Medical Imaging

MVA 2024-2025

http://www-sop.inria.fr/teams/asclepios/cours/MVA/

X. Pennec

Statistics on Riemannian manifolds and Lie groups

Inria Epione e-patient / e-medicine

Epione team
2004, route des Lucioles B.P. 93
06902 Sophia Antipolis Cedex

http://www-sop.inria.fr/epione

Medical Image Analysis – MVA 2024-2025

Course notes : http://www-sop.inria.fr/teams/asclepios/cours/MVA/

- Tue. Oct 1 2024, 14:00 ENS 1Z25 [XP] Introduction to Medical Image Acquisition & Image Registration
- Tue. Oct 8 2024, 14:00 ENS 1Z25 [XP] Riemannian Geometry and Statistics
- Tue. Oct 15 2024, 14:00 ENS 1Z25 [HD] Image Filtering & Segmentation
- Tue. Oct 22 2024: 14:00 ENS 1Z25 [HD] Image Segmentation based on Clustering and Markov Random Fields
- Tue. Nov 5 2024: 14:00 ENS 1Z25 [XP] Analysis in the space of Covariance Matrices
- Tue. Nov 12 2024: 14:00 ENS 1Z25 [HD] Shape constrained image segmentation
- Tue. Nov 19 2024: 14:00 ENS 1Z25 [XP] Diffeomorphic Registration and Computational Anatomy
- Tue. Nov 26 2024: 14:00 ENS 1Z25 [HD] Biophysical Modeling
- Tue. Dec 3, 2024, 14:00 (Visio) [XP & HD] Exam

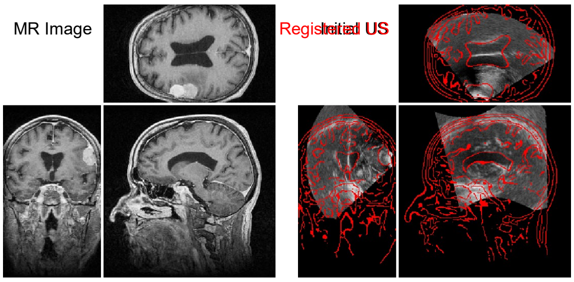
Course overview

Introduction
- Why do we need statistics on manifolds?

Simple statistics on interesting manifolds
- The Riemannian Geometric framework
- Simple Statistics
  - Mean, Covariance, Gaussian, t-tests

Application to registration
- Statistics on spine shapes
- Evaluation of registration accuacy

Per-operative registration of MR/US images



Performance Evaluation?

Variability of a registration algorithm

Hidden parameters
- Anatomy
- Pathologies

Geometric transformation
- **Rigid**
- Deformations

Image 1
- Distortions
- Noise

Registration algorithm

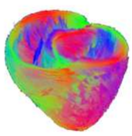
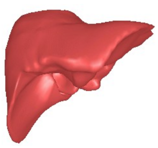
Image 2
- Distortions
- Noise

Estimated transformation

Quantify the statistical Variability of the transformation:

- Expected value (bias)
- Covariance matrix, std dev. (accuracy, precision)
  - On the transformation (rotation σ_r [rad], translation σ_t [mm])
  - Propagate on target points (TRE σ_x)

Computational Anatomy

***Computational Anatomy, an emerging discipline*, P. Thompson, M. Miller, NeuroImage special issue 2004**
***Mathematical Foundations of Computational Anatomy*, X. Pennec, S. Joshi, MICCAI workshop, 2006**

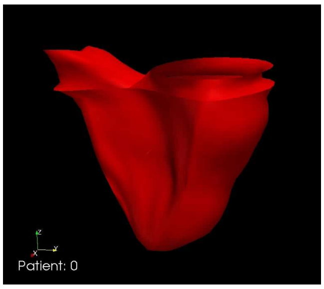
Modeling and Analysis of the Human Anatomy

- Estimate representative / average organ anatomies
- Model organ development across time
- Establish normal variability
- To detect and classify of pathologies from structural deviations
- To adapt generic (atlas-based) to patients-specific models

Methods of computational anatomy

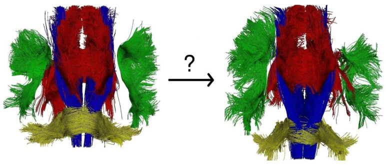
Remodeling of the right ventricle of the heart in tetralogy of Fallot

- Mean shape
- Shape variability
- Correlation with clinical variables
- Predicting remodeling effect



Shape of RV in 18 patients

Shapes: forms & deformations

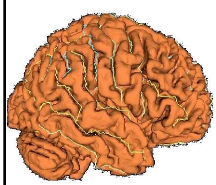


"Shape space" embedding [Kendall]

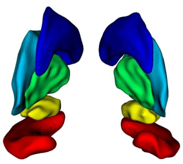
- Shape = what remains from the object when we remove all transformations from a given group
  - Transformation (rigid, similarity, affine) = nuisance factor
  - Shape manifold = quotient of the Object manifold by the group action
- Quotient spaces are non-linear (e.g. R^n / scaling = S^n)
- Kendall size & shape space: $(R^n)^d / SO_n$

Statistics on shape spaces?

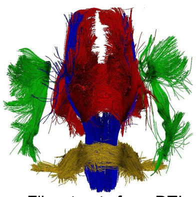
Anatomical structures segmented in Brain Images



Sulcal lines at the surface of the cortex



Surface of deep brain structures



Fiber tracts from DTI

How to measure the variability across subjects?
Generic framework to deal with all object types?

