Medical Imaging : Image Filtering & Segmentation

Hervé Delingette

Epione Team

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Course teachers

- Hervé Delingette 
- Xavier Pennec 

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1. Biomedical Image/Data Analysis, Machine Learning
2. Imaging & Phenomics, Biostatistics
3. Computational Anatomy, Geometric Statistics
4. Computational Physiology & Image-Guided Therapy
5. Computational Cardiology & Image-Based Interventio

Course Schedule

https://www-sop.inria.fr/asclepios/cours/MVA/

- Tuesday Oct 1, 2024, 14:00-17:15 (ENS Saclay, salle 1Z25) Introduction to Medical Image Acquisition, Image Registration [Xavier Pennec]
- Tuesday Oct 8, 2024, 14:00-17:15 (ENS Saclay, salle 1Z25) Riemanian Geometry and Statistics [Xavier Pennec]
- Tuesday Oct 15, 2024, 14:00-17:15 (ENS Saclay salle 1Z25) Image Filtering & Segmentation [Hervé Delingette]
- Tuesday Oct 22, 2024: 14:00-17:15 (ENS Saclay, salle 1Z25) Image Segmentation based on Clustering and Markov Random Fields [Hervé Delingette]
- Tuesday Nov 5, 2024: 14:00-17:15 (ENS Saclay, salle 1Z25) Analysis in the space of Covariance Matrices [Xavier Pennec]
- Tuesday Nov 12, 2024: 14:00-17:15 (ENS Saclay, salle 1Z25) Shape constrained image segmentation [Hervé Delingette]
- Tuesday Nov 19 2024: 14:00-17:15 (ENS Saclay, salle 1Z25) Diffeomorphic Registration and Computational Anatomy [Xavier Pennec]
- Tuesday Nov 26 2024: 14:00-17:15 (ENS Saclay, salle 1Z25) Biophysical Modeling [Hervé Delingette]
- Tuesday Dec 3, 2024, 14:00-17:15 (Visio) Exam [Hervé Delingette, Xavier Pennec]

Course Exam

- 4 components :
  - Scientific Article Study :
    - 10 min oral presentation
    - 10 min Questions & Answers
    - 5-6 page report presenting the paper and putting it in perspective.
    - Implementation (optional)
    - May be performed in pairs or triplets depending on class size
  - Multiple choice Quizz : 10-15 questions

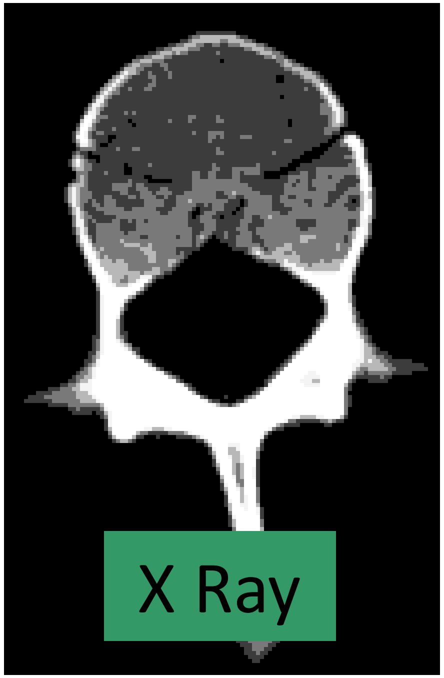
1. Medical Image Representation & Visualization

➡ 1.1 Image representation : discrete or continuous

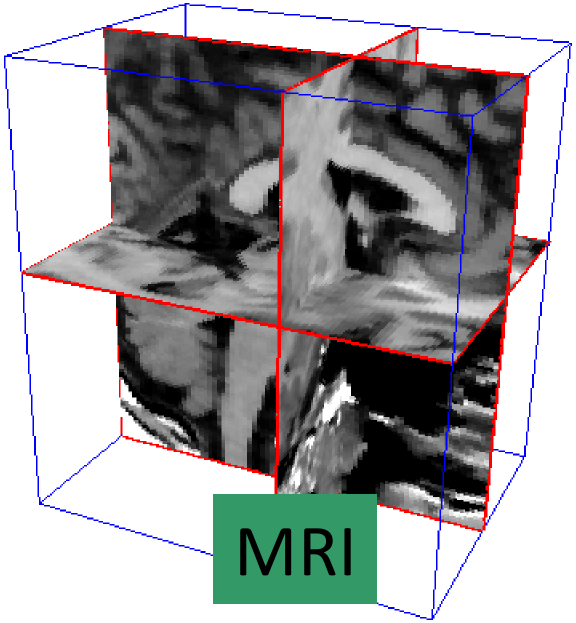
1.2 Image Visualization

Medical Imaging Classification (1)

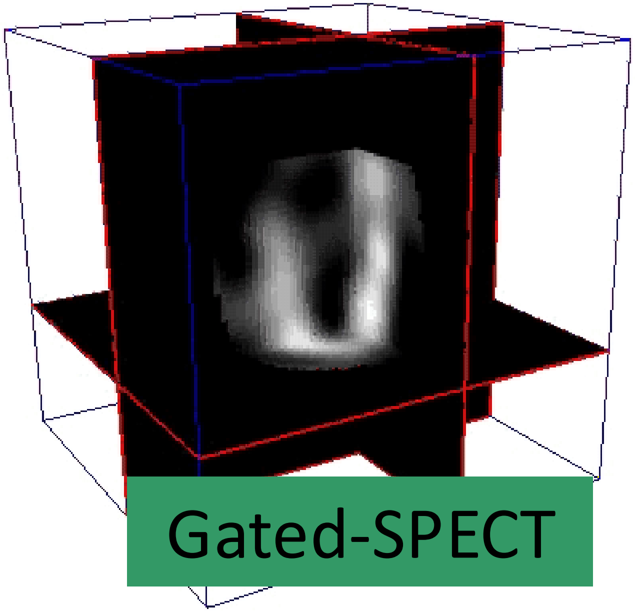
- Dimensionality



2D



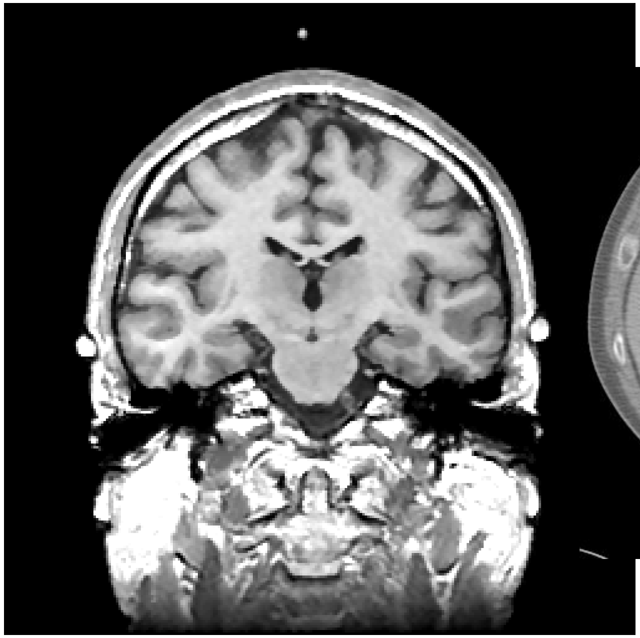
3D



4D (3D+T)

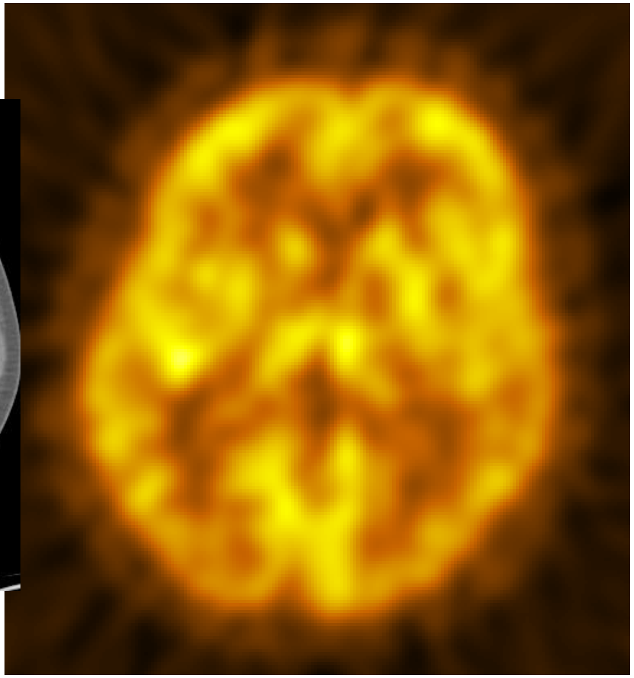
Medical Imaging Classification (2)

- Anatomical vs functional Imagery



MRI

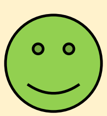
CT with contrast agent



PET scan

Anatomical → Functional

Medical Image Processing
vs
Computer Vision

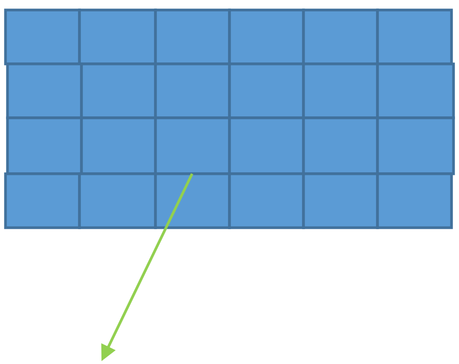
	Computer Vision	Medical Image Processing
	Projective Geometry Occluding Objects Intensity depends on lighting	Complex Image Formation Large Datasets Patient Images
	Easy to acquire Low dimensionality	Cartesian Geometry Statistics Information Intensity links to physics Patient Images

Discrete Image Representation (1)

- Domain is considered as a 2D/3D regular grid

2D Array I

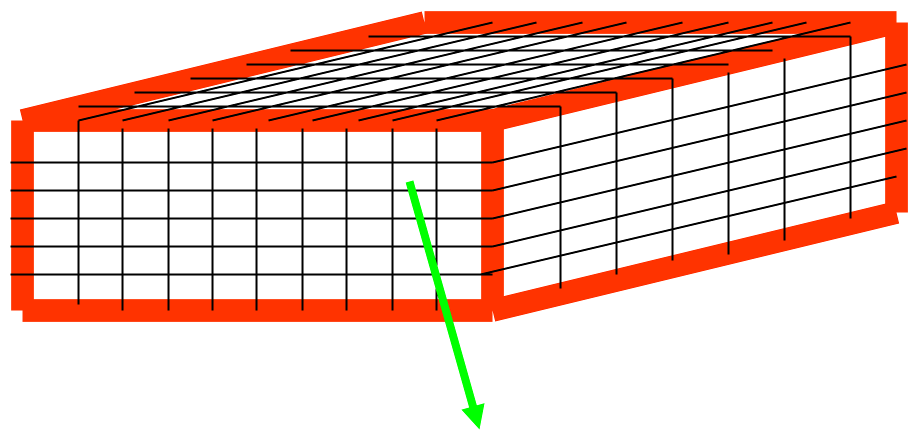
I[col][row]



Pixel=picture element

3D Array I

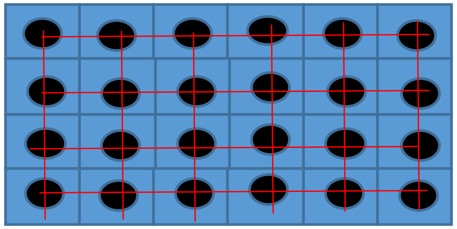
I[plane][col][row]



Voxel=
Volume Element

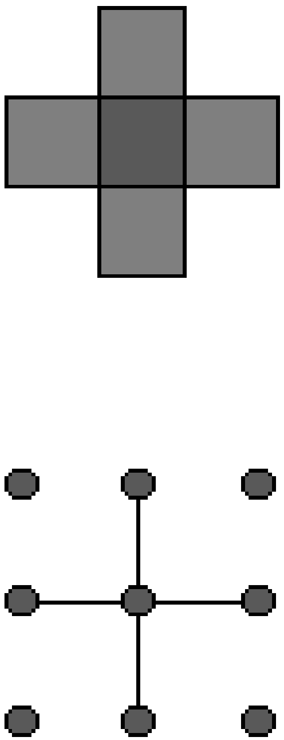
Discrete Image Representation (2)

- Pixel / Voxel values can be :
  - Discrete :
    - Integer : char (MRI), signed short (CT-scan)
    - Labels of structures
  - Continuous :
    - Float / double
- Images can be seen as a graph
  - Nodes are pixel / voxel centers
  - Edges between adjacent elements
  - Grid Duality

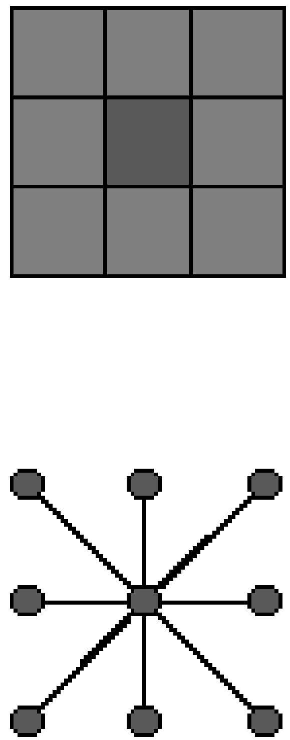


Neighborhood

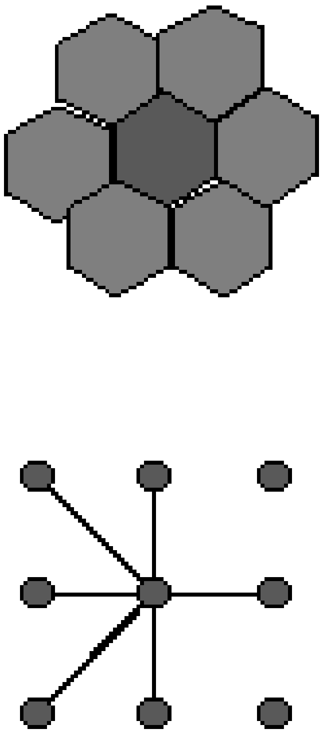
- Different types of neighborhood in 2D



4 -neighborhood



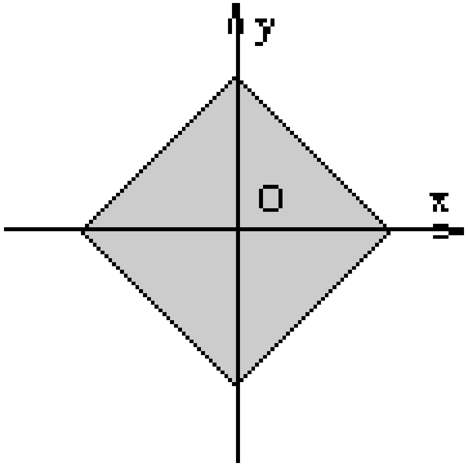
8 -neighborhood

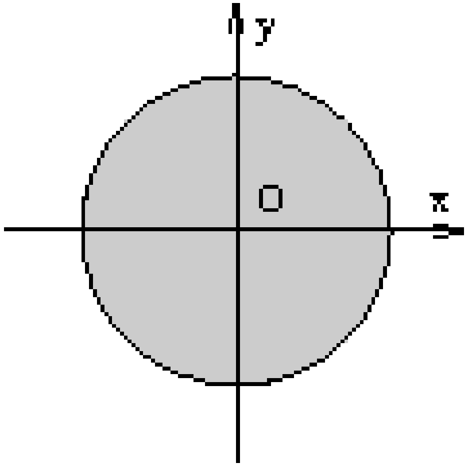


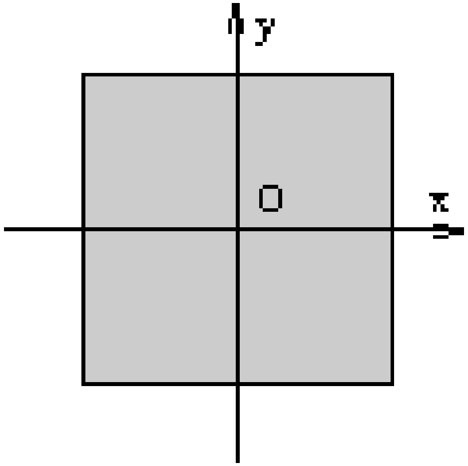
6 -neighborhood

Neighborhood

- Some generalizes to higher dimensions
- Corresponds to a choice of metric norm



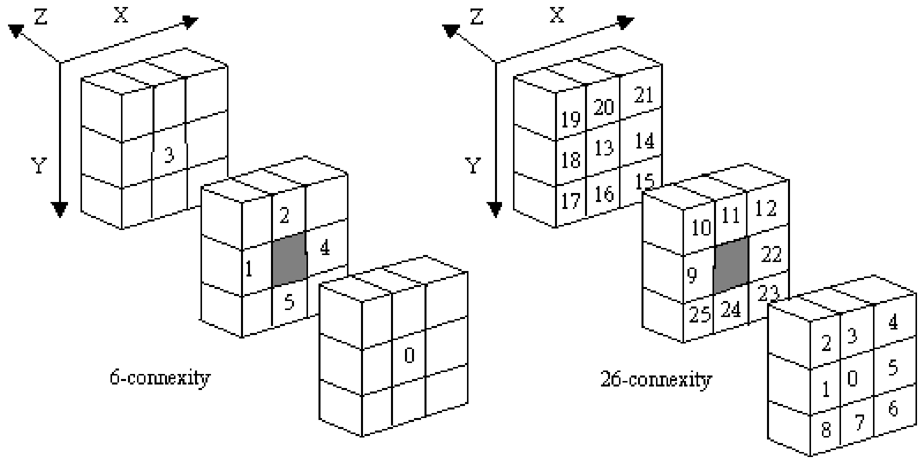




$$D_1(x,y)=\sum_{i=1}^{n}|y_i-x_i| \qquad D_2(x,y)=\sqrt{\sum_{i=1}^{n}(y_i-x_i)^2} \qquad D_\infty(x,y)=\max_{i=1\ldots n}|y_i-x_i|$$

Neighborhood

- 3 types of neighborhood for a 3D image :
  - 6-neighborhood : adjacency through faces
  - 18-neighborhood : adjacency through faces and edges
  - 26-neighborhood : adjacency through faces and edges and vertices



Continuous Image representation

- Image seen as 2D or 3D Fields : $I(x), x \in \mathbb{R}^n, n = 2,3$
- Requires to define interpolation functions

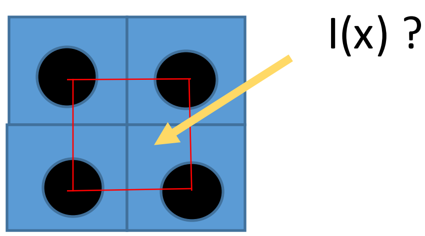
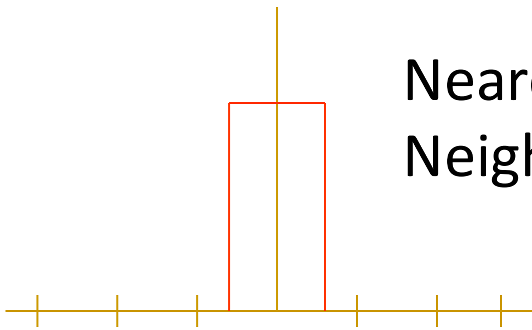


