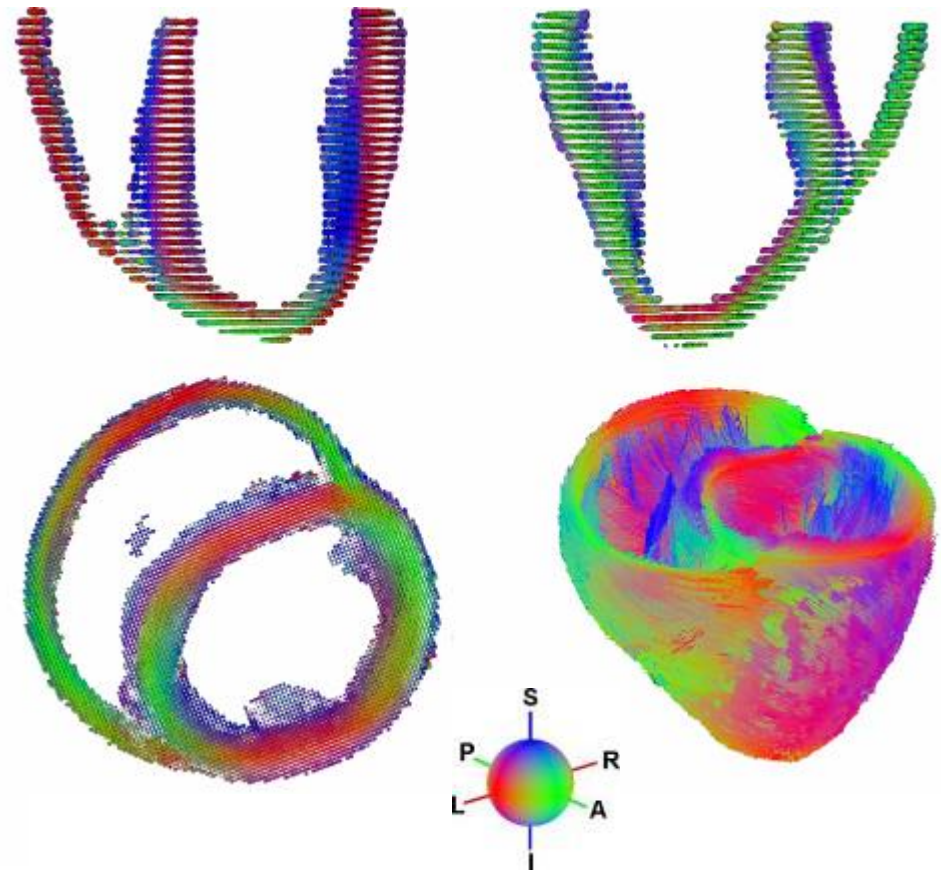
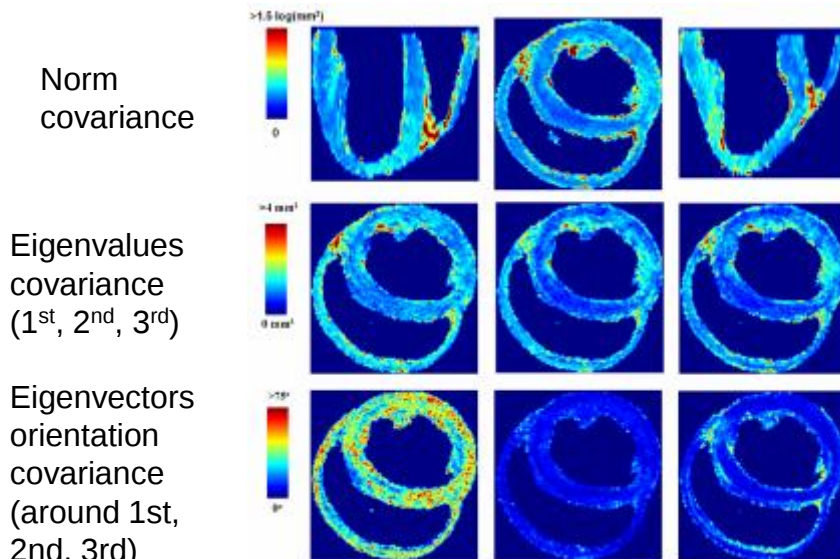
[J.M. Peyrat, et al., MICCAI'06, TMI 26(11), 2007]

Database

- 7 canine hearts from JHU
- Anatomical MRI and DTI

Method

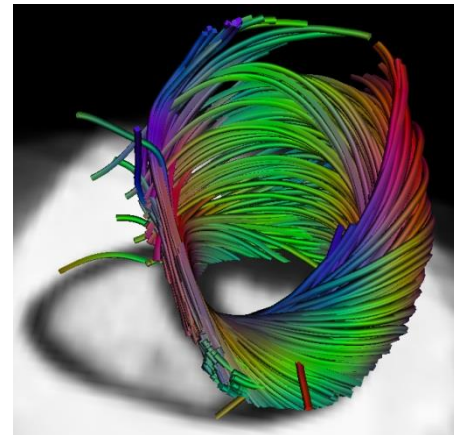
- Normalization based on aMRIs
- Log-Euclidean statistics of Tensors