Diffusion Tensor Imaging

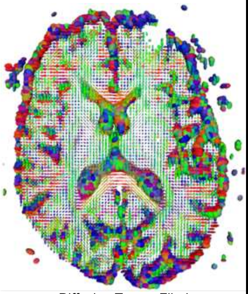
Covariance of the Brownian motion of water
-> Architecture of axonal fibers

Very noisy data

- Tensor image processing
  - Robust estimation
  - Filtering, regularization
  - Interpolation / extrapolation
- Information extraction (fibers)

Symmetric positive definite matrices

- Convex operations are stable
  - mean, interpolation
- More complex operations are not
  - PDEs, gradient descent…



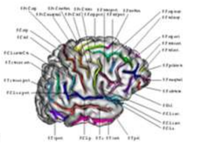
Diffusion Tensor Filed
(slice of a 3D volume)

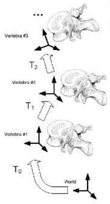
Intrinsic computing on Manifold-valued images?

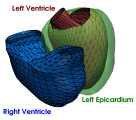
Statistical Analysis of Geometric Features

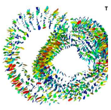
Geometric features belong to manifolds

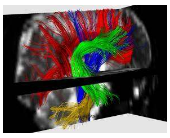
- Curves, tracts
- Surfaces
- Tensors, covariance matrices
- Transformations / deformations











Algorithms for statistics on geometric manifolds

- Definition of mean / covariance / PCA / distributions of geometric features?
- Mathematical structure = algorithmic bases

Basic probabilities and statistics

Measure: random vector x of pdf $p_{\mathbf{x}}(z)$

Approximation: $\mathbf{x} \sim (\bar{\mathrm{x}}, \mathbf{\Sigma}_{\mathbf{xx}})$

- Mean: $\bar{\mathrm{x}} = \mathrm{E}(\mathbf{x}) = \int z.p_{\mathbf{x}}(z).dz$
- Covariance: $\Sigma_{\mathbf{xx}} = \mathrm{E}\left[(\mathbf{x}-\bar{\mathrm{x}}).(\mathbf{x}-\bar{\mathrm{x}})^T\right]$

Propagation:

$$\mathbf{y} = h(\mathbf{x}) \sim \left(h(\bar{\mathrm{x}}),\ \frac{\partial h}{\partial \mathrm{x}}.\mathbf{\Sigma}_{\mathbf{xx}}.\frac{\partial h}{\partial \mathrm{x}}^{\mathrm{T}} \right)$$

Noise model: additive, Gaussian...

Principal component analysis

Statistical distance: Mahalanobis and χ^2

Some problems with geometric features

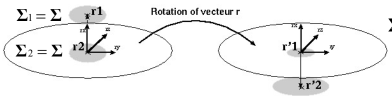
Extrinsic means of 3D rotations: invariance w.r.t. the chart

$$\underline{\mathbf{R}} = \frac{1}{n}\sum_i \mathbf{R}_i \qquad \underline{q} = \frac{1}{n}\sum_i q_i \qquad \underline{r} = \frac{1}{n}\sum_i r_i$$

Noise on 3D rotations: invariance w.r.t. the transformation group

$\Sigma_1 = \Sigma$

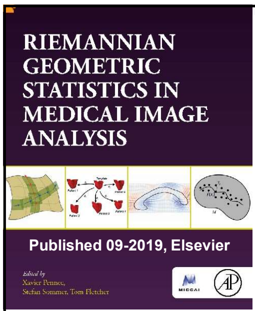
$\Sigma_2 = \Sigma$



$\Sigma'_1 = \frac{8}{\pi^2}\Sigma$

$\Sigma'_2 = \frac{\pi^2}{8}\Sigma$

$$\Sigma = \sigma^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Part 1: Foundations
- **1: Riemannian geometry [Sommer, Fetcher, Pennec]**
- **2: Statistics on manifolds [Fletcher]**
- **3: Manifold-valued image processing with SPD matrices [XP]**
- 4: Riemannian Geometry on Shapes and Diffeomorphisms [Marsland, Sommer]
- **5: Beyond Riemannian: the affine connection setting for transformation groups [Pennec, Lorenzi]**

Part 2: Statistics on Manifolds and Shape Spaces
- 6: Object Shape Representation via Skeletal Models (s-reps) and Statistical Analysis [Pizer, Maron]
- 7: Inductive Fréchet Mean Computation on S(n) and SO(n) with Applications [Chakraborty, Vemuri]
- 8: Statistics in stratified spaces [Ferage, Nye]
- 9: Bias in quotient space and its correction [Miolane, Devilier,Pennec]
- 10: Probabilistic Approaches to Statistics on Manifolds: Stochastic Processes, Transition Distributions, and Fiber Bundle Geometry [Sommer]
- 11: Elastic Shape Analysis, Square-Root Representations and Their Inverses [Zhang, Klassen, Srivastava]

Part 3: Deformations, Diffeomorphisms and their Applications
- 13: Geometric RKHS models for handling curves and surfaces in Computational Anatomy : currents, varifolds, f-shapes, normal cycles [Charlie, Charon, Glaunes, Gori, Roussillon]
- 14: A Discretize-Optimize Approach for LDDMM Registration [Polzin, Niethammer, Vialad, Modezitski]
- 15: Spatially varying metrics in the LDDMM framework [Vialard, Risser]
- 16: Low-dimensional Shape Analysis In the Space of Diffeomorphisms [Zhang, Fleche, Wells, Golland]
- 17: Diffeomorphic density matching, Bauer, Modin, Joshi]

Course overview

Introduction
- Why do we need statistics on manifolds?