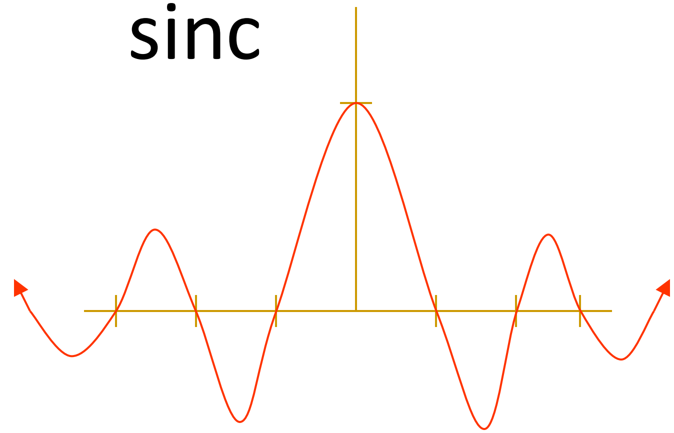
Image Domain / Image Value	**Discrete**	**Continuous**
Discrete	Array of Int	~~Field of Integer~~
Continuous	Array of Float	Field of Float

1D Interpolation functions



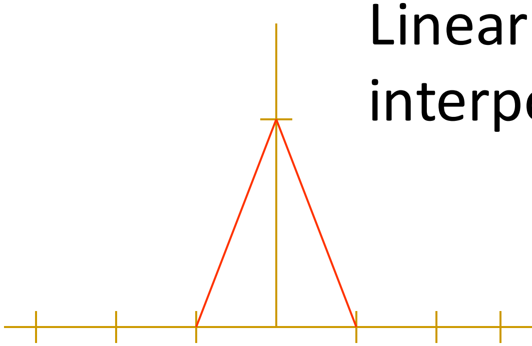
Nearest Neighbor

Discontinuous



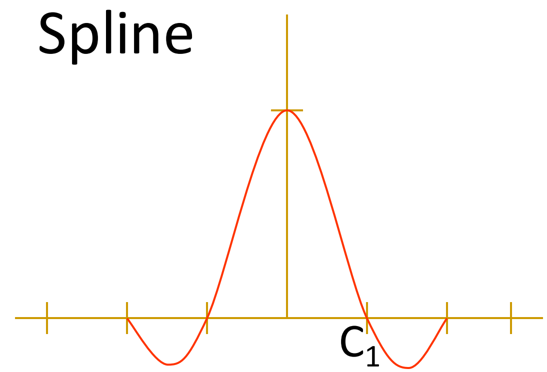
sinc

C_{00} continuity



Linear interpolation

C_0 continuity



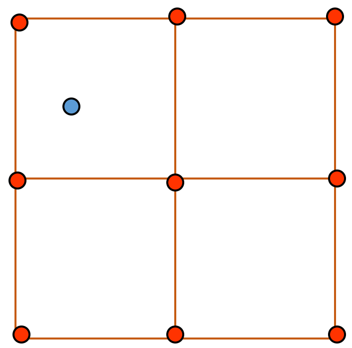
Spline

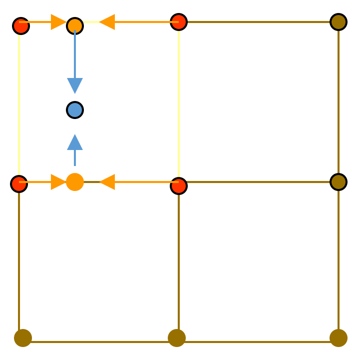
C_1 continuity

Source : Joe Michael Kniss – Direct Volume Rendering

Bilinear Interpolation (2D Field)

- Bilinear Interpolation : 3 linear interpolations



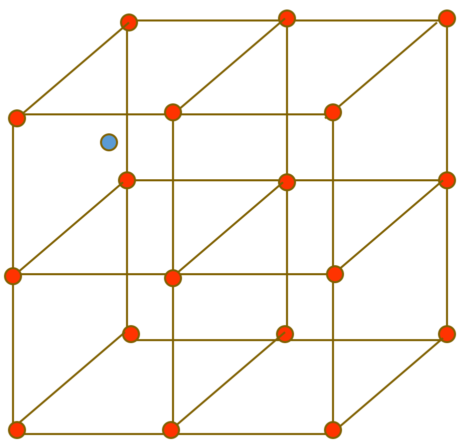


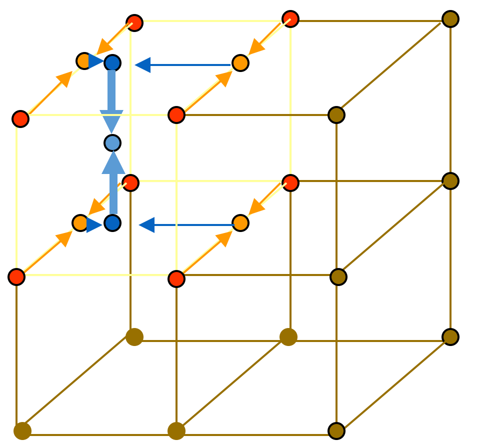
$$I(u,v) = (1-u)(1-v)I_{i,j} + u\,v\,I_{i+1,j+1} + (1-u)v\,I_{i,j+1} + u\,(1-v)I_{i+1,j}$$

Source : Joe Michael Kniss – Direct Volume Rendering

Trilinear Interpolation (3D Field)

- 7 linear interpolations

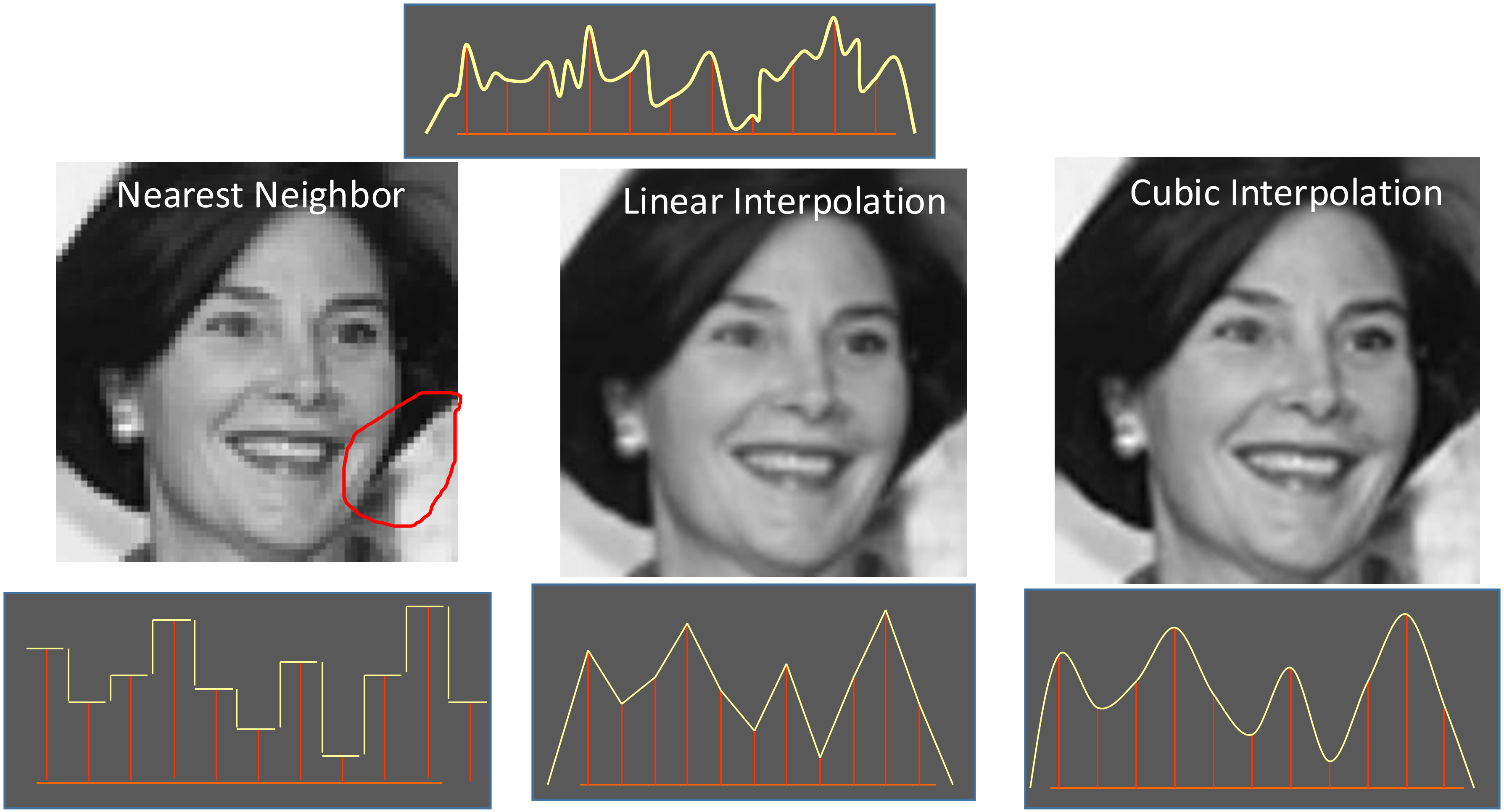




$$I(u,v,w) = (1-u)(1-v)(1-w)I_{i,j,k} + u\,v\,w\,I_{i+1,j+1,k+1} + (1-u)v\,w\,I_{i,j+1,k+1} + u\,(1-v)w\,I_{i+1,j,k+1} + (1-u)v\,(1-w)I_{i,j+1,k} + u\,(1-v)(1-w)I_{i+1,j,k} + u\,v\,(1-w)\,I_{i+1,j+1,k} + (1-u)\,(1-v)w\,I_{i,j,k+1}$$

Source : Joe Michael Kniss – Direct Volume Rendering

Image interpolation



Source : Joe Michael Kniss – Direct Volume Rendering

Medical Image Format

- Industrial standard :
  - **DICOM** : Digital Imaging and COmmunications in Medicine
  - More a communication standard for interoperability than an image format
- Academic standard :
  - Must support volumetric images, generic voxel format (short, double, array of double), voxel size, metadata
  - ITK based : MHA, MHD
  - NIFTI : Neuroimaging Informatics Technology Initiative

1. Medical Image Representation & Visualization

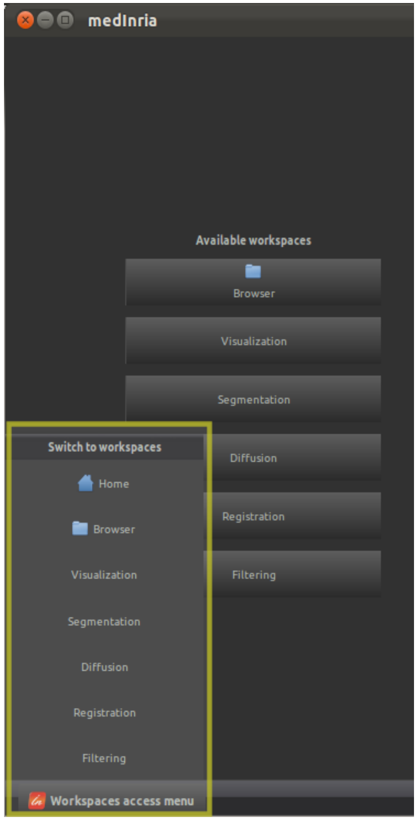
1.1 Image representation : discrete or continuous

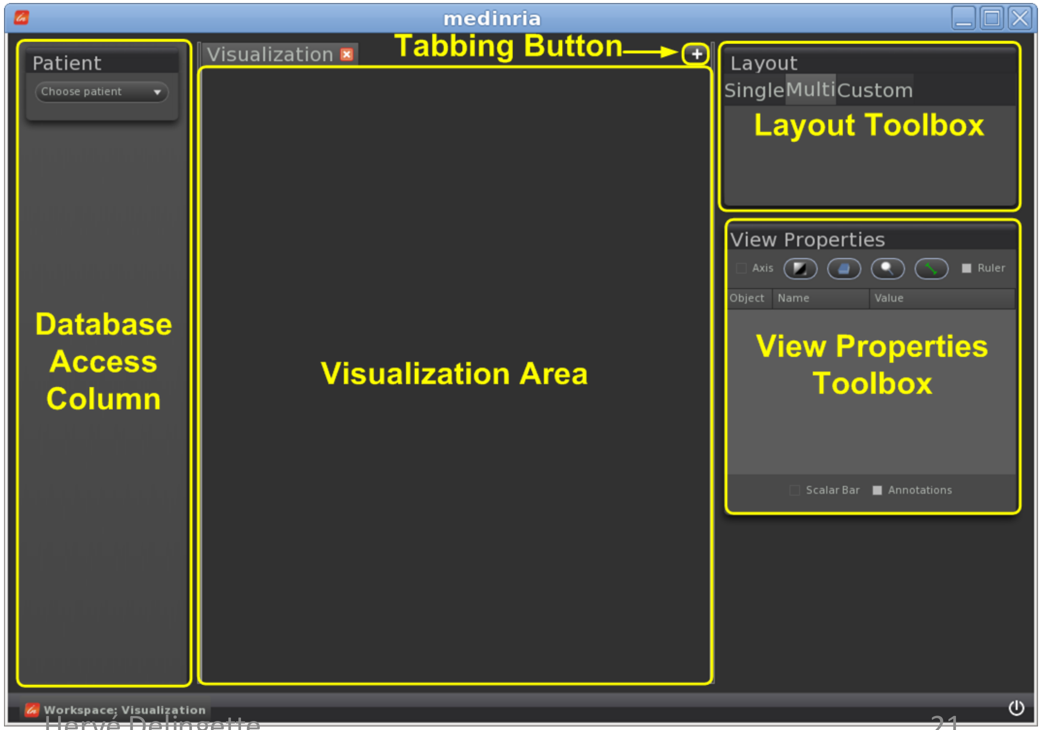
➡ 1.2 Image Visualization

Visualizing Medical Images

- MedInria : https://med.inria.fr
  - Free & Multiplatform, plugin based.

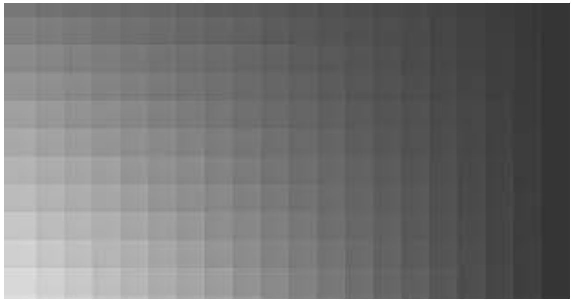
Quick Access Menu



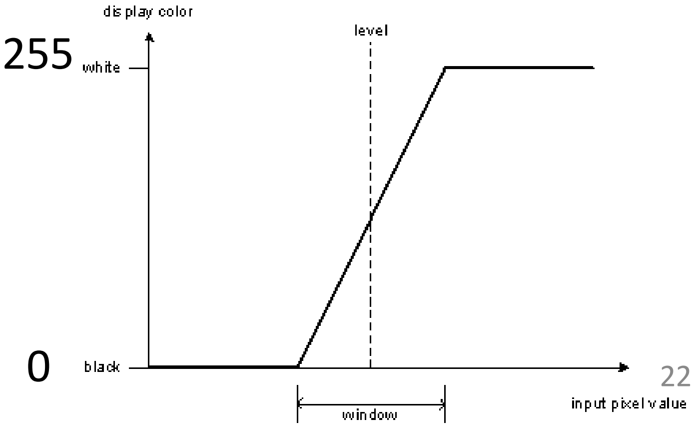


Visualizing Medical Images

- Windowing
  - CT images are coded on 2 bytes (2^{16} values).
  - The human eye can only see a limited (200 ?) number of shades of grey !

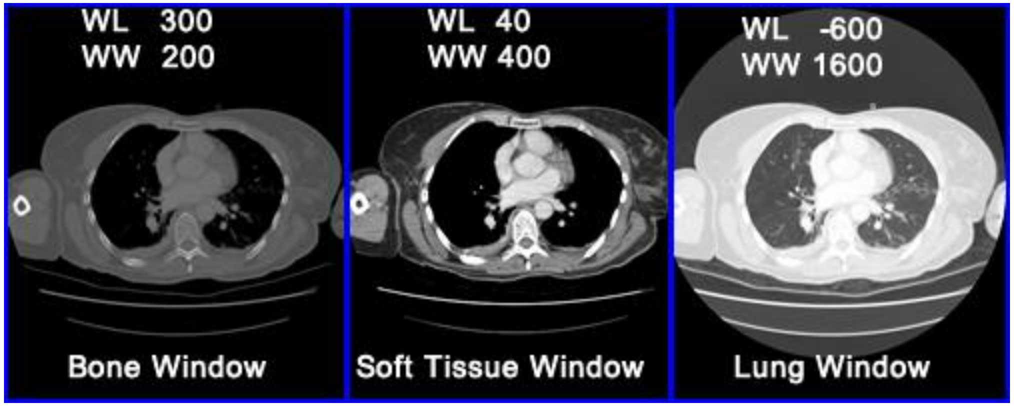


- Need to perform windowing i.e. map a range of intensity values in the [0,255] range

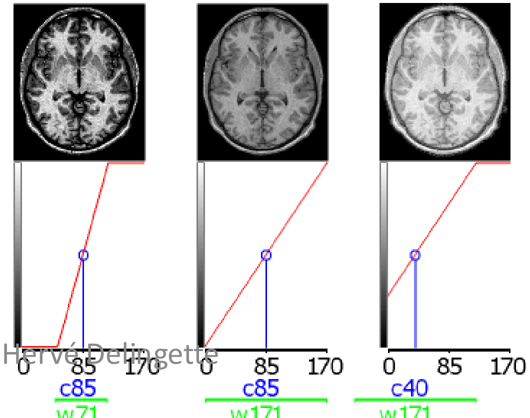


Visualizing Medical Images

- Windowing
  - Predefined windows on CT as Hounsfield units are absolute



Brain MRI Image

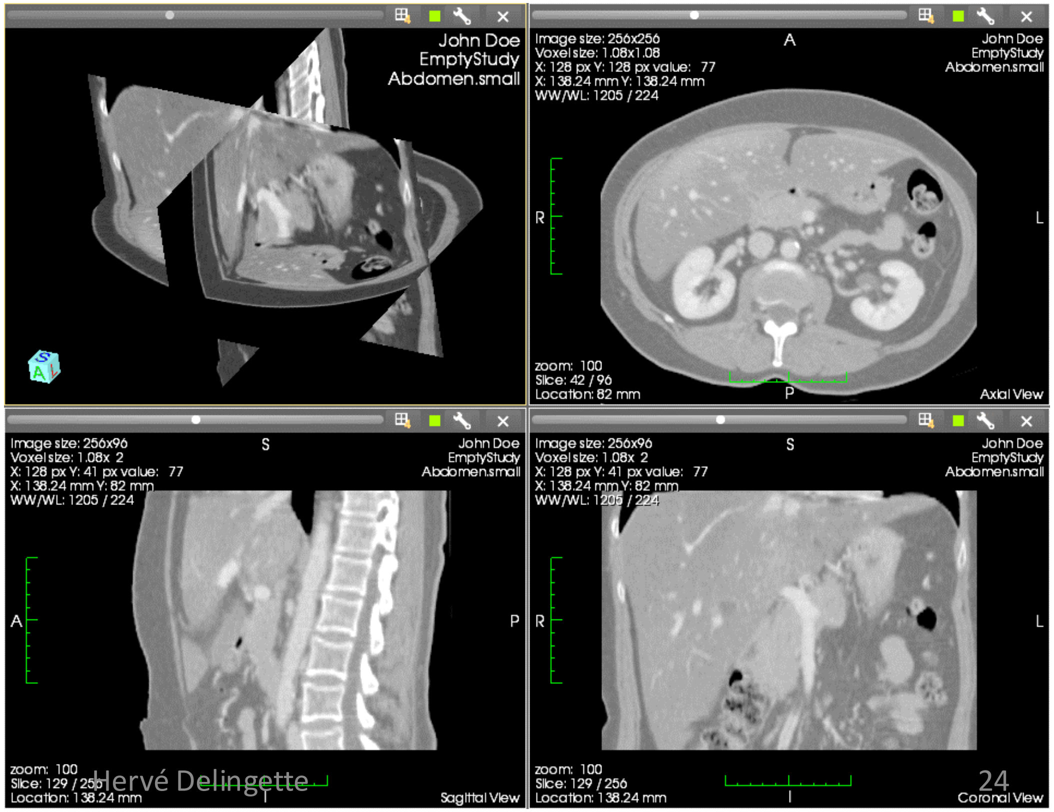


Visualizing Medical Images

- Volumetric Images

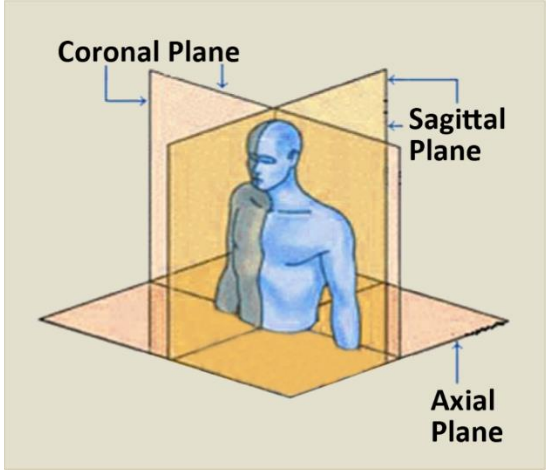
How to visualize a 3D image on a 2D screen ?