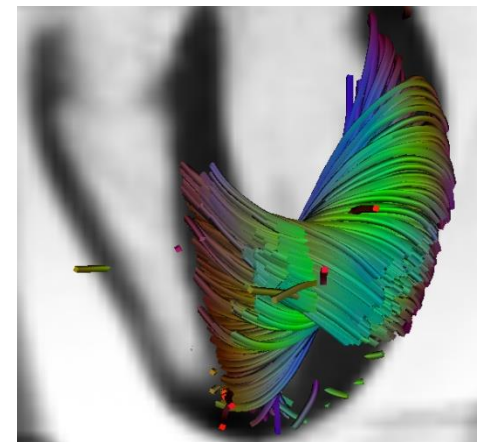
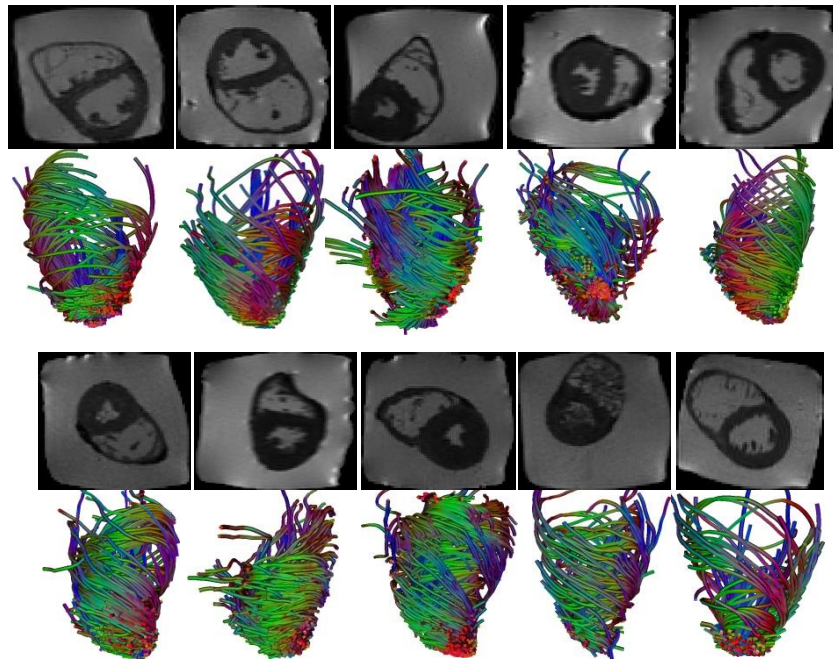
Diffusion model of the human heart

10 human ex vivo hearts (CREATIS-LRMN, Lyon, France)

- Classified as healthy (controlling weight, septal thickness, pathology examination)
- Acquired on 1.5T MR Avento Siemens
 - bipolar echo planar imaging, 4 repetitions, 12 gradients
- Volume size: $128 \times 128 \times 52$, 2 mm resolution