Simple statistics on interesting manifolds
- The Riemannian Geometric framework
- Simple Statistics
  - Mean, Covariance, Gaussian, t-tests

Application to registration
- Statistics on spine shapes
- Evaluation of registration accuacy

Riemannian geometry is a powerful structure to build consistent statistical computing algorithms

Shape spaces & directional statistics
- [Kendall StatSci 89, Small 96, Dryden & Mardia 98]

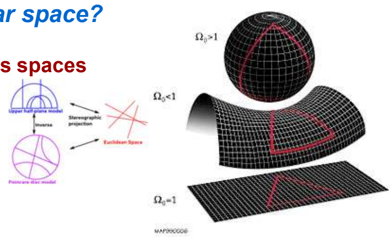
Numerical integration, dynamical systems & optimization
- [Helmke & Moore 1994, Hairer et al 2002]
- Matrix Lie groups [Owren BIT 2000, Mahony JGO 2002]
- Optimization on Matrix Manifolds [Absil, Mahony, Sepulchre, 2008]

Information geometry (statistical manifolds)
- [Amari 1990 & 2000, Kass & Vos 1997]
- [Oller Annals Stat. 1995, Battacharya Annals Stat. 2003 & 2005]

Statistics for image analysis
- Rigid body transformations [Pennec PhD96]
- General Riemannian manifolds [Pennec JMIV98, NSIP99, JMIV06]
- PGA for M-Reps [Fletcher IPMI03, TMI04]
- Planar curves [Klassen & Srivastava PAMI 2003]

Which non-linear space?

Constant curvatures spaces
- Sphere,
- Euclidean,
- Hyperbolic



Homogeneous spaces, Lie groups and symmetric spaces

Riemannian or affine connection spaces

Towards non-smooth quotient and stratified spaces

Definition of a manifold

Intuitive idea
- "*A manifold is a topological space which is locally Euclidean*"
- "*A Manifold is a topological space for which the neighborhood of each point is homeomorphic to the euclidean space*"
  - Homeomorphism: F is bijective and F and F^{-1} are continuous (no folding or tearing transformation)

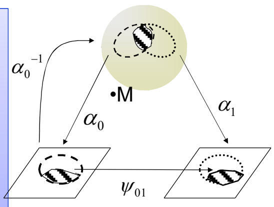
Definition: C^k *manifold of dimension* n (N ≥ 1, k ≥ 1, or k = ∞)
- A topological space M, together with a C^k atlas on M.
- Any equivalence class of atlases is called a *differentiable structure of class* C^k (*and dimension* n)

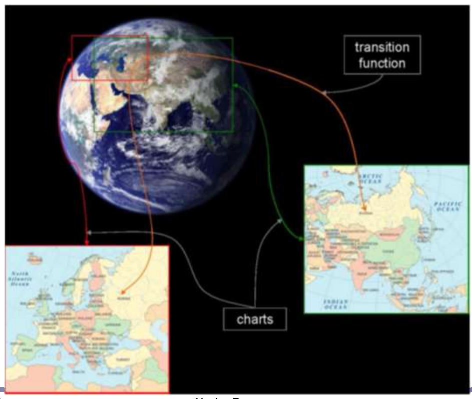
Differentiable manifolds
- When k = ∞, we say that M is a *smooth manifold*
- When k=1, M is a *differential manifold*
- When k=0, M is a *topological manifold*

Traditional atlas definition

Given: Manifold *M*
Construct: Atlas *A*
Chart
- Region U_c in M (open disk)
- Region c in R^n (open disk)
- Function α_c taking U_c to c
  - Inverse

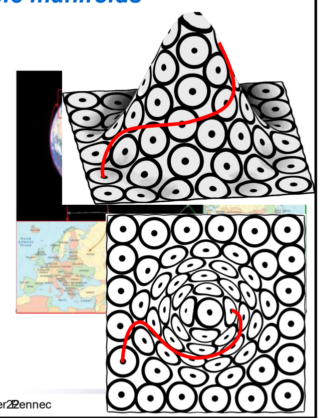
Atlas is collection of charts
- Every point in M in at least one chart
- Overlap regions
- Transition functions: $\psi_{01} = \alpha_1 \circ \alpha_0^{-1}$ smooth

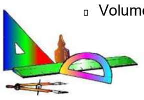




Differentiable manifolds

Computing on a manifold
- Extrinsic
  - Embedding in $\mathbb{R}^n$
- Intrinsic
  - Coordinates : charts
- Measuring?
  - Lengths
  - Straight lines
  - Volumes