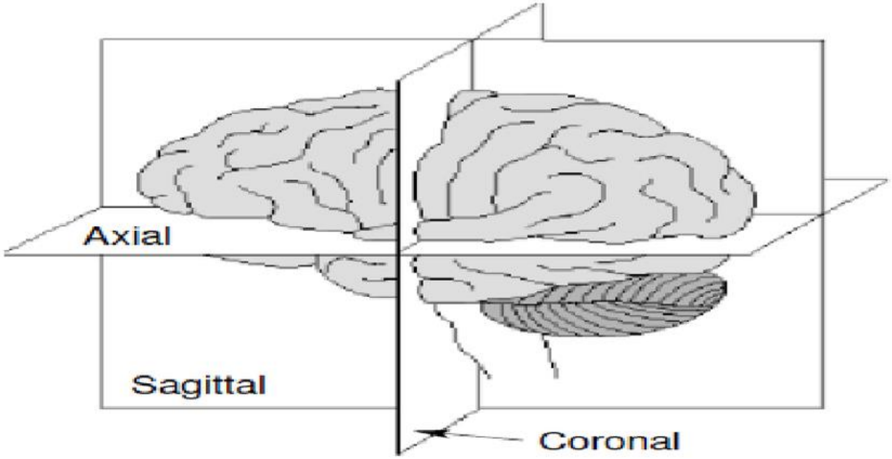
Multiplanar Reformating



Visualizing Medical Images

- Volumetric Images
  - Radiological Convention to name the 3 orthogonal slices.
  - Axial, coronal and saggital





2.0 Image Filtering

- 2.1 **Taxonomy of Filters**
  - Linear vs Non-Linear
  - Separable vs Non-Separable
  - Recursive Filters
- 2.2 Smoothing Filters :
- 2.3 Gradient Filters :
- 2.4 Image Contour Extraction
- 2.5 Mathematical Morphology

2.0 Image Filtering

- 2.1 Taxonomy of Filters
  - Linear vs Non-Linear
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- 2.2 Smoothing Filters :
  - Isotropic vs Anisotropic
  - Recursive
  - Non-Local
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  - Discrete vs Continuous
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Image Filtering

- Image Filtering consists in extracting local information : basis for feature extraction
- Different types of filters:
  - Linear : = Convolution with a function (filter)
  - Non-linear
  - Separable : a *n*D filter may be written as the composition of several 1D filters
  - Non-separable

Image Convolution

- convolution in the continuous domain

$$(I * f)(M_0) = \int_{\text{Image}} I(M) f(M_0 - M) dM$$

- convolution in the discrete domain

$$(I * f)(M_0) \approx \sum_{\text{Image}} I(M) f(M_0 - M)$$

- Notation

Matrix of size (2M+1)x(2M+1)

$$\bar{I} = I * \begin{pmatrix} A_{00} & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & A_{2M,2M} \end{pmatrix}$$

$$\bar{I}(x,y) = \sum_{i=-M}^{M} \sum_{j=-M}^{M} I(x+i, y+j) * A(M+i, M+i)$$

Properties of Convolution

- Transitive combinaison of convolution filters

$$I * (f * g) = (I * f) * g$$

- Derivatives of convolution filters

$$\begin{aligned}(I * f)^{(n)} &= I^{(n)} * f \\ &= I^{(k)} * f^{(n-k)} = I * f^{(n)}\end{aligned}$$

- This property is not verified for discrete convolutions

Separable Convolution Filters

- Separable filters lead to efficient implementation
  - In dimension *n*, the application of a convolution filter f of size $\mathrm{p}^{\mathrm{n}} = p \times p \times \cdots \times p$ on an image *I* of size d^n, requires :
    - ❑ number of multiplications : $d^n p^n$
    - ❑ number of additions : $d^n(p^n-1)$
  - if *f* is separable along *n* directions, it requires :
    - ❑ nombre de multiplications : $n(d^n p)$
    - ❑ nombre d'additions : $n(d^n(p-1))$

Convolution

- Derivation and Separation

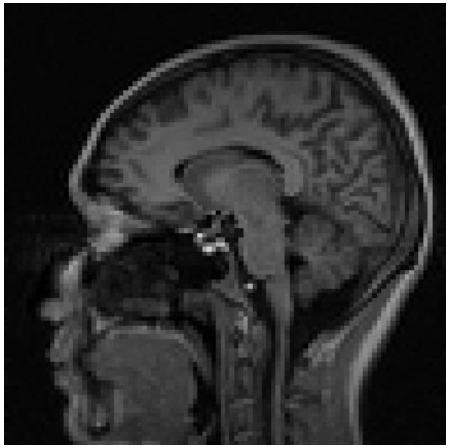
$$\frac{\partial\left(I * \left(f^x(x) * f^y(y)\right)\right)}{\partial x} = I * \frac{\partial\left(f^x(x) * f^y(y)\right)}{\partial x}$$

$$= I * f_x^x(x) * f^y(y)$$

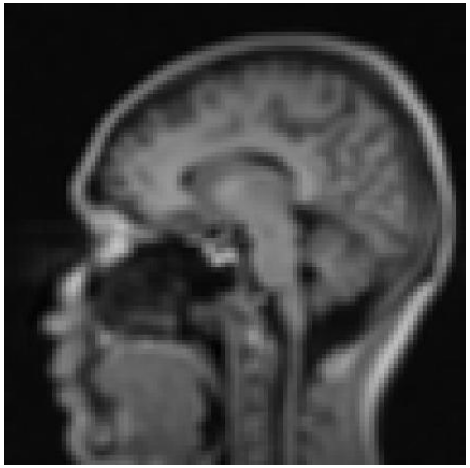
Example : Mean Filter

$$\bar{I}(x,y) = \frac{1}{9}\sum_{i=-1}^{1}\sum_{j=-1}^{1} I\left(x+i, y+j\right)$$

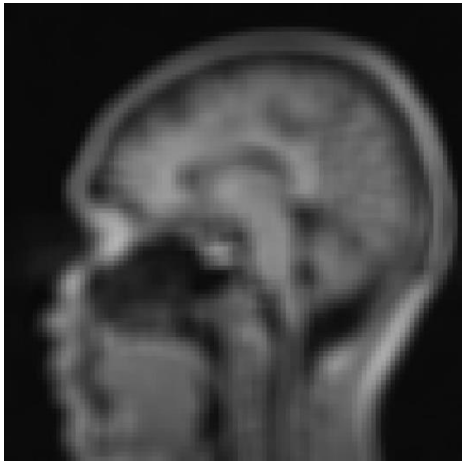
- Can be written as : $\bar{I} = I * \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$



Input MR Image



MR Image with 3x3 Mean Filter



MR Image with 5x5 Mean Filter

Example : Mean Filter

- Separable

$$\bar{I}(x,y) = \frac{1}{3}\sum_{i=-1}^{1}\left(\frac{1}{3}\sum_{j=-1}^{1} I(x+i, y+j)\right)$$

- Can be written as

$$\bar{I} = \left(I * \frac{1}{3}\begin{pmatrix}1 & 1 & 1\end{pmatrix}\right) * \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Non Linear Separable Filters

- maximum and minimum filters

$$I \oplus \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} = \max_{i=-1\ldots1,\ j=-1\ldots1} I\left(x+i, y+j\right)$$

$$= \max_{j=-1\ldots1}\left(\max_{i=-1\ldots1} I\left(x+i, y+j\right)\right)$$

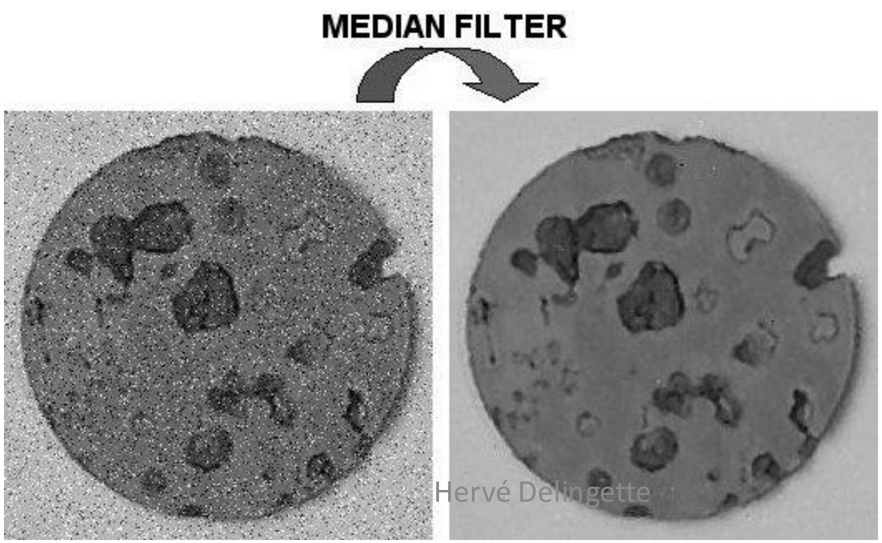
Non Linear Non Separable Filters

- Median Filter

$\Rightarrow$ Remove Impulse noise

- For a given voxel *M*, values of its *n* neighbors are sorted
- Output value for M is the value of rank *n*/2

Use moving window with fast sort algorithms (*quicksort*, …)



Recursive Filters

- Limitation of discrete convolution :
  - Finite support
  - Computation load increases with support size
- Discrete 1D convolution filter of size M implemented as:

$$y[n] = \sum_{m=n-M}^{n+M} a[n-m]x[m]$$

aka Finite Impulse Response
FIR Filter

- Recursive Filters lead to infinite support filters

$$y[n] = \sum_{m=n-M}^{n} a[n-m]x[m] + \sum_{m=n-M}^{n-1} b[n-m]y[m]$$

  - Need to combine causal and anti-causal filters

aka Infinite Impulse Response
IIR Filter

2.0 Image Filtering

- 2.1 Taxonomy of Filters
- 2.2 **Smoothing Filters**
  - Isotropic vs Anisotropic
  - Recursive
  - Non-Local
- 2.3 Gradient Filters :
  - Discrete vs Continuous
- 2.4 Image Contour Extraction
- 2.5 Mathematical Morphology

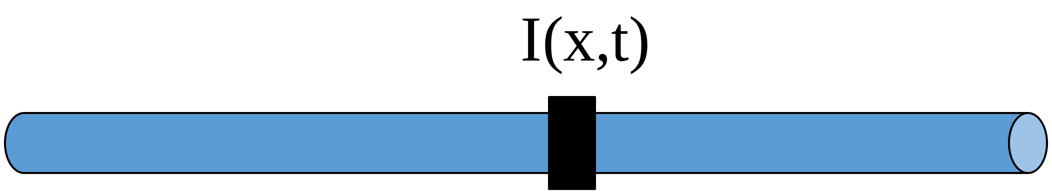
Image Smoothing

- Mean Filter, Triangular Filter

- Gaussian Filter G(μ;σ):
  - infinite support therefore inefficient computation
  - Equivalent to solving an isotropic diffusion problem
    - 1D problem

$$\frac{\partial I}{\partial t} = k \frac{\partial^2 I}{\partial x^2}$$

Minimize $E = \frac{1}{2} k \int \left\| \nabla I \right\|^2 d\Omega$



$$I(x,t) \approx \int \exp\left(\frac{(x-y)^2}{4kt} \right) I(y,0)\, dy = G\left(\sqrt{4kt}\right) * I(x,0)$$

Binomial Filters

$$\left[\frac{1}{2} \quad \frac{1}{2}\right]^n$$

- Separable and finite support
- Iterative convolution of a simple filter

n=2 $\frac{1}{4}\begin{bmatrix}1 & 2 & 1\end{bmatrix}$ n=3 $\frac{1}{8}\begin{bmatrix}1 & 3 & 3 & 1\end{bmatrix}$ n=4 $\frac{1}{16}\begin{bmatrix}1 & 4 & 6 & 4 & 1\end{bmatrix}$

- Coefficient of rank k in a filter of size n given by (see Pascal Triangle)

$$\frac{1}{2^n}\binom{n}{k} = \frac{n!}{2^n k!(n-k)!}$$

- Converge towards a Gaussian distribution $G\left(\frac{n}{2}; \frac{\sqrt{n}}{2}\right)$
- Separable Filters

$$\frac{1}{4}\begin{bmatrix}1 & 2 & 1\end{bmatrix} * \frac{1}{4}\begin{bmatrix}1 \\ 2 \\ 1\end{bmatrix} = \frac{1}{16}\begin{bmatrix}1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1\end{bmatrix}$$

Gaussian Recursive Filtering

- Approximate Gaussian Filter with recursive filters
- Two approaches
  - Sum of causal & anti-causal filters

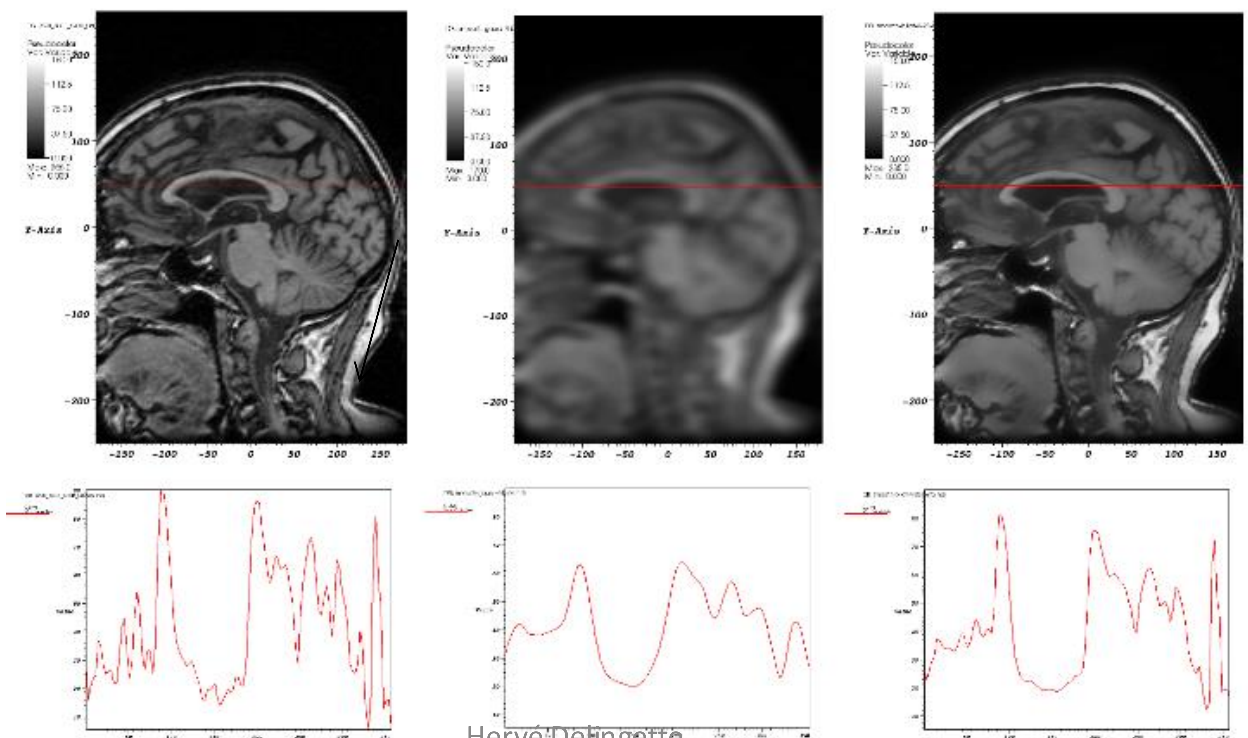
    Deriche, R., 1992, Recursively implementing the Gaussian and its derivatives: Proceedings of the 2nd International Conference on Image Processing, Singapore, p. 263–267
  - Product of causal & anti-causal filters

van Vliet, L., Young, I., and Verbeek, P. 1998, Recursive Gaussian derivative filters: Proceedings of the International Conference on Pattern Recognition, Brisbane, p. 509–514.