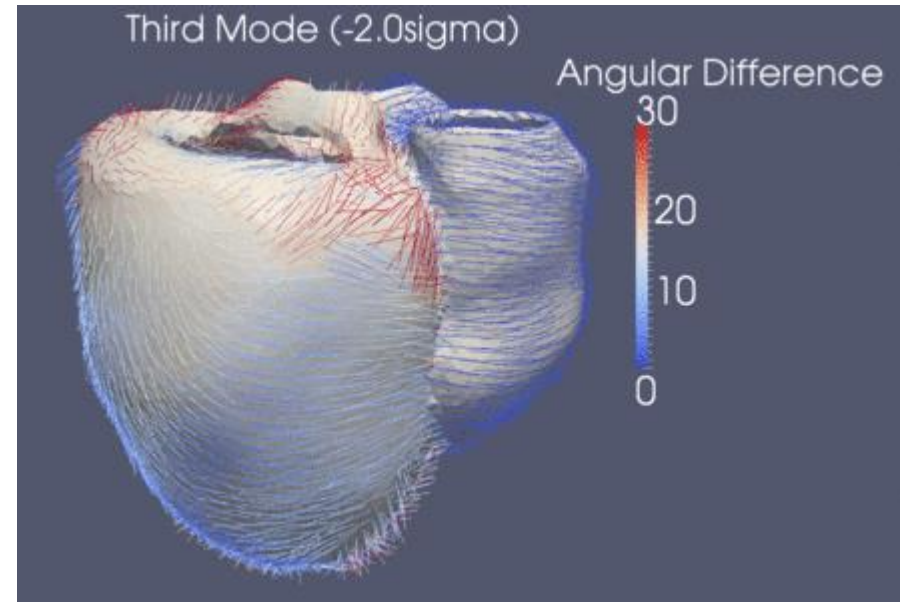
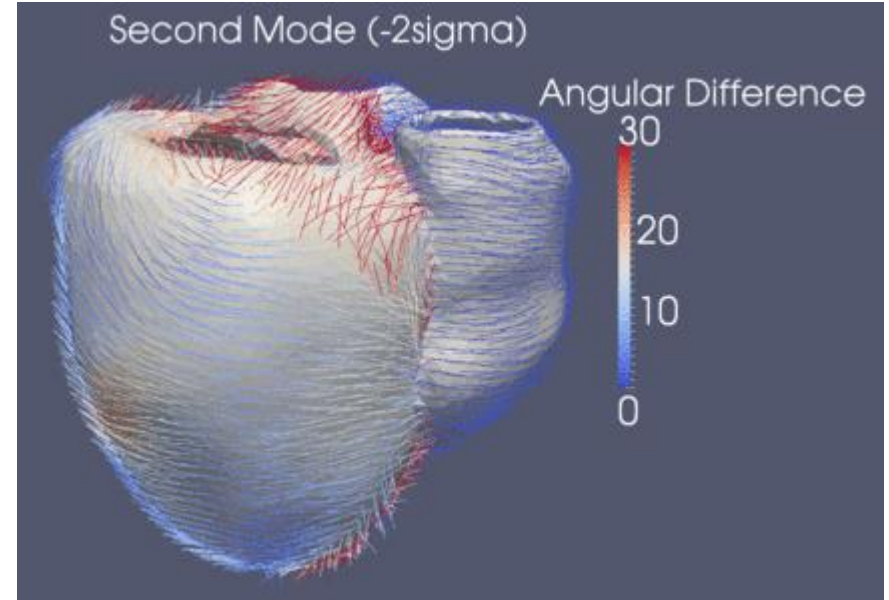
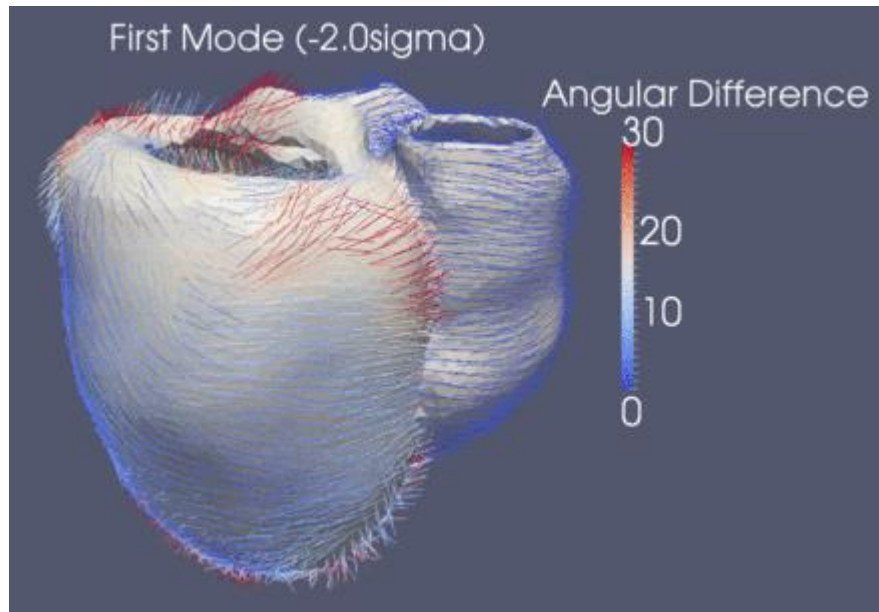
Fiber tractography in the left ventricle



Helix angle highly correlated to the transmural distance

[H. Lombaert Statistical Analysis of the Human Cardiac Fiber Architecture from DT-MRI, ISMRM 2011, FIMH 2011]

A Statistical Atlas of the Cardiac Fiber Structure



10 human ex vivo hearts (CREATIS-LRMN, France)

- Classified as healthy (controlling weight, septal thickness, pathology examination)
- Acquired on 1.5T MR Avento Siemens
 - bipolar echo planar imaging, 4 repetitions, 12 gradients
- Volume size: 128×128×52, 2 mm resolution

[R. Mollero, M.M Rohé, et al,
to appear in FIMH 2015]

Conclusion

The Riemannian computing framework

- Integral or sum in M : minimize an intrinsic functional
 - Fréchet / Karcher mean
 - Filtering, convolution through weighted means
- The exponential chart is the basic tool
 - Gradient descent is geodesic walking
 - Intrinsic numerical scheme for Laplace Beltrami

Extrapolation of sparse tensor measurements

- Easy with log-Euclidean metrics
- Directional discontinuities (faults) can be imposed
 - Anisotropic filtering with structure tensor