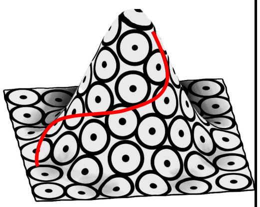




Measuring extrinsic distances

Basic tool: the scalar product

$$< v, w > = v^t w$$

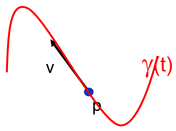


- Norm of a vector

$$\|v\| = \sqrt{< v, v >}$$

- Length of a curve

$$L(\gamma) = \int \|\dot{\gamma}(t)\|\, dt$$

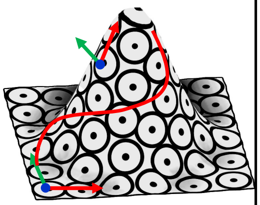


Measuring extrinsic distances

Basic tool: the scalar product

Bernhard Riemann
1826-1866

$$< v, w >_p = v^t G(p) w$$

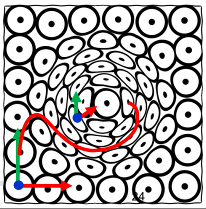


- Norm of a vector

$$\|v\|_p = \sqrt{< v, v >_p}$$

- Length of a curve

$$L(\gamma) = \int \|\dot{\gamma}(t)\|\, dt$$

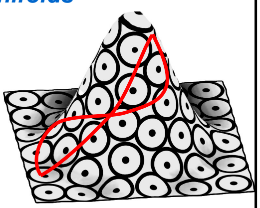


Riemannian manifolds

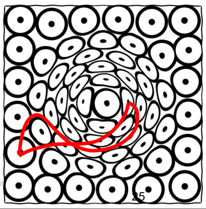
Basic tool: the scalar product

Bernhard Riemann
1826-1866

$$< v, w >_p = v^t G(p) w$$



- Geodesics
  - Shortest path between 2 points
  - Calculus of variations (E.L.) : 2nd order differential equation (specifies acceleration)
  - Free parameters: initial speed and starting point
- Length of a curve



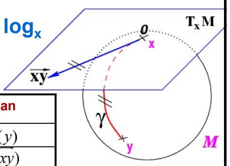
Bases of Algorithms in Riemannian Manifolds

Exponential map (Normal coordinate system):

- **Exp_x** = geodesic shooting parameterized by the initial tangent
- **Log_x** = unfolding the manifold in the tangent space along geodesics
  - Geodesics = straight lines with Euclidean distance
  - Geodesic completeness: covers M \ Cut(x)

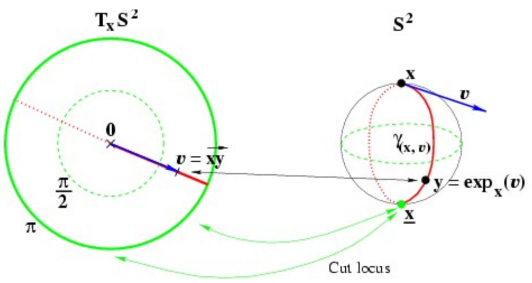
Reformulate algorithms with exp_x and log_x

Vector -> Bi-point (no more equivalence classes)



Operation	Euclidean space	Riemannian
Subtraction	$\overrightarrow{xy} = y - x$	$\overrightarrow{xy} = Log_x(y)$
Addition	$y = x + \overrightarrow{xy}$	$y = Exp_x(\overrightarrow{xy})$
Distance	$\text{dist}(x, y) = \lVert y - x \rVert$	$\text{dist}(x, y) = \lVert \overrightarrow{xy} \rVert_x$
Gradient descent	$x_{t+\varepsilon} = x_t - \varepsilon \nabla C(x_t)$	$x_{t+\varepsilon} = Exp_{x_t}(-\varepsilon \nabla C(x_t))$

Cut locus



Metric choice

Transformations (Lie group):

- Left (or right) invariant $\quad \text{dist}(g, h) = \text{dist}(f \circ g, f \circ h) = \lVert f^{(-1)} \circ g \rVert_{Id}$
- Practical computations $\quad \exp_f(\overrightarrow{\delta f}) = f \circ \overrightarrow{\delta f} \qquad \overrightarrow{fg} = f^{(-1)} \circ g$
- *No bi-invariant metric*

Homogeneous manifolds

$$\text{dist}(x, y) = \text{dist}(g * x, g * y)$$

- Invariance wrt the isotropy group
- Practical computations $\quad \exp_x(\overrightarrow{\delta x}) = f_x * \overrightarrow{\delta x} \qquad \overrightarrow{xy} = f_x^{(-1)} * \vec{y}$

General Riemannian manifolds

- Exp and log through numerical optimization / integration

Example on 3D rotations

Space of rotations SO(3):
- Manifold: $R^t.R=Id$ and $det(R)=+1$
- Lie group:
  - Composition: R_1 o R_2 = $R_1.R_2$
  - Inversion: $R^{(-1)} = R^t$

Tangent space
- At Identity (skew symmetric matrices)
- At any point by left or right translation

Metrics on SO(3)
- Left / right invariant metrics
- Induced by the ambient space: bi-invariance

Example on 3D rotations

Group exponential
- One parameter subgroups
- Matrix exponential and Rodrigue's formula

Exponential map for the bi-invariant metric
- Geodesic starting at identity = one parameter subgroups
- Geodesic everywhere by left (or right) translation

More details in the memo on rotations on the web

Course overview

Introduction
- Why do we need statistics on manifolds?

Simple statistics on interesting manifolds
- The Riemannian Geometric framework
- Simple Statistics
  - Mean, Covariance, Gaussian, t-tests

Application to registration
- Statistics on spine shapes
- Evaluation of registration accuacy

Basic probabilities and statistics

Measure: random vector x of pdf $p_{\mathbf{x}}(z)$

Approximation: $\mathbf{x} \sim (\bar{\mathrm{x}}, \boldsymbol{\Sigma}_{\mathbf{xx}})$

- Mean: $\bar{\mathrm{x}} = \mathrm{E}(\mathbf{x}) = \int z.p_{\mathbf{x}}(z).dz$
- Covariance: $\Sigma_{\mathbf{xx}} = \mathrm{E}\left[(\mathbf{x}-\bar{\mathrm{x}}).(\mathbf{x}-\bar{\mathrm{x}})^T\right]$

Propagation:

$$\mathbf{y} = h(\mathbf{x}) \sim \left(h(\bar{\mathrm{x}}),\ \frac{\partial h}{\partial \mathrm{x}} . \boldsymbol{\Sigma}_{\mathbf{xx}} . \frac{\partial h}{\partial \mathrm{x}}^{\mathrm{T}} \right)$$

Noise model: additive, Gaussian...