Anisotropic Diffusion

- Isotropic diffusion blurs edges in images

Input Image | Isotropic Diffusion | Anisotropic Diffusion



Source vis.lbl.go

Anisotropic Diffusion

- Isotropic diffusion in 3D

$$\frac{\partial I}{\partial t} = k\left(\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} + \frac{\partial^2 I}{\partial z^2}\right) = k\,\Delta I = k\,div\,\nabla I$$

- Anisotropic diffusion in 3D

$$\frac{\partial I}{\partial t} = div\,c(x,y)\nabla I = \nabla c \cdot \nabla I + c(x,y)\Delta I$$

  - *c(x,y)* is the diffusion coefficient

Diffusion coefficient

- Choose $c(x, y, z) = c(\|\nabla I\|)$
- Minimize $E = \frac{1}{2}\int g\left(\|\nabla I\|^2\right) d\Omega$ $\quad c(x, y, z) = \frac{dg(\|\nabla I\|)}{d\|\nabla I\|}$
- Choice of the diffusion coefficient :
  - Should be high in homogeneous regions
  - Show be low around edges

- Perona-Malik

$$c(\|\nabla I\|) = e^{-\left(\frac{\|\nabla I\|}{K}\right)^2} \qquad c(\|\nabla I\|) = \frac{1}{1+\left(\frac{\|\nabla I\|}{K}\right)^2}$$

Non-Local Means

- Denoising by computing the voxel value as the weighted sum of neighboring pixels

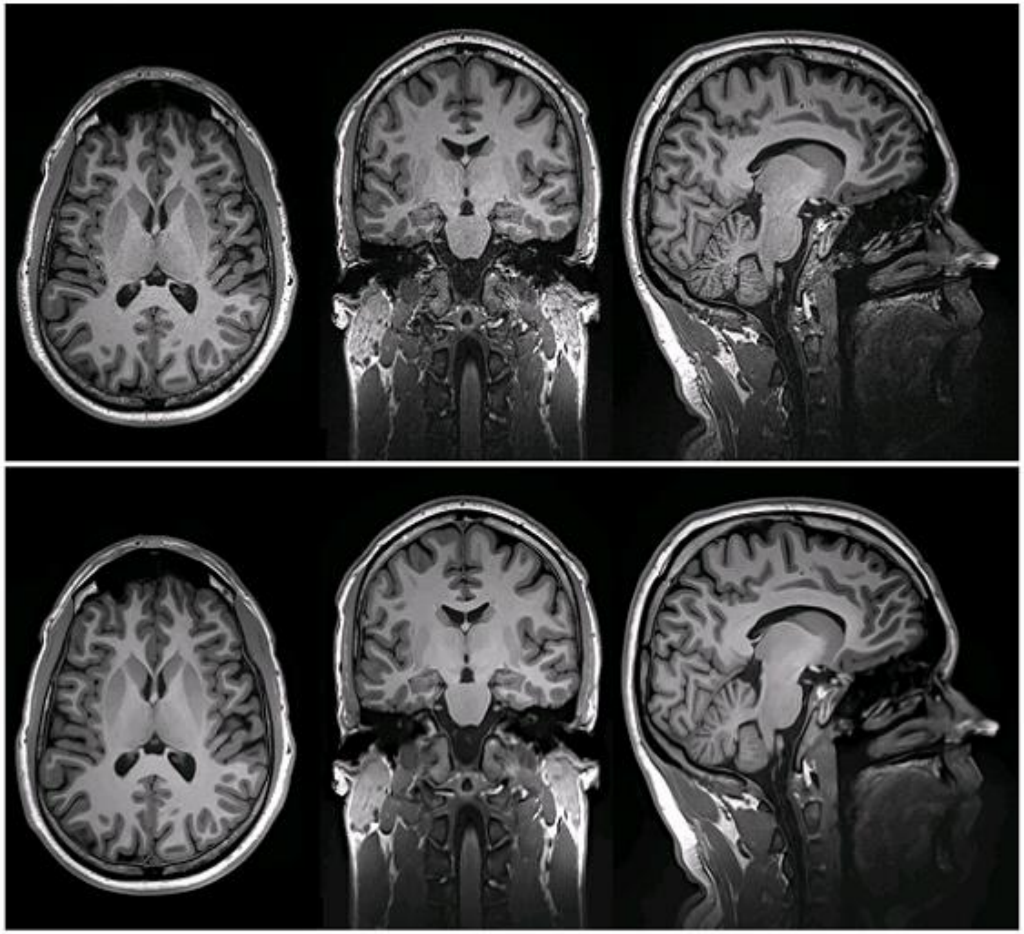
$$\tilde{I}(x,y,z)=\frac{1}{c(x,y,z)}\sum_{i=-n}^{n}\sum_{j=-n}^{n}\sum_{k=-n}^{n}w_{ijk}(x,y,z)\,I(x+i,y+j,z+k)$$

$$c(x,y,z)=\sum_{i=-n}^{n}\sum_{j=-n}^{n}\sum_{k=-n}^{n}w_{ijk}(x,y,z) \qquad w_{ijk}(x,y,z)=\exp\left(-\frac{\left|I(x,y,z)-I(x+i,y+j,z+k)\right|^2}{h^2}\right)$$

- Can be performed at the block level

Buades, B. Coll., and J. Morel, “A non local algorithm for image denoising,” In Proc. Int. Conf. Computer Vision and Pattern Recognition (CVPR), vol. 2, 2005, pp. 60–65.

Adaptive Non Local Means



Manjón JV, Coupé P, Martí-Bonmatí L, Collins DL, Robles M. Adaptive non-local means denoising of MR images with spatially varying noise levels. J Magn Reson Imaging. 2010 Jan;31(1):192-203

2.0 Image Filtering

- 2.1 Taxonomy of Filters
- 2.2 Smoothing Filters
- 2.3 **Gradient Filters**
  - Discrete vs Continuous
- 2.4 Image Contour Extraction
- 2.5 Mathematical Morphology

Computing Image Gradient

- Through Finite Differences

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$\Rightarrow$ First Order (Gradient) 1D Filter (h = 1) $\frac{1}{2}\begin{bmatrix}-1 & 0 & 1\end{bmatrix}$

- Gradient 2D Filters along x and y directions

$$\frac{1}{6}\begin{bmatrix}-1 & 0 & 1\\ -1 & 0 & 1\\ -1 & 0 & 1\end{bmatrix}$$

Separable filters based on 1D gradient and mean filters

$$\frac{1}{6}\begin{bmatrix}-1 & -1 & -1\\ 0 & 0 & 0\\ 1 & 1 & 1\end{bmatrix}$$

Sobel Filters

- Sobel Filters
  - Separable Filters
  - Reduce noise by convolving with low-pass filter (remove high frequencies) : binomial filters
  - First Order along x

$$\frac{1}{2}\begin{bmatrix}-1 & 0 & 1\end{bmatrix} * \frac{1}{4}\begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-1 & 0 & 1\\-2 & 0 & 2\\-1 & 0 & 1\end{bmatrix}$$

  - First Order along y

$$\frac{1}{4}\begin{bmatrix}1 & 2 & 1\end{bmatrix} * \frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-1 & -2 & -1\\0 & 0 & 0\\1 & 2 & 1\end{bmatrix}$$

Computing 2nd Derivatives

$$f''(x) \approx \frac{\dfrac{f(x+h_1+h_2)-f(x+h_1-h_2)}{2h_2} - \dfrac{f(x-h_1+h_2)-f(x-h_1-h_2)}{2h_2}}{2h_1}$$

$\Rightarrow$Second order filters for $h_1 = h_2 = 1/2$ $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$

- Laplacian Filters

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

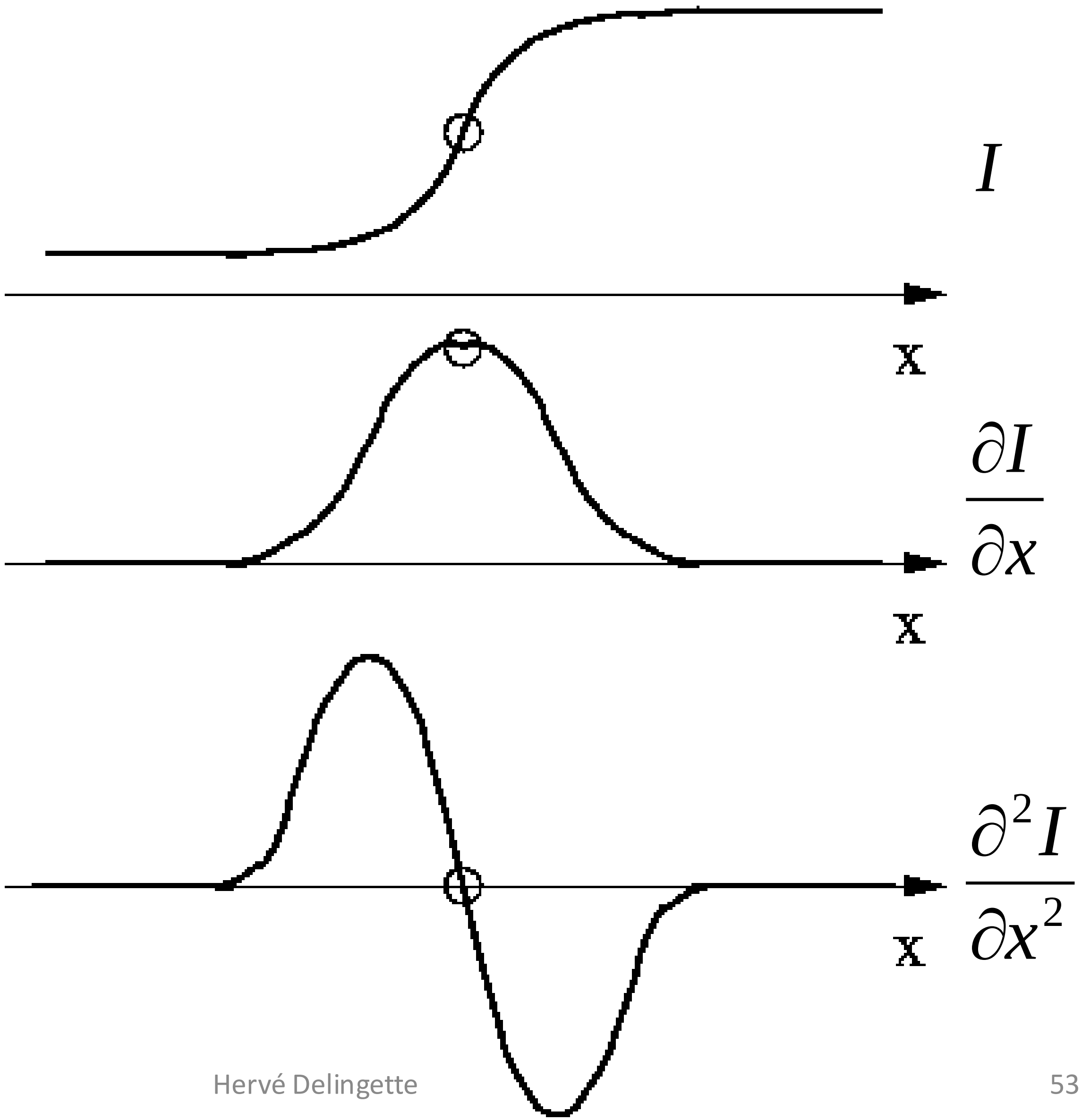
2.0 Image Filtering

- 2.1 Taxonomy of Filters
- 2.2 Smoothing Filters
- 2.3 Gradient Filters
- 2.4 **Image Contour Extraction**
  - Definition of visible contours
  - Implementation of contour detection
  - Optimal contour Detection
- 2.5 Mathematical Morphology

Contour Detection

- Qualitatively :

Find points where *I*(*x*,*y*) varies very quickly with (x,y).



Contour Detection

- 2 detection criteria
  - Find Maxima of gradient in the gradient direction

$$\|\nabla I\| = \sqrt{\left(\frac{\partial I}{\partial x}\right)^2 + \left(\frac{\partial I}{\partial y}\right)^2} \quad \text{maximum in the direction } \overrightarrow{\nabla I}$$

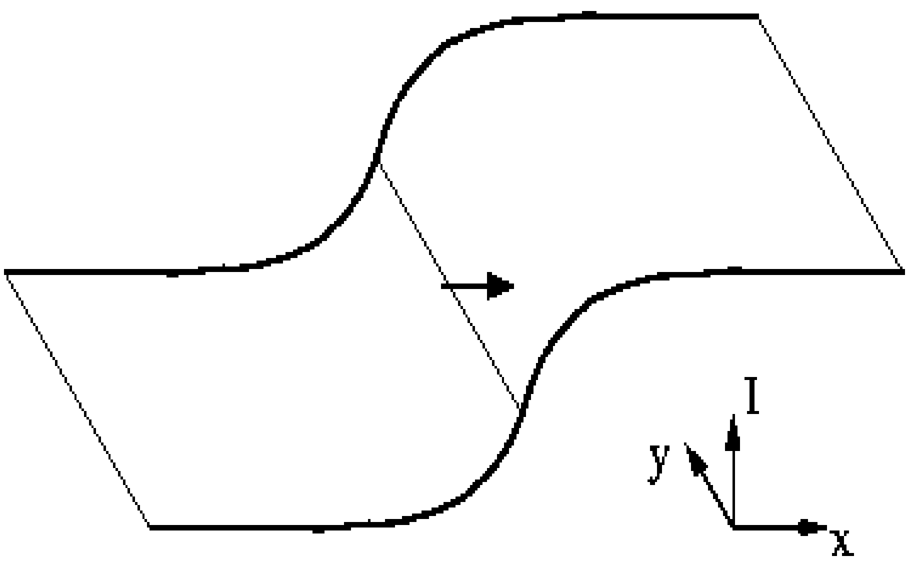
  - Zero crossing of Laplacian

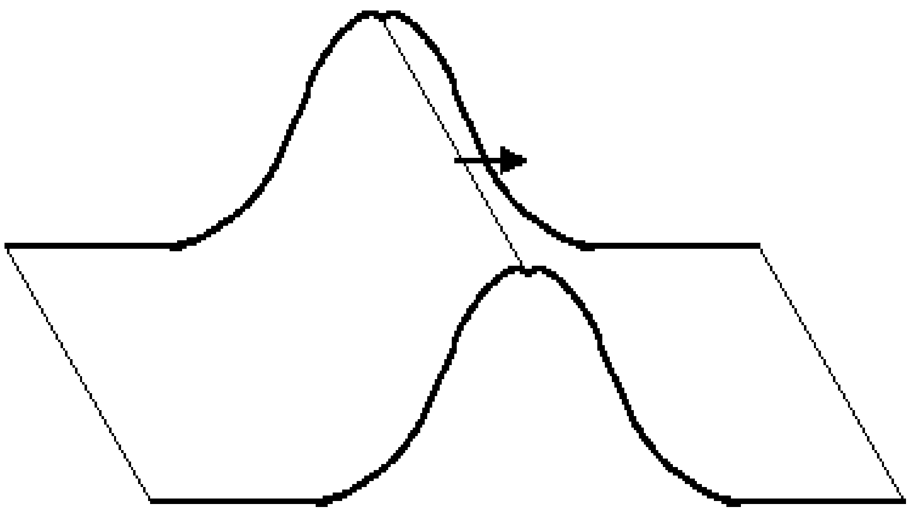
$$\Delta I = \nabla^2 I = I_{xx} + I_{yy} = 0$$

Gradient Maxima

- Find Maxima of gradient in the gradient direction

$$\frac{\partial \|\nabla I\|}{\partial \vec{n}} = 0 \text{ with } \vec{n} = \frac{\vec{\nabla} I}{\|\vec{\nabla} I\|}$$





Gradient Maxima

- Find Maxima of gradient in the gradient direction

$$\frac{\partial \|\nabla I\|}{\partial \vec{v}} = \vec{v}.\vec{\nabla}\|\nabla I\|$$

$$\vec{\nabla}\|\nabla I\| = \begin{pmatrix} \dfrac{I_x I_{xx} + I_y I_{yx}}{\|\nabla I\|} \\ \dfrac{I_x I_{xy} + I_y I_{yy}}{\|\nabla I\|} \end{pmatrix} = \frac{H(i)\vec{\nabla} I}{\|\nabla I\|}$$

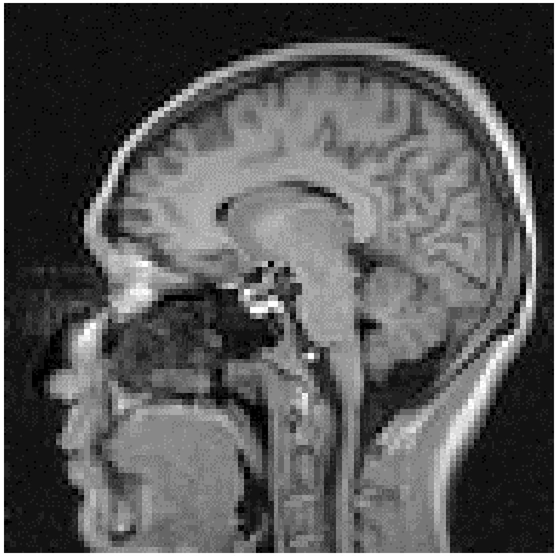
Contour Detection

- Find Maxima of gradient in the gradient direction

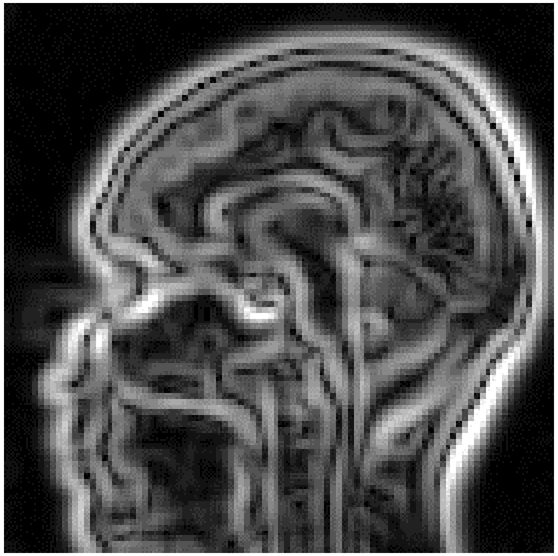
$$\frac{\partial \|\nabla I\|}{\partial \vec{n}} = \frac{\vec{\nabla} I . H(I) \vec{\nabla} I}{\|\nabla I\|^2}$$

$$\frac{\partial \|\nabla I\|}{\partial \vec{n}} = 0 \quad \leftrightarrow \vec{\nabla} I . H(I) \vec{\nabla} I = 0$$

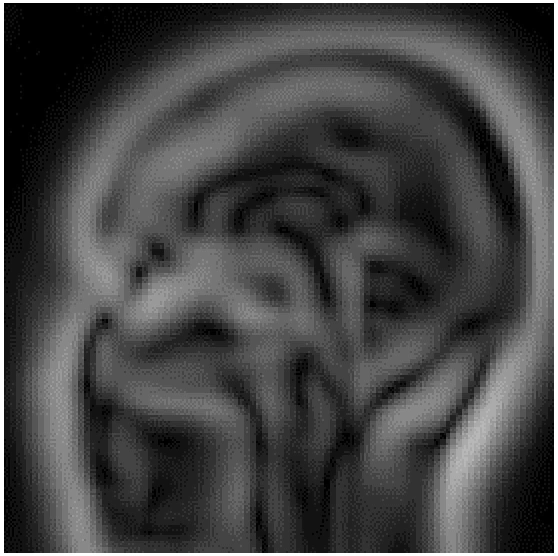
- Laplacian $\Delta I = 0 \quad \Leftrightarrow \quad \mathrm{trace}\big(H(I)\big) = 0$