Principal component analysis

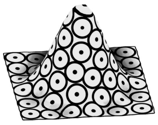
Statistical distance: Mahalanobis and χ^2

Random variable in a Riemannian Manifold

Intrinsic pdf of x

- For every set H

$$P(\mathbf{x} \in H) = \int_H p(y) dM(y)$$



- ~~Lebesgue's measure~~

→ Uniform Riemannian Mesure $dM(y) = \sqrt{\det(G(y))}\, dy$

Expectation of an observable in M

- $\boldsymbol{E}_{\mathbf{x}}[\phi] = \int_M \phi(y) p(y) dM(y)$
- $\phi = dist^2$ (variance) : $\boldsymbol{E}_{\mathbf{x}}[dist(.,y)^2] = \int_M dist(y,z)^2 p(z) dM(z)$
- $\phi = \log(p)$ (information) : $\boldsymbol{E}_{\mathbf{x}}[\log(p)] = \int_M p(y) \log(p(y)) dM(y)$
- ~~$\phi = x$ (mean) : $\boldsymbol{E}_{\mathbf{x}}[\mathbf{x}] = \int_M y\, p(y) dM(y)$~~
  - **Integral only valid in Hilbert/Wiener spaces [Fréchet 44]**

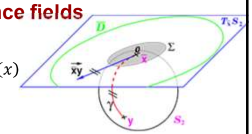
Statistical tools: Moments

Tensor moments of a random point on M

- $\mathfrak{M}_1(x) = \int_M \overrightarrow{xz}\, dP(z)$ — Tangent mean: (0,1) tensor field
- $\mathfrak{M}_2(x) = \int_M \overrightarrow{xz} \otimes \overrightarrow{xz}\, dP(z)$ — (0,2) tensor field
- $\mathfrak{M}_k(x) = \int_M \overrightarrow{xz} \otimes \overrightarrow{xz} \otimes \cdots \otimes \overrightarrow{xz}\, dP(z)$ — k-contravariant tensor field
- $\sigma^2(x) = Tr_g(\mathfrak{M}_2(x)) = \int_M dist^2(x,z)\, dP(z)$ — **Mean square distance**

Tangent mean and tangent covariance fields

- Tg mean: $\mathfrak{M}_1(x) = \int_M \overrightarrow{xz}\, dP(z)$
- Tg cov: $Cov(x) = \mathfrak{M}_2(\bar{x}) - \mathfrak{M}_1(x) \otimes \mathfrak{M}_1(x)$



Several definitions of the mean

Tensor moments of a random point on M

- $\mathfrak{M}_1(x) = \int_M \overrightarrow{xz}\, dP(z)$ Tangent mean: (0,1) tensor field
- $\mathfrak{M}_2(x) = \int_M \overrightarrow{xz} \otimes \overrightarrow{xz}\, dP(z)$ (0,2) tensor field
- $\mathfrak{M}_k(x) = \int_M \overrightarrow{xz} \otimes \overrightarrow{xz} \otimes \cdots \otimes \overrightarrow{xz}\, dP(z)$ k-contravariant tensor field
- $\boldsymbol{\sigma^2(x) = Tr_g(\mathfrak{M}_2(x)) = \int_M dist^2(x,z)\, dP(z)}$ **Mean square distance**

Fréchet mean set

- **Frechet mean** [1948] = global minima of MSD
- **Karcher mean** [1977] = local minima of MSD
- **Exponential barycenters** [Emery & Mokobodzki 1991]
  $\mathfrak{M}_1(\bar{x}) = \int_M \overrightarrow{\bar{x}z}\, dP(z) = 0$ (critical points if P(C) =0))
  sometimes called Riemannian center of mass
  [Groove & Karcher 1973-1976] under uniqueness assumptions


Maurice Fréchet (1878-1973)

Fréchet expectation (1948)

Minimizing the variance

$$E[\mathbf{x}] = \underset{y \in M}{\text{argmin}} \left(E\left[\text{dist}(y, \mathbf{x})^2 \right] \right)$$

Existence

- Finite variance at one point

Characterization as an exponential barycenter (P(C)=0)

$$\text{grad}\left(\sigma_{\mathbf{x}}^2(y)\right) = 0 \quad \Rightarrow \quad E\left[\overrightarrow{\bar{x}\mathbf{x}}\right] = \int_M \overrightarrow{\bar{x}\mathbf{x}}.p_{\mathbf{x}}(z).dM(z) = 0$$

Uniqueness Karcher 77 / Kendall 90 / Afsari 10 / Le 10

- Unique Karcher mean (thus Fréchet) if distribution has support in a regular geodesic ball with radius $r < r^* = \frac{1}{2}\min(inj(M), \pi/\sqrt{\kappa})$ (k upper bound on sectional curvatures on M)
- Empirical mean: a.s. uniqueness [Arnaudon & Miclo 2013]

Other central primitives

$$E^{\alpha}[\mathbf{x}] = \underset{y \in M}{\text{argmin}} \left(E\left[\text{dist}(y, \mathbf{x})^{\alpha} \right] \right)^{1/\alpha}$$

A gradient descent (Gauss-Newton) algorithm

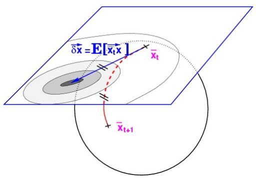
Vector space

$$f(x+v) = f(x) + \nabla f^T.v + \tfrac{1}{2} v^T.H_f.v$$

$$x_{t+1} = x_t + v \quad \text{with} \quad v = -H_f^{(-1)}.\nabla f$$

Manifold

$$f(\exp_x(v)) = f(x) + \nabla f(v) + \tfrac{1}{2} H_f(v,v)$$



$$\nabla\left(\sigma_{\mathbf{x}}^2(y)\right) = -2\, E\left[\overrightarrow{y\mathbf{x}}\right] = \frac{-2}{n}\sum_i \overrightarrow{y x_i}$$

$$H_{\sigma_{\mathbf{x}}^2} \approx 2Id$$

Geodesic marching

$$\bar{x}_{t+1} = \exp_{\bar{x}_t}(v) \quad \text{with} \quad v = E\left[\overrightarrow{y\mathbf{x}}\right]$$

Algorithms to compute the mean

Karcher flow (gradient descent)

$$\bar{x}_{t+1} = \exp_{\bar{x}_t}(\epsilon_t\, v_t) \;\; with \;\; v_t = \mathrm{E}(\overrightarrow{\mathbf{yx}}) = \frac{1}{\mathrm{n}} \sum_i \log_{\bar{x}_t}(x_i)$$

- Usual algorithm with $\epsilon_t = 1$ can diverge on SPD matrices [Bini & Iannazzo, Linear Algebra Appl., 438:4, 2013]
- Convergence for non-negative curvature (p-means) [Afsari, Tron and Vidal, SICON 2013]

Inductive / incremental weighted means

- $\bar{x}_{k+1} = \exp_{\bar{x}_k}\left(\frac{1}{k}\, v_k\right) \;\; with \;\; v_k = \log_{\bar{x}_k}(x_{k+1})$
- On negatively curved spaces [Sturm 2003], BHV centroid [Billera, Holmes, Vogtmann, 2001]
- On non-positive spaces [G. Cheng, J. Ho, H. Salehian, B. C. Vemuri 2016]

Stochastic algorithm

- [Arnaudon & Miclo, Stoch. Processes and App. 124, 2014]

Example on 3D rotations

Space of rotations SO(3):

- Manifold: $R^T.R=Id$ and $\det(R)=+1$
- Lie group ($R_1 \circ R_2 = R_1.R_2$ & Inversion: $R^{(-1)} = R^T$)

Metrics on SO(3): compact space, there exists a bi-invariant metric

- Left / right invariant / induced by ambient space $\langle X, Y\rangle = Tr(X^T Y)$

Group exponential

- One parameter subgroups = bi-invariant Geodesic starting at Id
  - Matrix exponential and Rodrigue's formula: R=exp(X) and X = log(R)
- Geodesic everywhere by left (or right) translation

$$Log_R(U) = R \log(R^T U) \qquad Exp_R(X) = R \exp(R^T X)$$

Bi-invariant Riemannian distance

- $d(R,U) = \|\log(R^T U)\| = \theta(R^T U)$

Example with 3D rotations

Principal chart: rotation vector : $r = \theta.n$

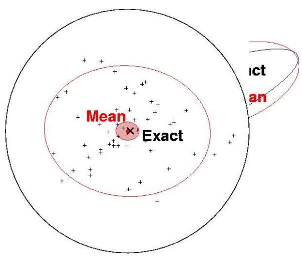
Distance: $\mathrm{dist}(R_1, R_2) = \left\| r_1^{(-1)} \circ r_2 \right\|$

Frechet mean:

$$\bar{R} = \arg \min_{R \in SO_3} \left(\sum_i \mathrm{dist}(R, R_i) \right)$$

Centered chart:

mean = barycenter



tPCA vs PGA

tPCA

- Generative model: Gaussian
- Find the subspace that best explains the variance
  - → Maximize the squared distance to the mean

PGA (Fletcher 2004, Sommer 2014)

- Generative model:
  - Implicit uniform distribution within the subspace
  - Gaussian distribution in the vertical space
- Find a low dimensional subspace (geodesic subspaces?) that minimizes the error
  - → Minimize the squared Riemannian distance from the measurements to that sub-manifold (no closed form)

Different models in curved spaces (no Pythagore thm)

Extension to BSA (Pennec 2018)

Distributions for parametric tests

Uniform density:

- maximal entropy knowing X $\quad p_{\mathbf{x}}(z) = \mathrm{Ind}_X(z) / \mathrm{Vol}(X)$

Generalization of the Gaussian density:

- Stochastic heat kernel p(x,y,t) [complex time dependency]
- Wrapped Gaussian [Infinite series difficult to compute]
- Maximal entropy knowing the mean and the covariance

$$N(y) = k.\exp\left(\left(\overrightarrow{\bar{\mathbf{x}}\mathbf{x}}\right)^{\mathrm{T}} .\boldsymbol{\Gamma}.\left(\overrightarrow{\bar{\mathbf{x}}\mathbf{x}}\right)/2\right) \qquad \begin{aligned} \boldsymbol{\Gamma} &= \boldsymbol{\Sigma}^{(-1)} - \tfrac{1}{3}\mathrm{Ric} + O(\sigma) + \varepsilon(\sigma / r) \\ k &= (2\pi)^{-n/2}.\det(\boldsymbol{\Sigma})^{-1/2}.\left(1 + O(\sigma^3) + \varepsilon(\sigma / r)\right) \end{aligned}$$