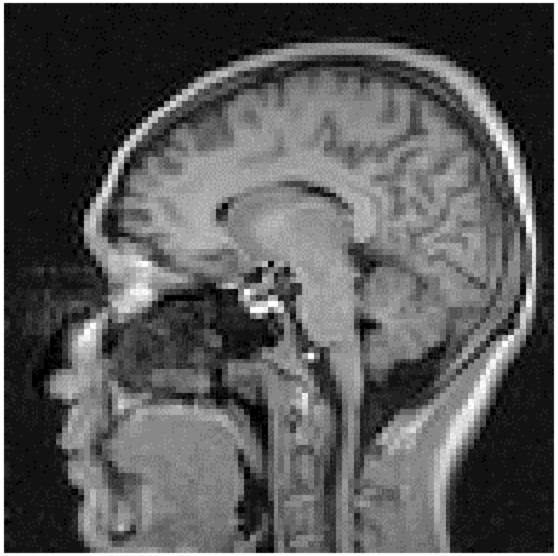
Original Image



Norm of Gradient $\sigma = 3$



Norm of Gradient $\sigma = 9$

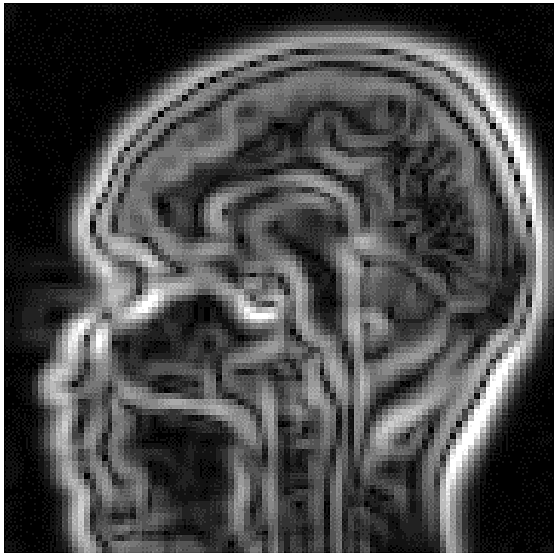


Original Image

Laplacian $\sigma = 3$

Laplacian $\sigma = 9$

Norm of Gradient $\sigma = 3$



Threshold= 51

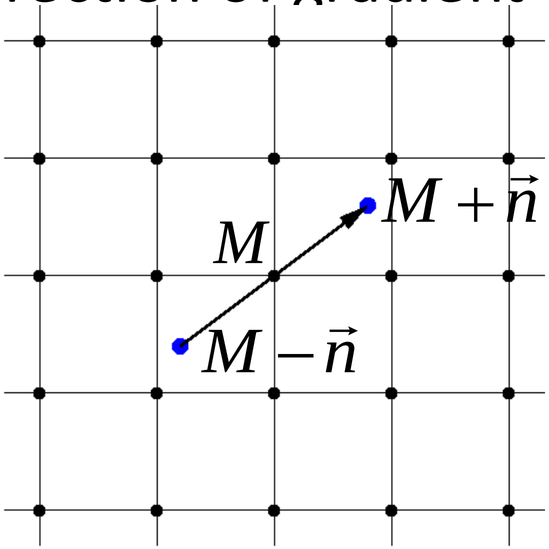
Threshold= 60

Gradient Extrema

1. Compute Image Gradient
2. Extract extrema gradient in the direction of gradient

$$M_1 = M + d \, \overrightarrow{\nabla I} / \|\nabla I\|$$

$$M_2 = M - d \, \overrightarrow{\nabla I} / \|\nabla I\|$$

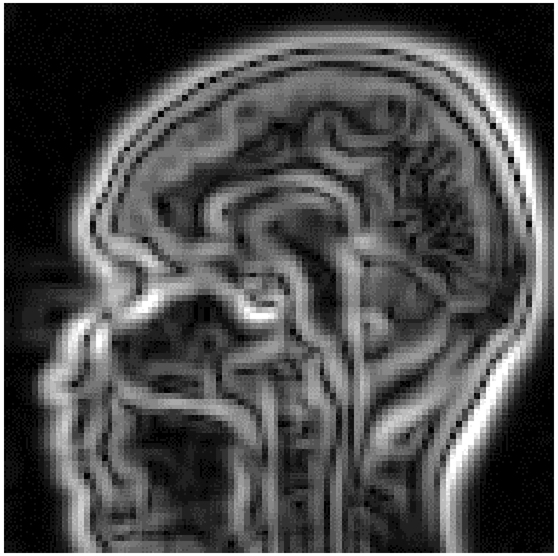


Extremum Criterion

3. Threshold Gradient Extrema

$$\|\nabla I(M)\| > \|\nabla I(M_2)\| \quad \text{et} \quad \|\nabla I(M)\| \geq \|\nabla I(M_1)\|$$

Norm of Gradient $\sigma = 3$



Threshold= 51

Extrema of Gradient

in Gradient direction

Thresholding

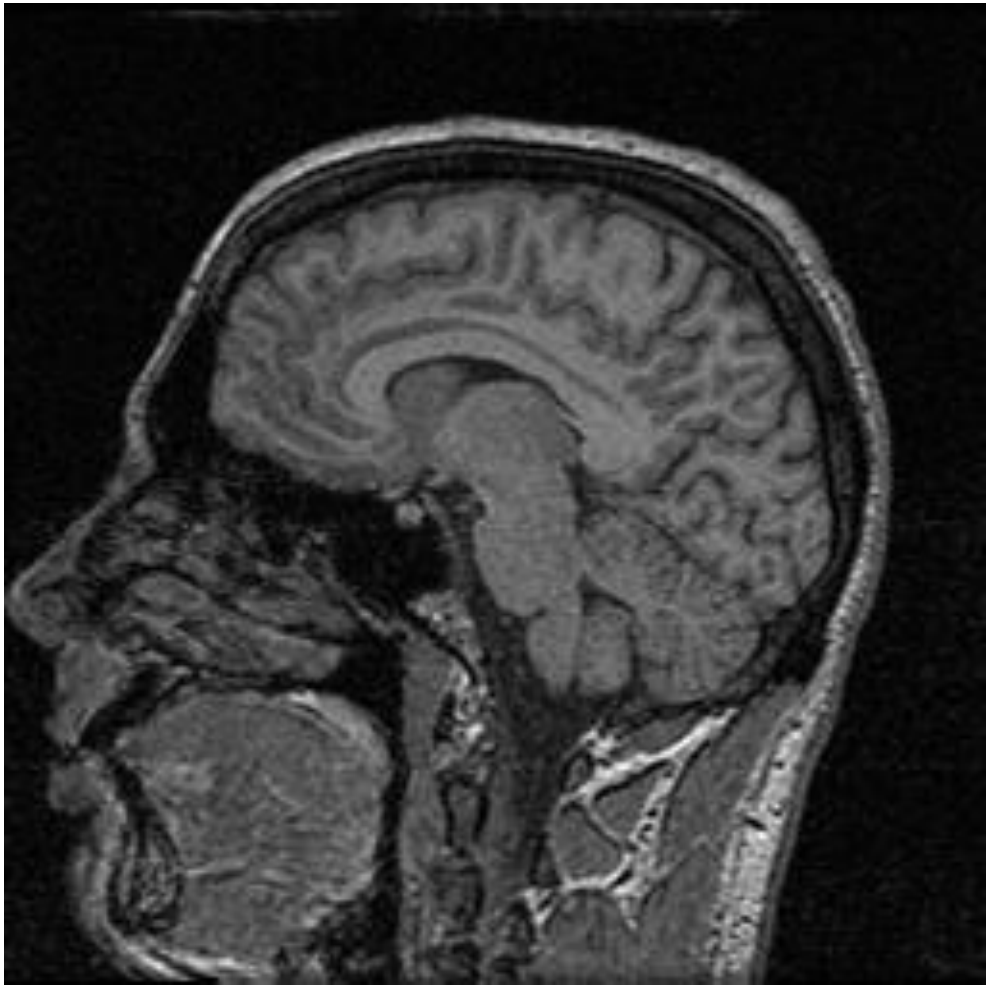
- Simple Thresholding $\left\|\overrightarrow{\nabla I}(M)\right\| \geq s$

- Hysteresis Thresholding
  - Select connected components for which
    - All points verify $\left\|\overrightarrow{\nabla I}(N)\right\| \geq s_{\text{bas}}$
    - At least one point P verify $\left\|\overrightarrow{\nabla I}(P)\right\| \geq s_{\text{haut}}$
    - (optional) Size is greater than t

Optimal Contour Detection

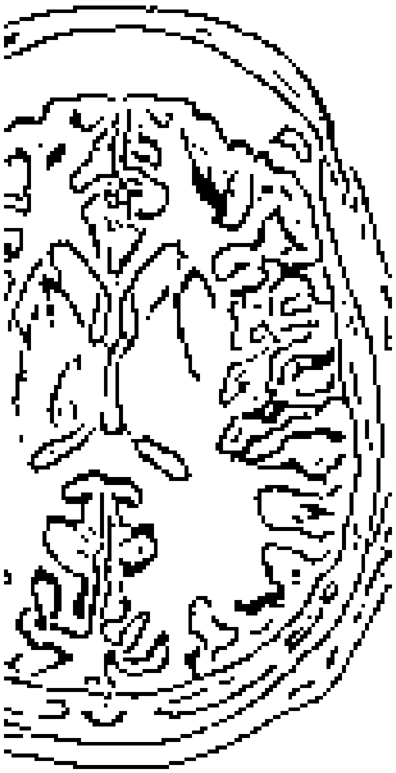
- Contours from previous approach are difficult to evaluate
- Define contours as maxima of the convolution of $I(x)$ with filter f
- Canny defined an ideal filter that should optimize the following 3 criteria on an ideal contour
  - **detection** : filter should be able to provide high values next to the contour
  - **localisation** : contour should be localised with accuracy
  - **unique response** : contour should provide a unique maximum around contour

Contour Detection 2D / 3D



Contour Detection 2D

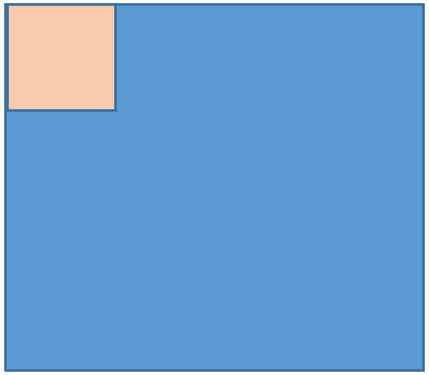
Contour Detection 3D



2.1 Mathematical Morphology

- Overview :
- Definitions of Neighborhood
- Transformations on Binary Images
  - Boolean
  - Hit or Miss
  - Erosion
  - Dilation
- Extension to Grey-level Images
- Distance maps

Mathematical Morphology Principles



- Similar principle as convolution (sliding window)
  - Apply on a binary or grey level image
  - Based on a structuring element (a small template)

- Iteratively center template on a pixel and compute resulting value

- but with non-linear operations:

Definition of Structuring Element

Definitions (Binary)

E_a Affine space

E_v Associated vector space

I : Binary Image : $K \subset E_a \rightarrow \{0,1\}$

X : **Foreground Object**

$$X = \{x \in K \mid I(x)=1\}$$

X^c : **Background Object** *complementary* of X *with respect to* E_a**,**

B : structuring element

= $\{ b \in E_v \}$ set of vectors

= $\{ x \in E_a / Ox = b \}$ set of points

Boolean Operations

- Union : $X \cup Y = \{ x / x \in X \text{ or } x \in Y \}$
- Intersection : $X \cap Y = \{ x / x \in X \text{ and } x \in Y \}$
- Inclusion : $X \subset Y \Leftrightarrow \forall x \in X \Rightarrow x \in Y$
- Non symmetric difference

$$X \setminus Y = \{x / x \in X \text{ and } x \notin Y\}$$

- Symmetric Difference

$$X \setminus\setminus Y = \{ x / x \in ((X \cup Y) \setminus (X \cap Y)) \}$$

- Complement

$$(X^c)_E = \{ x / x \in E \text{ and } x \notin X \}$$

Translated & Symmetric

- Translated of *X* by vector *b*

$$X_b = \{x + \vec{b}\ ; x \in X\}$$

- *B* centered in *x* (a.k.a translated of B by *x*)

$$B_x = \{x + \vec{b}\ ; \vec{b} \in B\}$$

- Symmetric of *B*

$$\breve{B} = \{-\vec{b}; \vec{b} \in B\}$$

Hit or miss transformation

- Partition of a structuring element B

$$B = B^1 \cup B^2 \quad \text{et} \quad B^1 \cap B^2 = \varnothing$$

- Hit or miss transformation of set X

$$X \otimes B = \{ x \,/\, B^1_x \subset X \text{ et } B^2_x \subset X^c \}$$

- $x \in X \otimes B$

$\Leftrightarrow$ B^1_x is in foreground and B^2_x is in background

$\Leftrightarrow$ B describe the local configuration of the background and foreground objects

2.1 Mathematical Morphology

- Overview :
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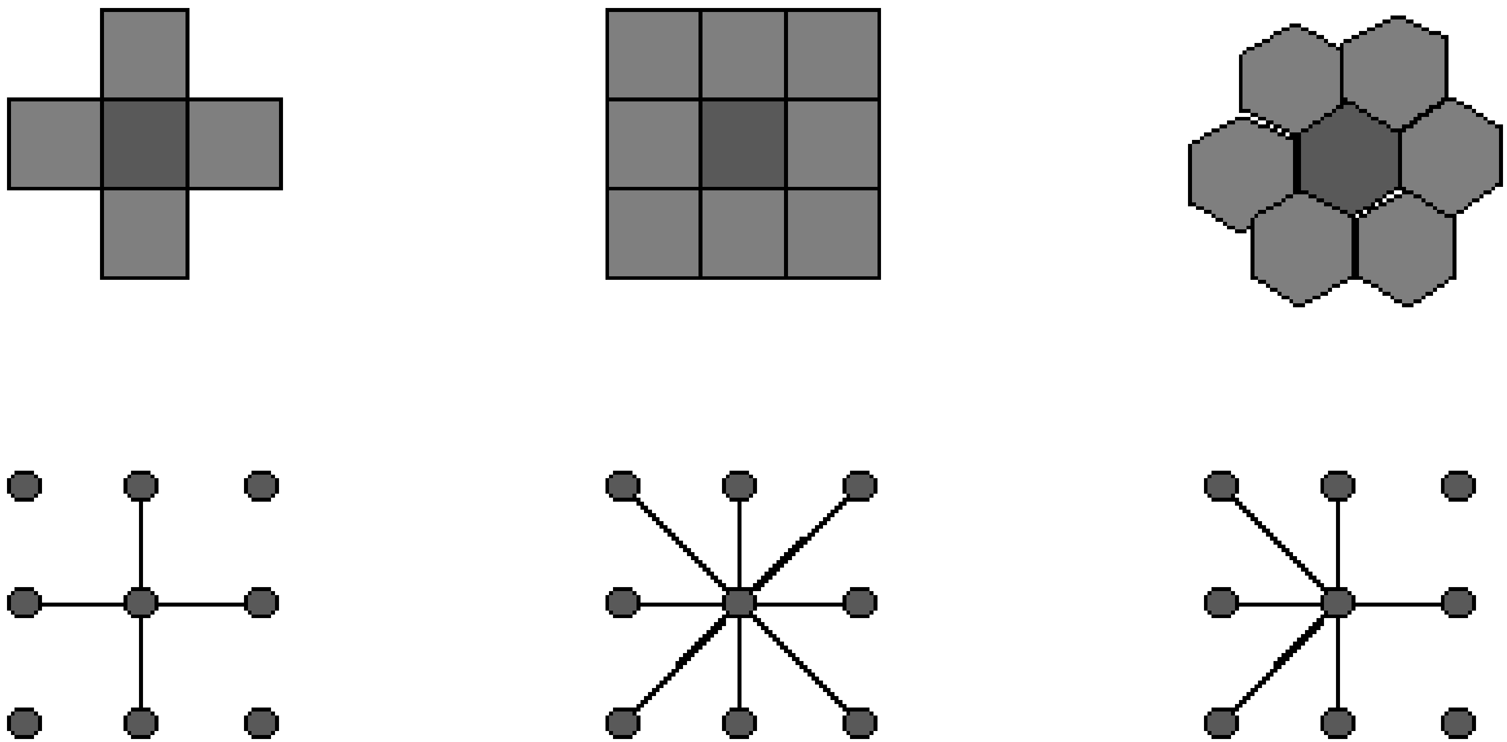
Neighborhood

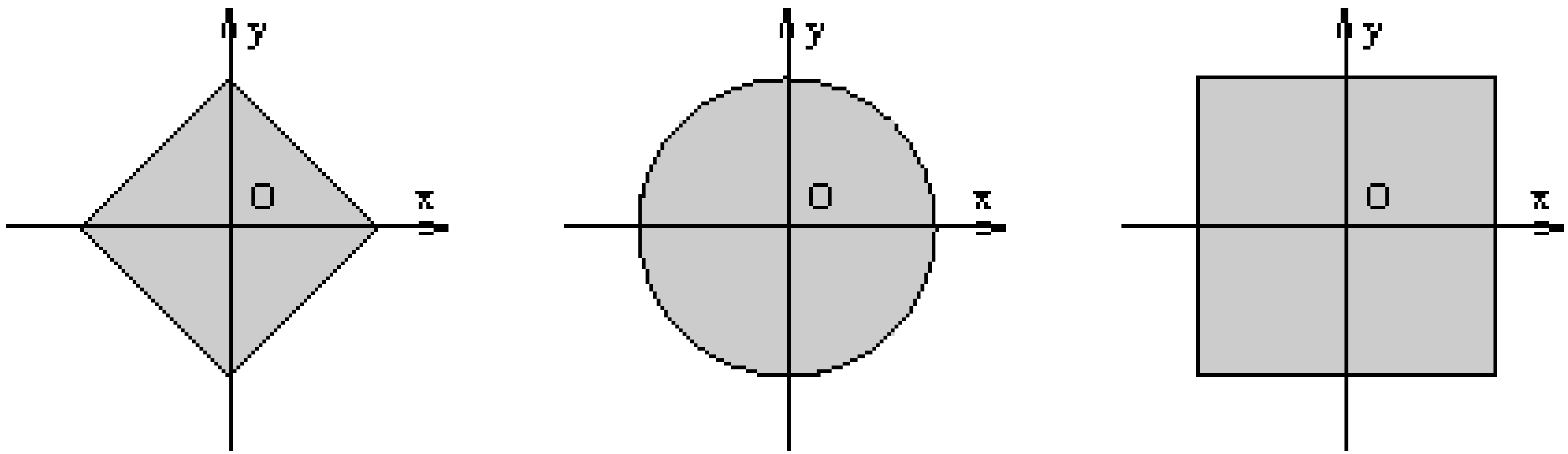
- Different types of neighborhood in 2D



4 -neighborhood **8 -neighborhood** **6 -neighborhood**

Neighborhood

- Some generalizes to higher dimensions
- Corresponds to a choice of metric norm



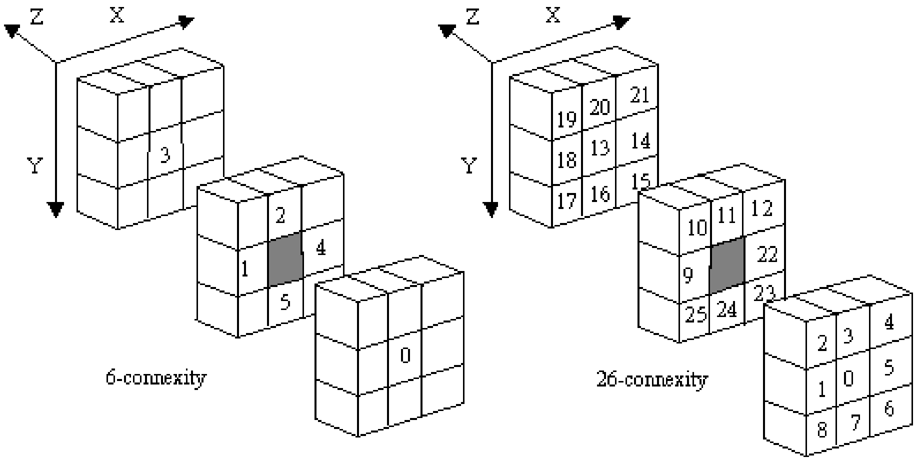
$$D_1(x,y) = \sum_{i=1}^{n} |y_i - x_i|$$

$$D_2(x,y) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$$

$$D_\infty(x,y) = \max_{i=1\ldots n} |y_i - x_i|$$

Neighborhood

- 3 types of neighborhood for a 3D image :
  - 6-neighborhood : adjacency through faces
  - 18-neighborhood : adjacency through faces and edges
  - 26-neighborhood : adjacency through faces and edges and vertices