Mahalanobis D2 distance / test: $\quad \mu^2_{\mathbf{x}}(\mathrm{y}) = \overrightarrow{\bar{\mathrm{x}}\mathrm{y}}^{\,t} .\Sigma^{(-1)}_{\mathbf{xx}} .\overrightarrow{\bar{\mathrm{x}}\mathrm{y}}$

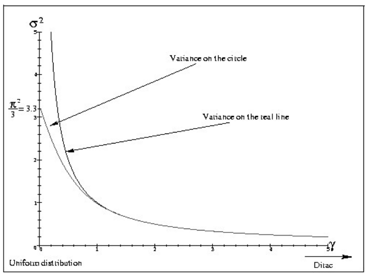
- Any distribution: $\quad \mathrm{E}\left[\mu^2_{\mathbf{x}}(\mathbf{x})\right] = n$
- Gaussian: $\quad \mu^2_{\mathbf{x}}(\mathbf{x}) \propto \chi^2_n + O(\sigma^3) + \varepsilon(\sigma / r)$

[Pennec, JMIV06, NSIP'99]

Gaussian on the circle

Exponential chart: $x = r\theta \in \left]-\pi.r\,;\pi.r\right[$

Gaussian: truncated standard Gaussian



- $r \to \infty$: standard Gaussian (Ricci curvature → 0)
- $\gamma \to 0$: uniform pdf with $\sigma^2 = (\pi.r)^2 / 3$ (compact manifolds)
- $\gamma \to \infty$: Dirac

Computing on manifolds: a summary

The Riemannian metric easily gives
- Intrinsic measure and probability density functions
- Expectation of a function from M into R (variance, entropy)

Integral or sum in M: minimize an intrinsic functional
- Fréchet / Karcher mean: minimize the variance
- Filtering, convolution: weighted means
- Gaussian distribution: maximize the conditional entropy

The exponential chart corrects for the curvature at the reference point
- Gradient descent: geodesic walking
- Covariance and higher order moments
- Laplace Beltrami for free

[Pennec, NSIP'99, JMIV 2006, Pennec et al, IJCV 66(1) 2006, Arsigny, PhD 2006]

Course overview

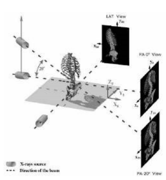
Introduction
- Why do we need statistics on manifolds?

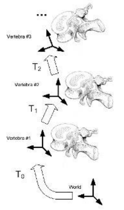
Simple statistics on interesting manifolds
- The Riemannian Geometric framework
- Simple Statistics
  - Mean, Covariance, Gaussian, t-tests

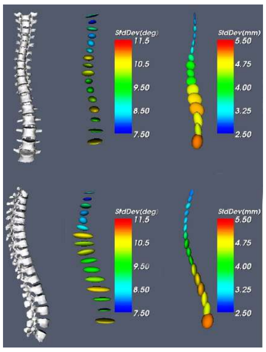
Application to registration
- Statistics on spine shapes
- Evaluation of registration accuacy

Statistical Analysis of the Scoliotic Spine

[J. Boisvert, X. Pennec, N. Ayache, H. Labelle, F. Cheriet,, ISBI'06]





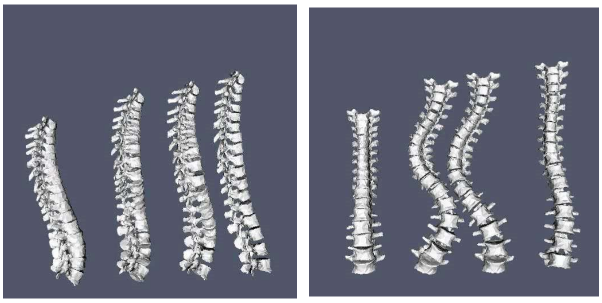


Database
- 307 Scoliotic patients from the Montreal's Sainte-Justine Hospital.
- 3D Geometry from multi-planar X-rays

Mean
- Main translation variability is axial (growth?)
- Main rotation var. around anterior-posterior axis

PCA of the Covariance
- 4 first variation modes have clinical meaning

Statistical Analysis of the Scoliotic Spine

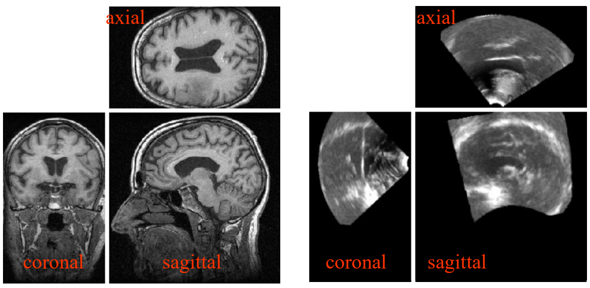


- Mode 1: King's class I or III
- Mode 2: King's class I, II, III
- Mode 3: King's class IV + V
- Mode 4: King's class V (+II)

MR-US Images

Pre - Operative MR Image

Per - Operative US Image

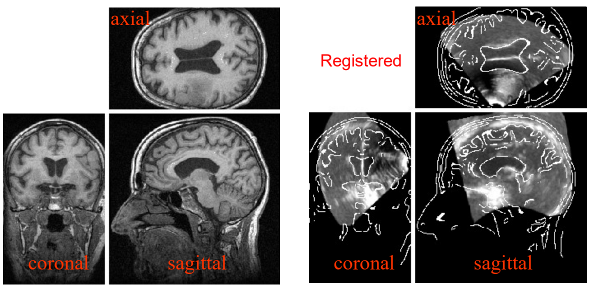


Acquisition of images : L. & D. Auer, M. Rudolf

Typical Registration Result with Bivariate Correlation Ratio

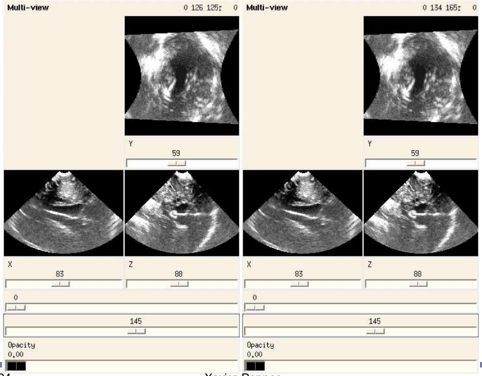
Pre - Operative MR Image

Per - Operative US Image

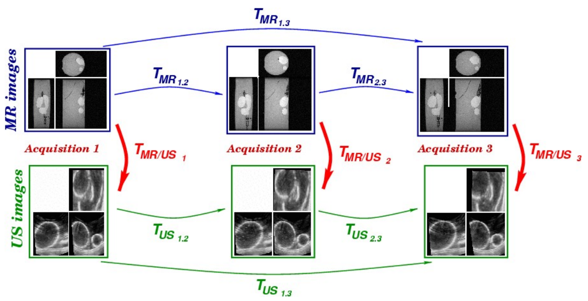


Acquisition of images : L. & D. Auer, M. Rudolf

US Intensity
MR Intensity and Gradient

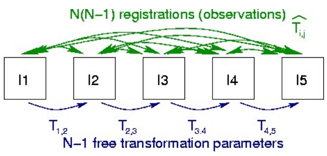


Accuracy Evaluation (Consistency)



$$\sigma^2_{loop} = 2\sigma^2_{MR/US} + \sigma^2_{MR} + \sigma^2_{US}$$

Validation using Bronze Standard



Best explanation of the observations (ML) : $C = \sum_{ij} d^2(T_{ij}, \hat{T}_{ij})$

- LSQ criterion
- Robust Fréchet mean $d^2(T_1, T_2) = \min\left(\mu^2(T_1, T_2), \chi^2\right)$
- Robust initialization and Newton gradient descent
- Grid scheduling for efficiency

[T. Glatard & al, MICCAI 2006, Int. Journal of HPC Apps, 2006]

Result $T_{i,j}, \sigma_{rot}, \sigma_{trans}$

Results on per-operative patient images

Data (per-operative US)

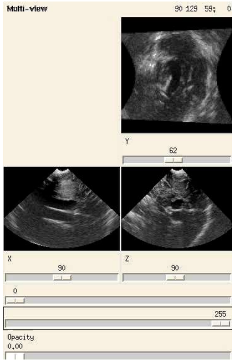
- 2 pre-op MR (0.9 x 0.9 x 1.1 mm)
- 3 per-op US (0.63 and 0.95 mm)
- 3 loops

Robustness and precision

	Success	var rot (deg)	var trans (mm)
MI	29%	0.53	0.25
CR	90%	0.45	0.17
BCR	**85%**	**0.39**	**0.11**

Consistency of BCR

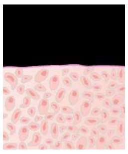
	var rot (deg)	var trans (mm)	var test (mm)
Multiple MR	0.06	0.06	0.10
Loop	2.22	0.82	2.33
MR/US	**1.57**	**0.58**	**1.65**

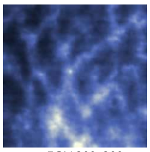


[Roche et al, TMI 20(10), 2001]
[Pennec et al, Multi-Sensor Image Fusion, Chap. 4, CRC Press, 2005]

Mosaicing of Confocal Microscopic in Vivo Video Sequences.

Cellvizio: Fibered confocal fluorescence imaging

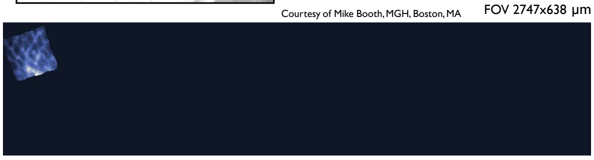




FOV 200x200 µm

Courtesy of Mike Booth, MGH, Boston, MA

FOV 2747x638 µm



[T. Vercauteren et al., MICCAI 2005, T.1, p.753-760]

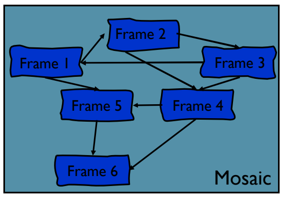
Mosaicing of Confocal Microscopic in Vivo Video Sequences.

Common coordinate system

- **Multiple rigid** registration
- Refine with **non rigid**

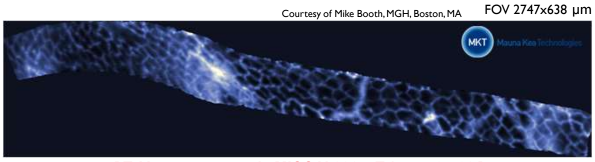
Mosaic image creation

- Interpolation / approximation with **irregular sampling**



Courtesy of Mike Booth, MGH, Boston, MA

FOV 2747x638 µm



[T. Vercauteren et al., MICCAI 2005, T.1, p.753-760]