Mathematical Morphology

- Technique to manipulate digital shapes
  - Unknown image
  - Known shape : **structuring element**
- Boolean operation between image & structuring element

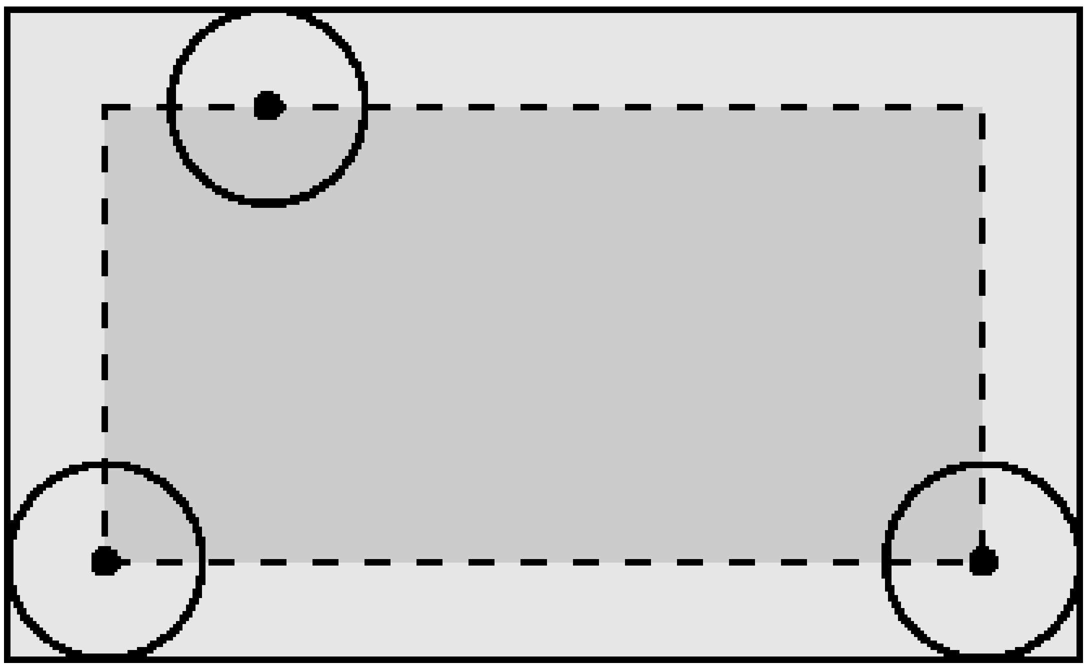
$$\cup \, , \cap \, , \subset \, , \in \, , \neq\emptyset \, , \ldots$$

- See work by Serra, Schmitt, ...

Erosion

- Specific case of HoM transformation when $B^2 = \varnothing$

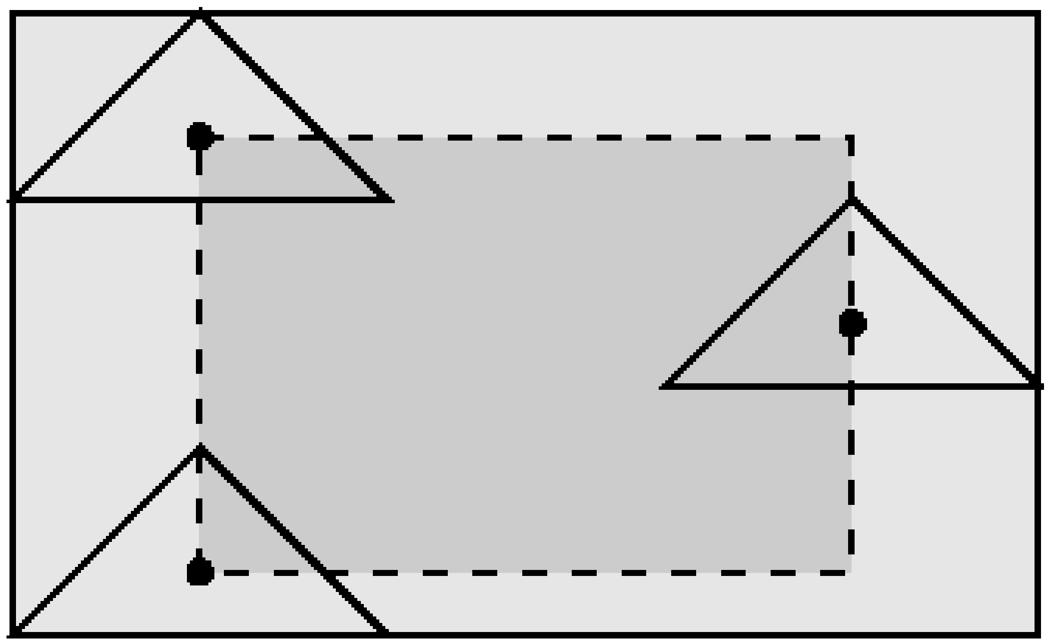
$$\text{Erosion of } X \text{ by } B = \mathrm{Ero}(X,B) = \{\, y \in E_a \,/\, B_y \subset X \,\}$$



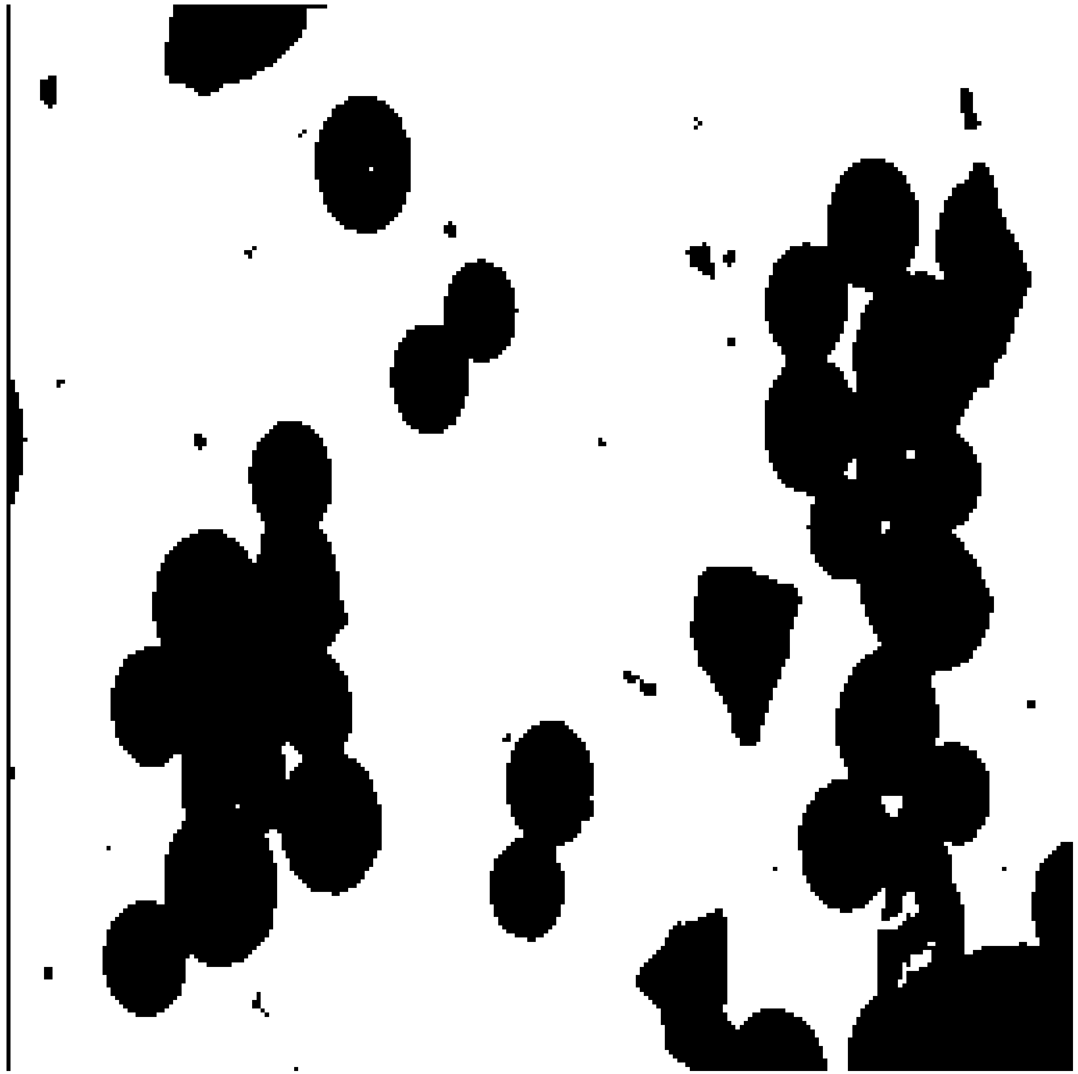
Erosion

- Specific case of HoM transformation when $B^2 = \emptyset$

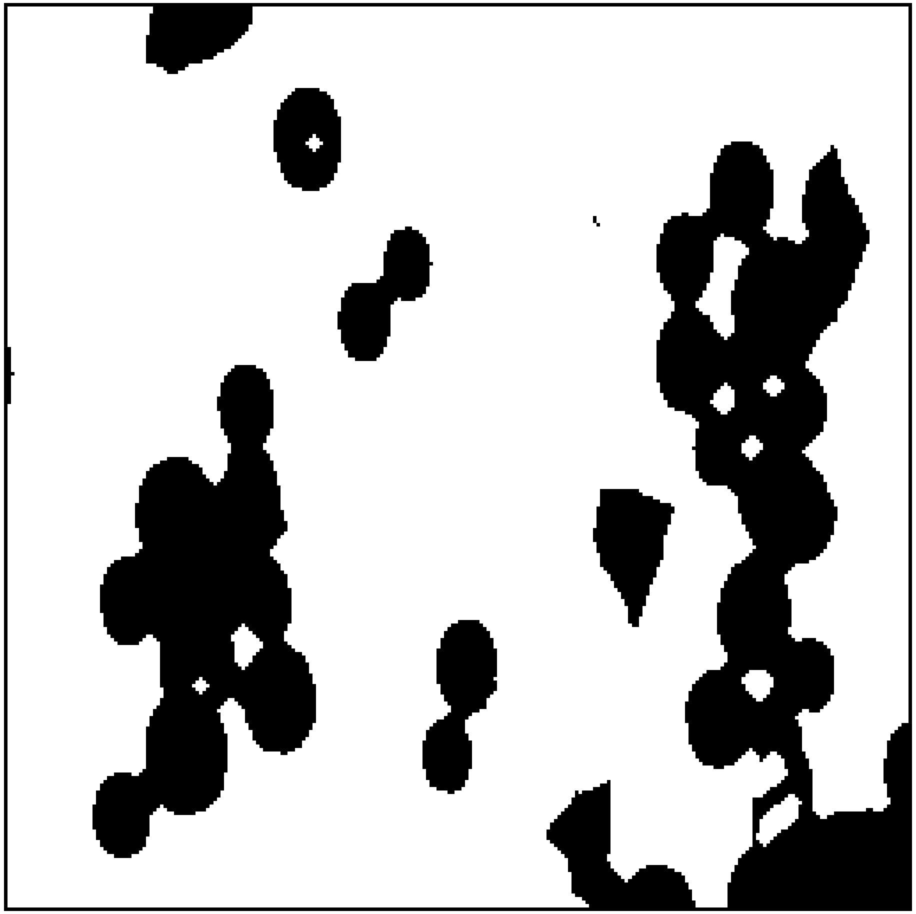
  Erosion of X by B = Ero(X,B) = $\{ y \in E_a / B_y \subset X \}$

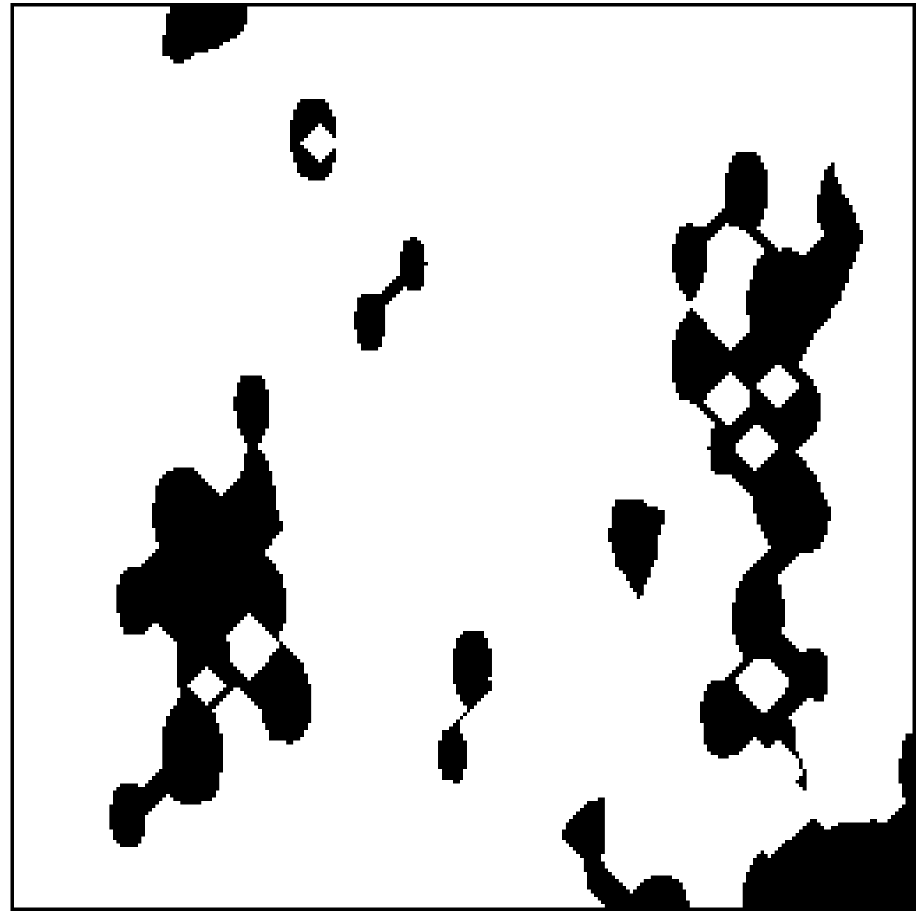


Definitions (Binary)



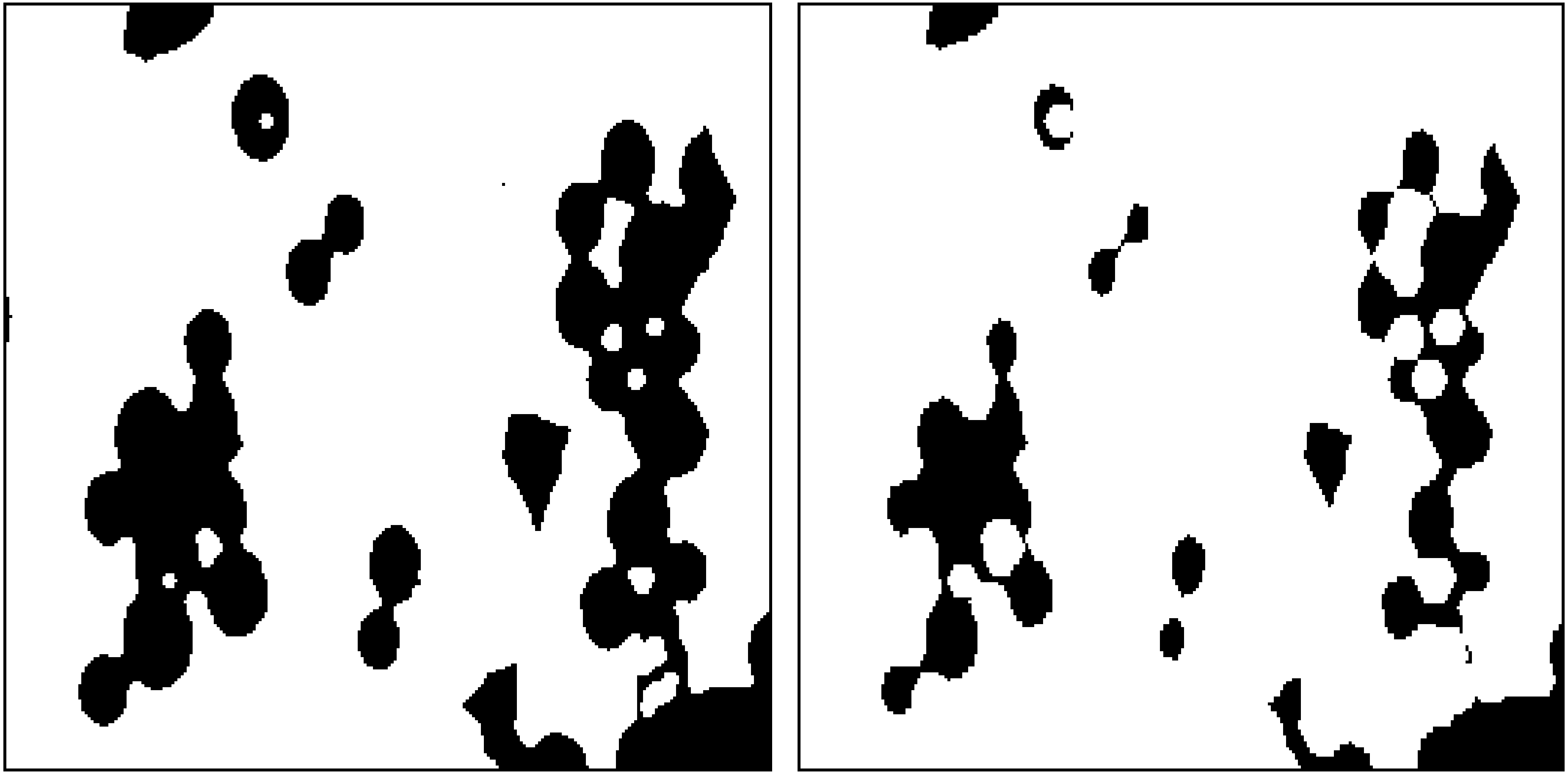
Erosion





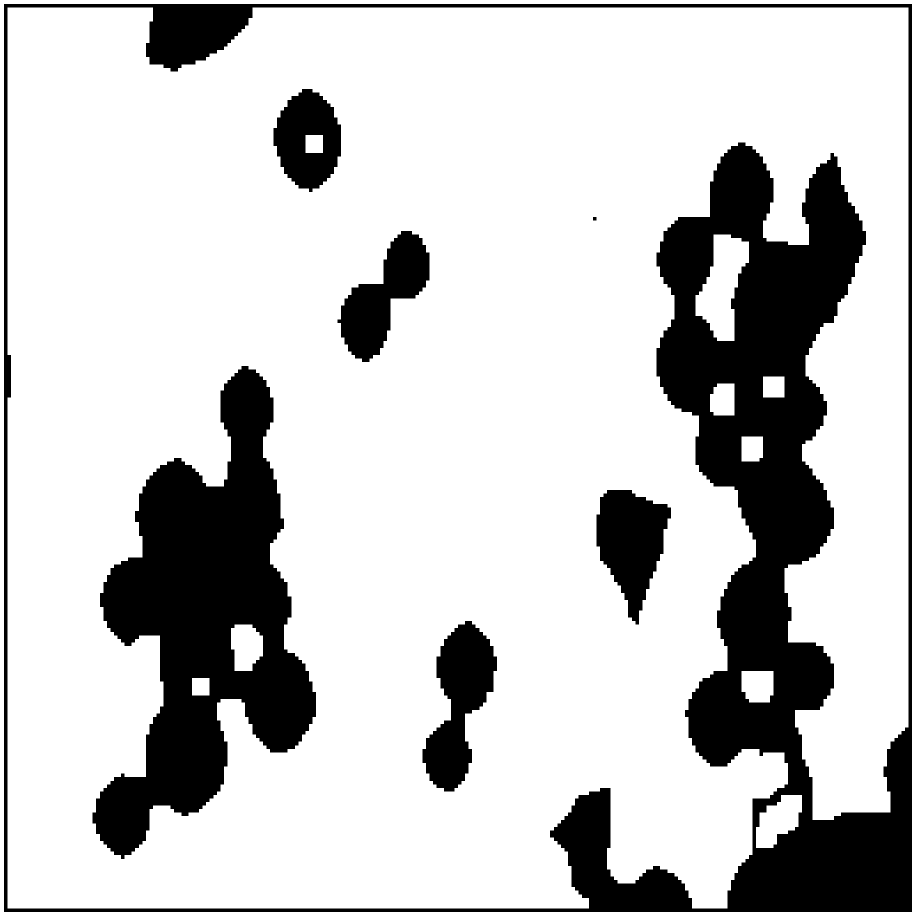
iterations = 2 | 4-structuring element | iterations = 5

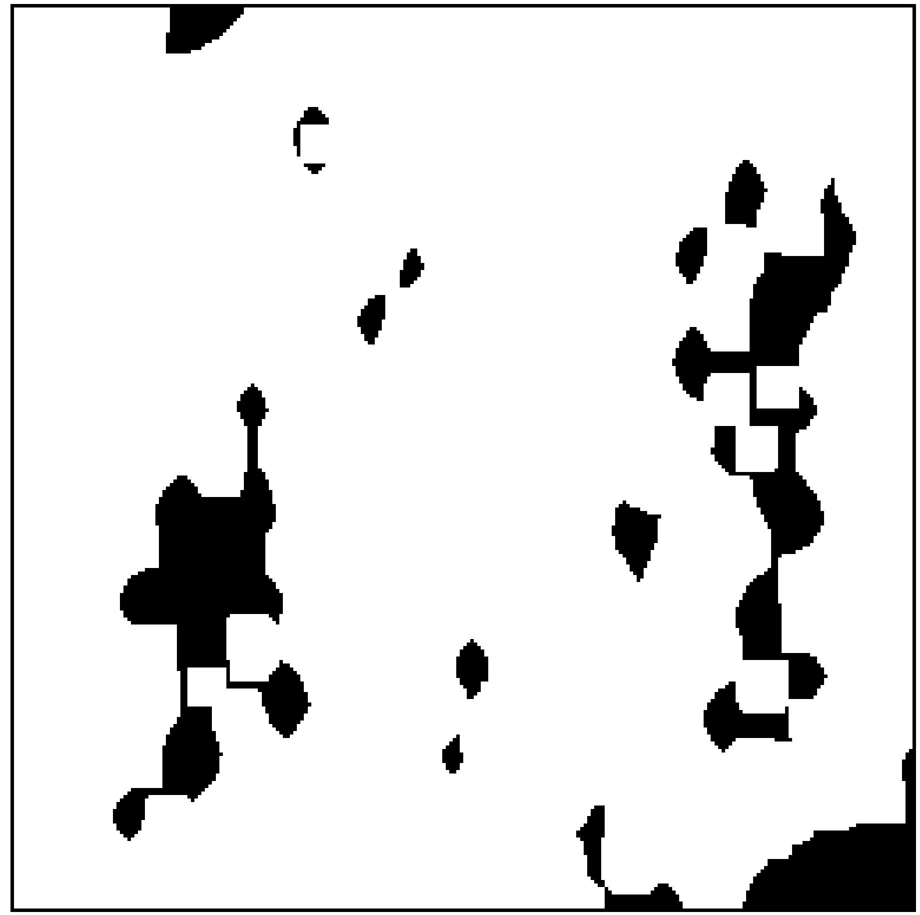
Erosion



itérations = 2 6-structuring element itérations = 5

Erosion



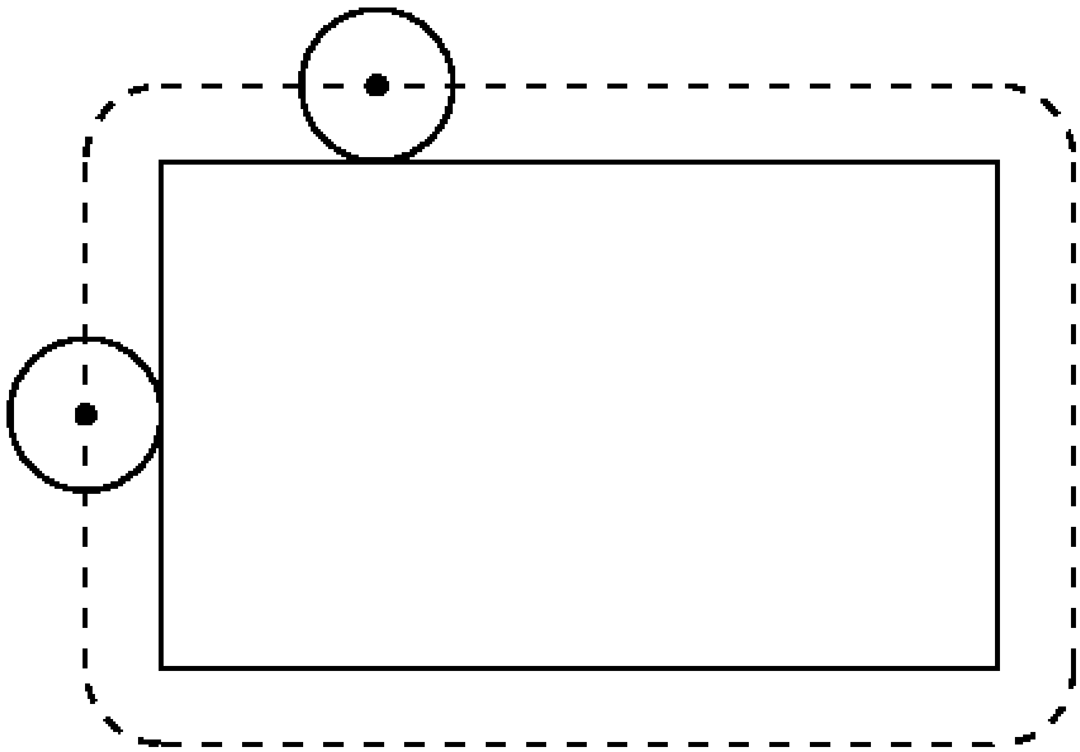


itérations = 2 8-structuring element itérations = 5

Dilation

- Dilation of *X* by *B*

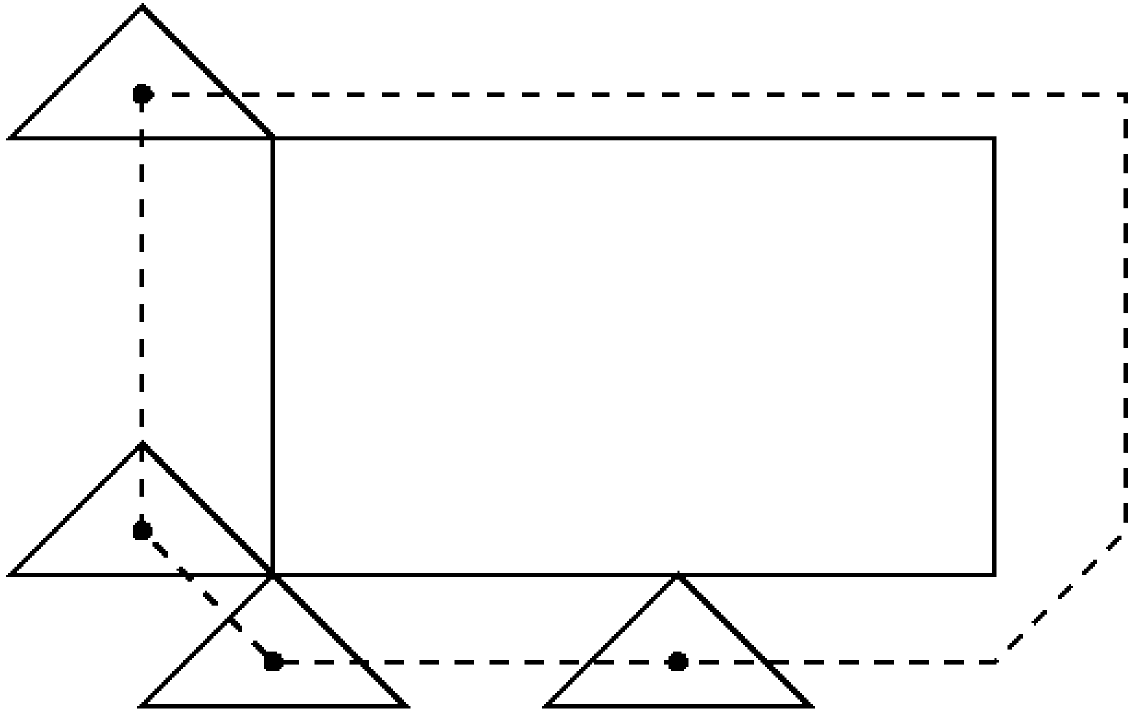
$$\text{Dil}(X,B) = \{ y \in E_a \,/\, B_y \cap X \neq \varnothing \}$$



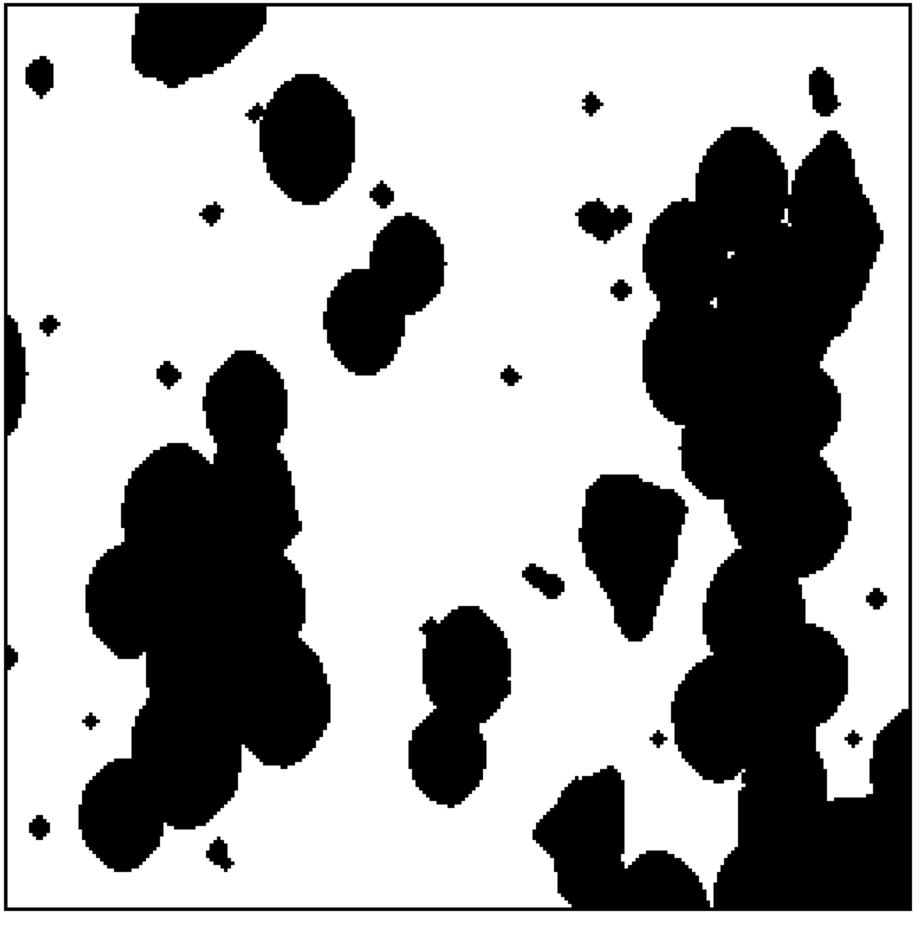
Dilation

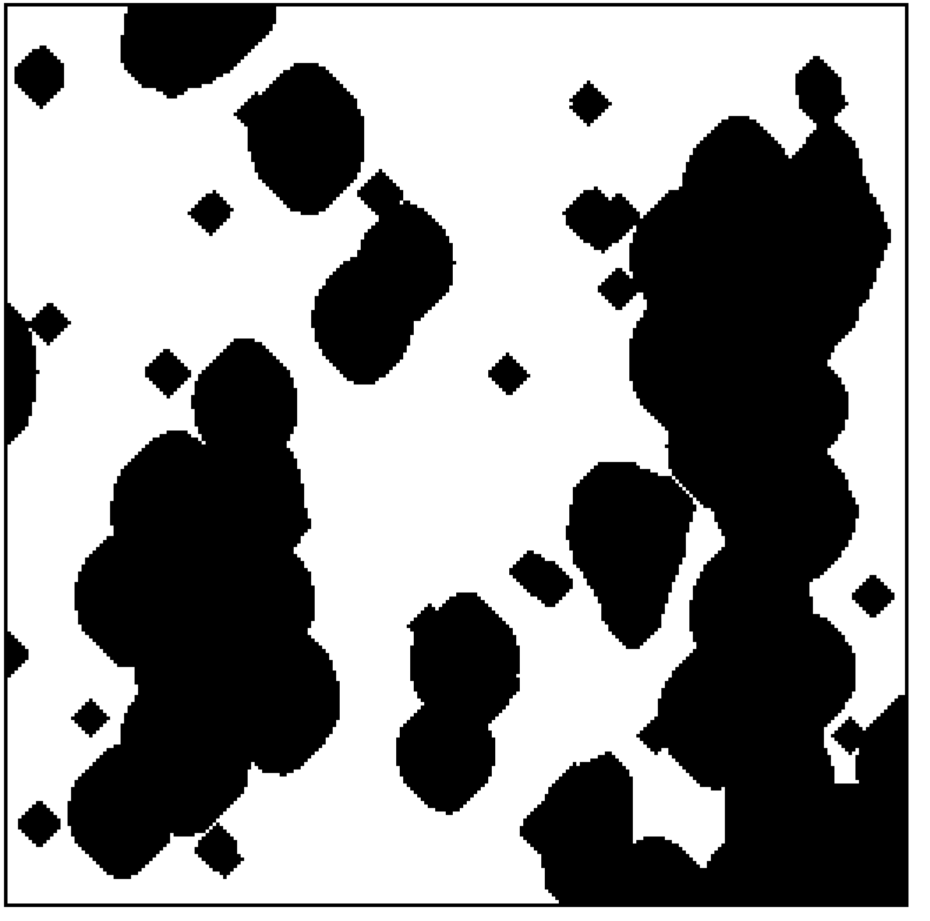
- Dilation of *X* by *B*

$$\mathrm{Dil}(X,B) = \{\, y \in E_a \,/\, B_y \cap X \neq \varnothing \}$$



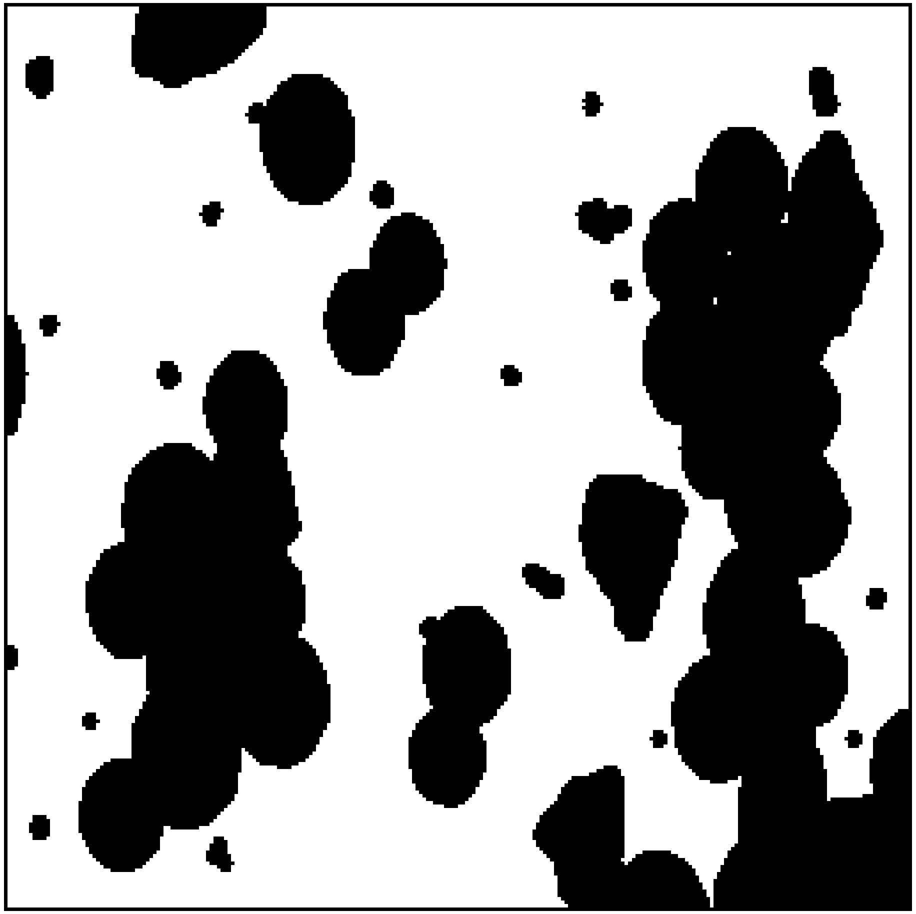
Dilation

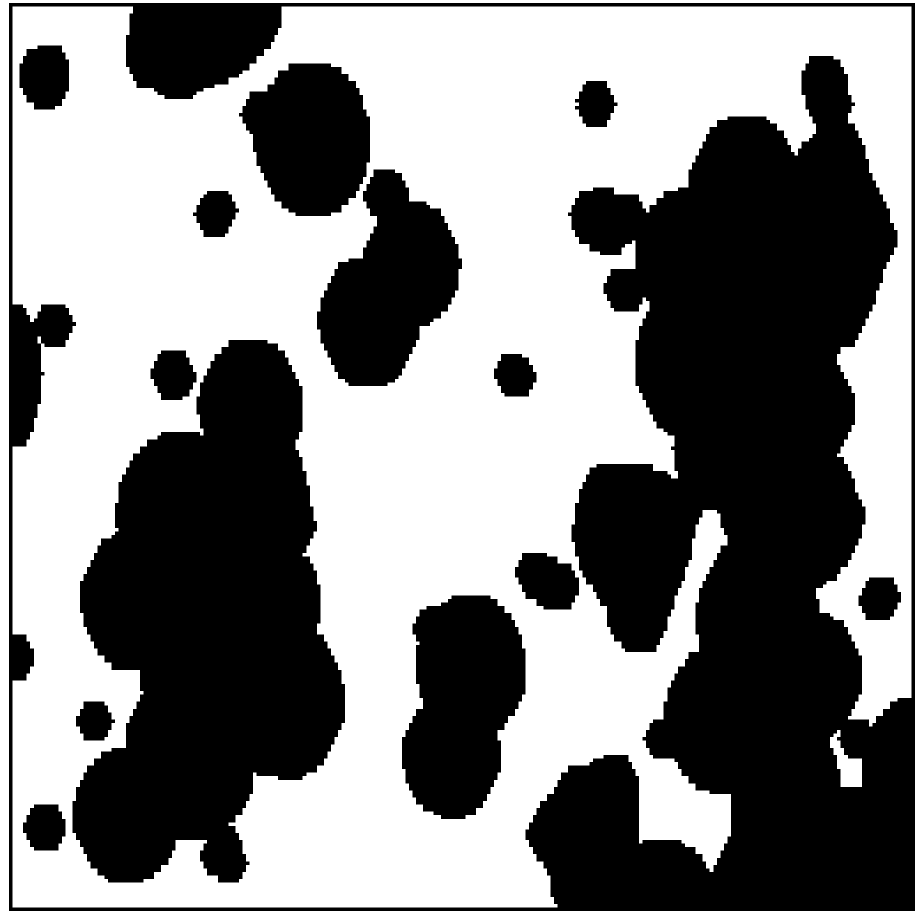




iterations = 2 4-structuring element iterations = 5

Dilation



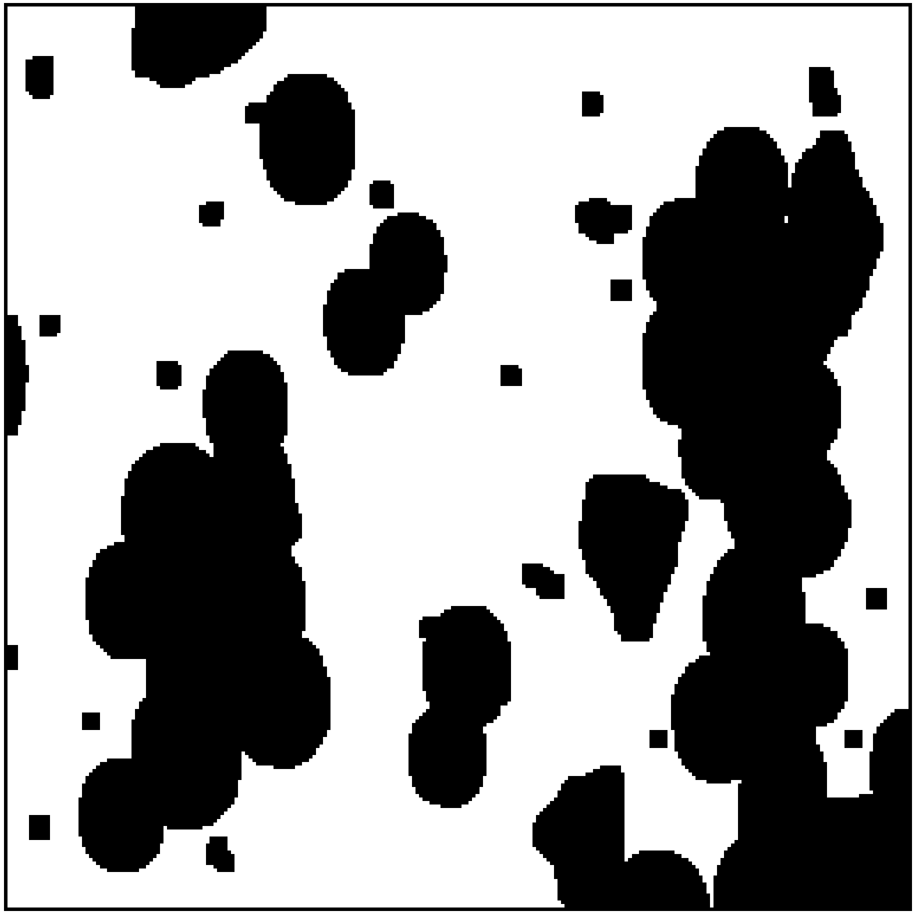


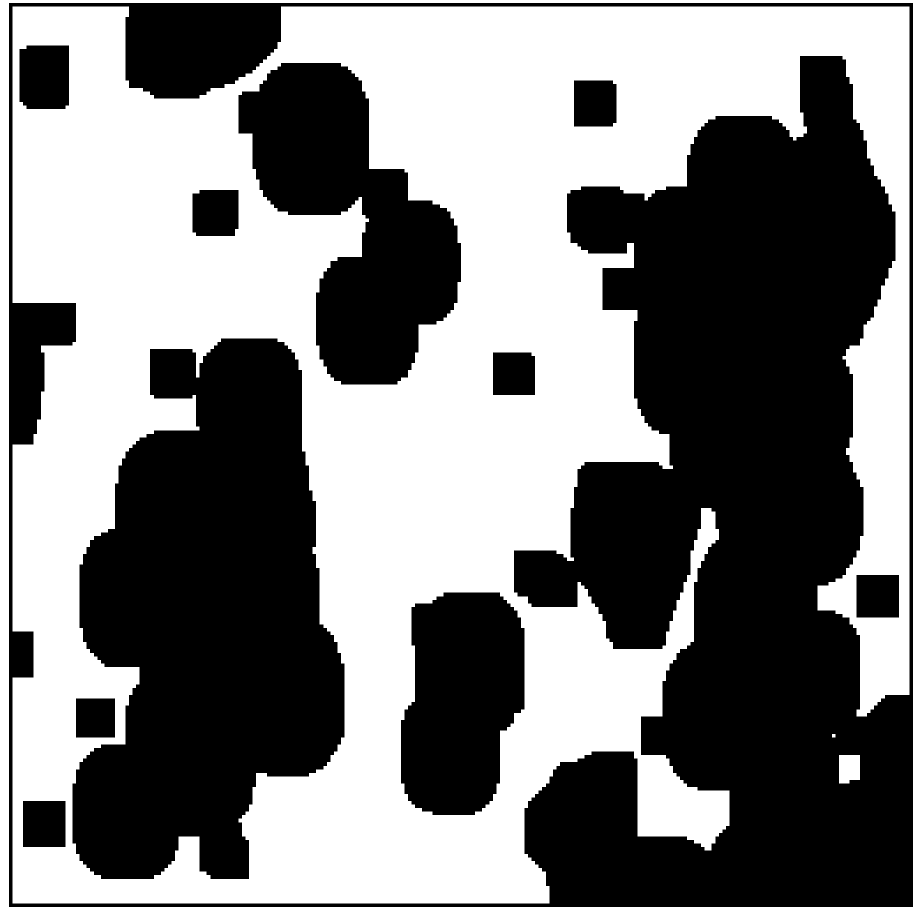
iterations = 2

6-structuring element

iterations = 5

Dilation





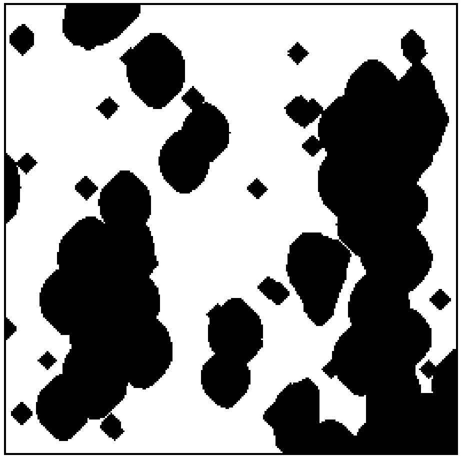
iterations = 2 8-element structurant iterations = 5

Duality Erosion / Dilation

$$\left(X \oplus \breve{B}\right)^{c} = X^{c} \ominus \breve{B}$$

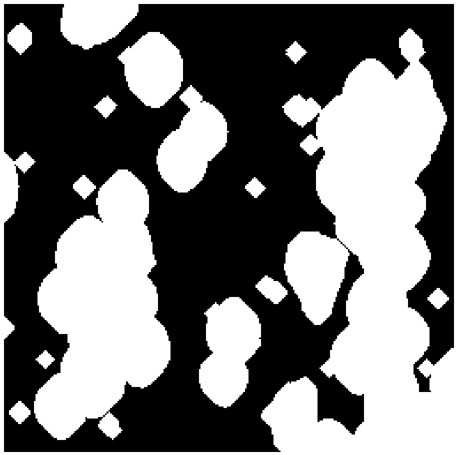
X

Original image



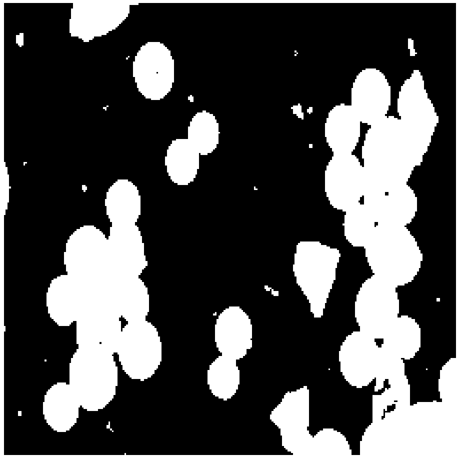
$X \oplus \breve{B}$

Dilated Image



$X^{c} \ominus \breve{B}$

Erosion of
The complement of X



X^{c}

The complement of

2.1 Mathematical Morphology

- Overview :
- Definitions of Neighborhood
- Transformations on Binary Images
  - Boolean
  - Hit or Miss
  - Erosion
  - Dilation
- Extension to Grey-level Images
- Distance maps

Extension to grey-level images

- Let f be a grey level image defined on K with values $\Re$.
- Let g be the structuring element.
- One writes $\breve{g}(x) = g(-x)$
- Dilation :

$$(f \oplus \breve{g})(x) = \max_{y \in B_x}\{f(y) + g(y - x)\}$$

- Erosion :

$$(f \ominus \breve{g})(x) = \min_{y \in B_x}\{f(\boldsymbol{y}) - g(y - x)\}$$

Extension to grey-level images

- Correspondance with binary mathematical morphology *B*

$$B(x) = \begin{cases} 0 & \text{si} \quad x \in B \\ -\infty & \text{si} \quad x \in B^c \end{cases}$$

- In this case, erosion and dilation write as :

$$\left(f \oplus \breve{B}\right)(x) = \max_{y \in B_x}\{f(x+y)\}$$

$$\left(f \ominus \breve{B}\right)(x) = \min_{y \in B_x}\{f(x+y)\}$$

- Dilation and Erosion are still dual of each other.

Extension to grey-level images

dilation

4-structuring element (2 iterations)

8-structuring element (2 iterations)

erosion

Morphological Gradient

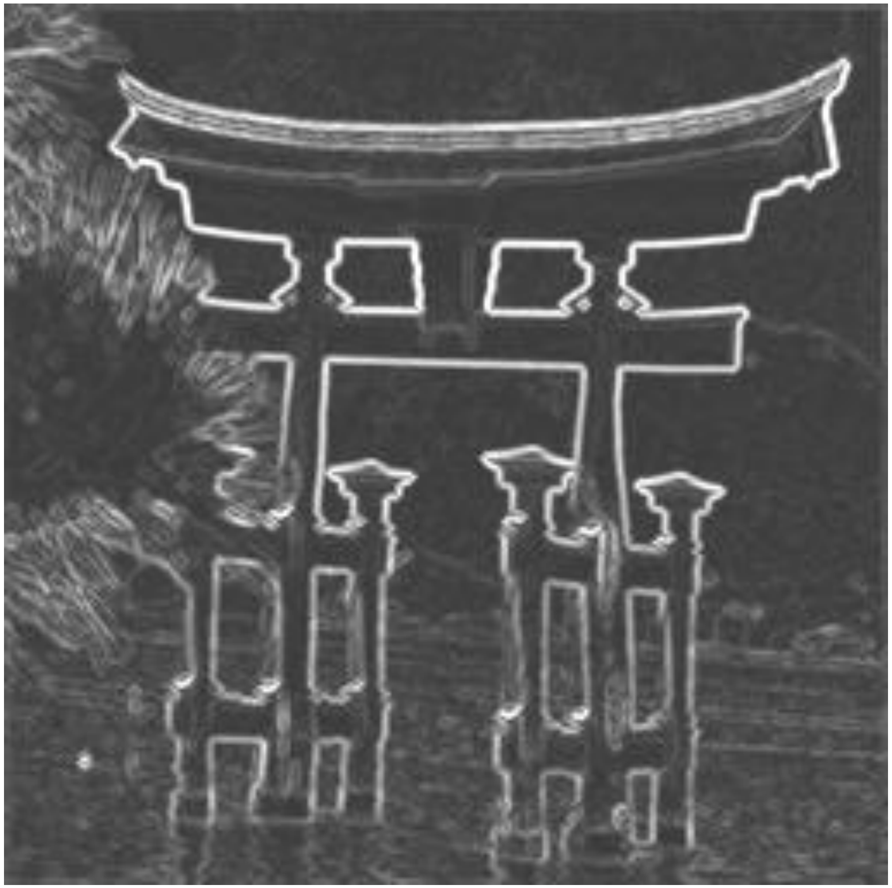
- Function gradient (when it exists) may be formalized as :

$$\lim_{r \to 0} \frac{f \oplus \breve{B}(r) - f \ominus \breve{B}(r)}{2r}$$

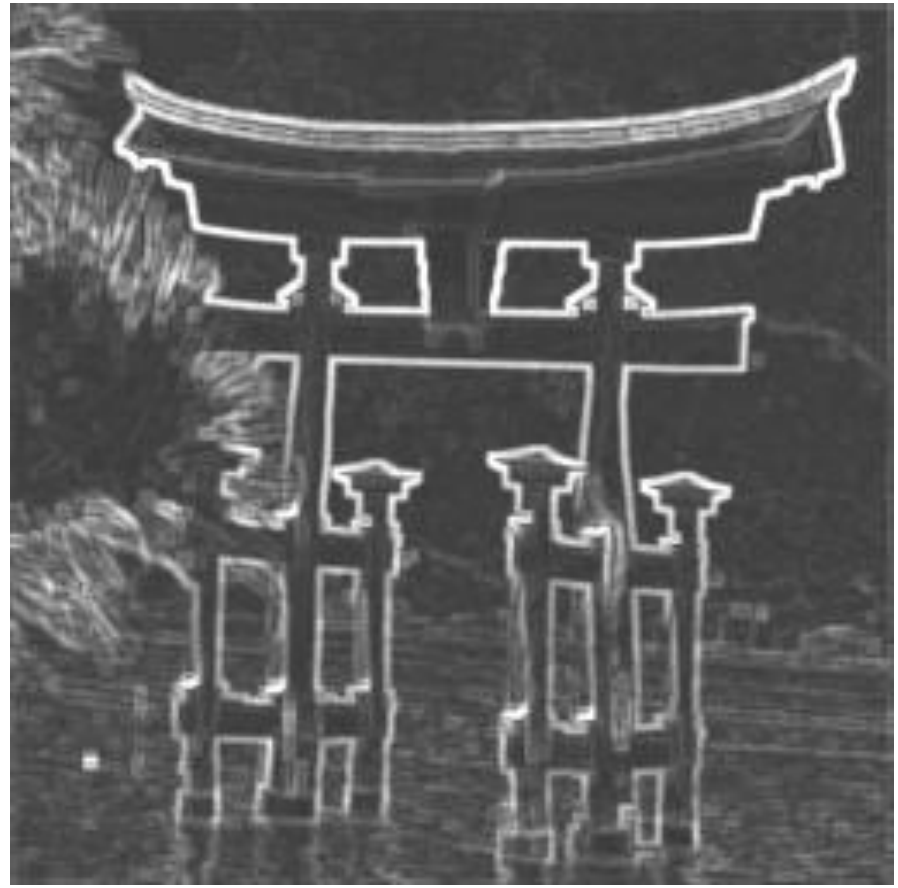
Where $B(r)$ is a bowl of radius r

- Morphological Gradient $B = \breve{B}$ $f \oplus \breve{B} - f \ominus \breve{B}$
- Superior Gradient $f \oplus \breve{B} - f$
- Inferior Gradient $f - f \ominus \breve{B}$

Morphological Gradient



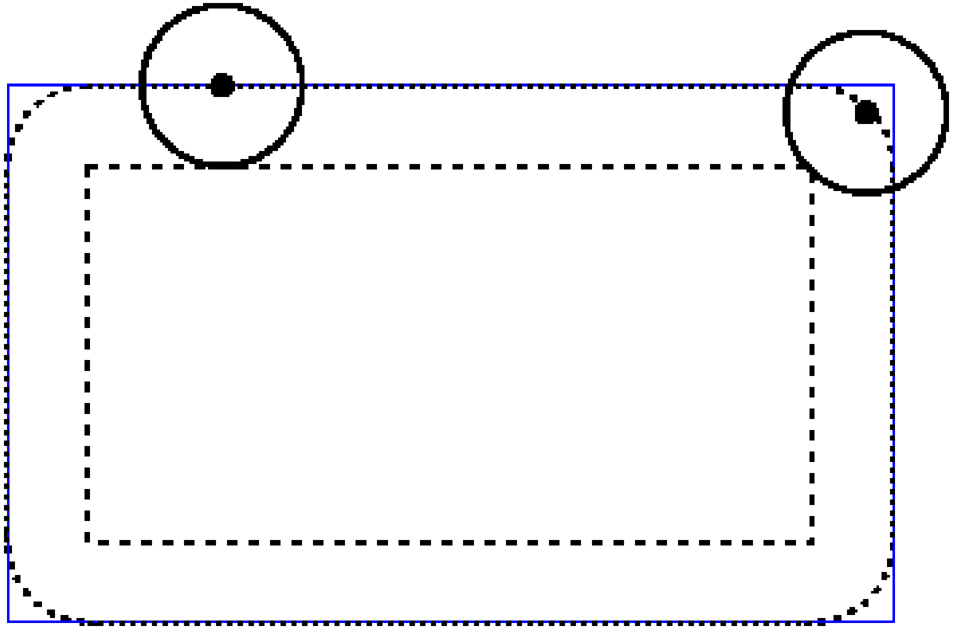
4-structuring element

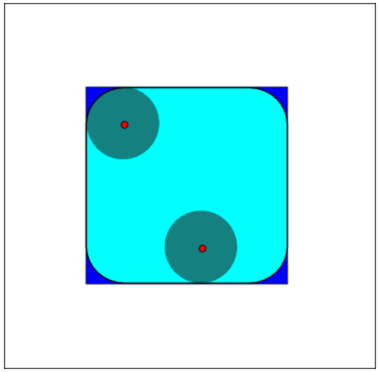


8-structuring element

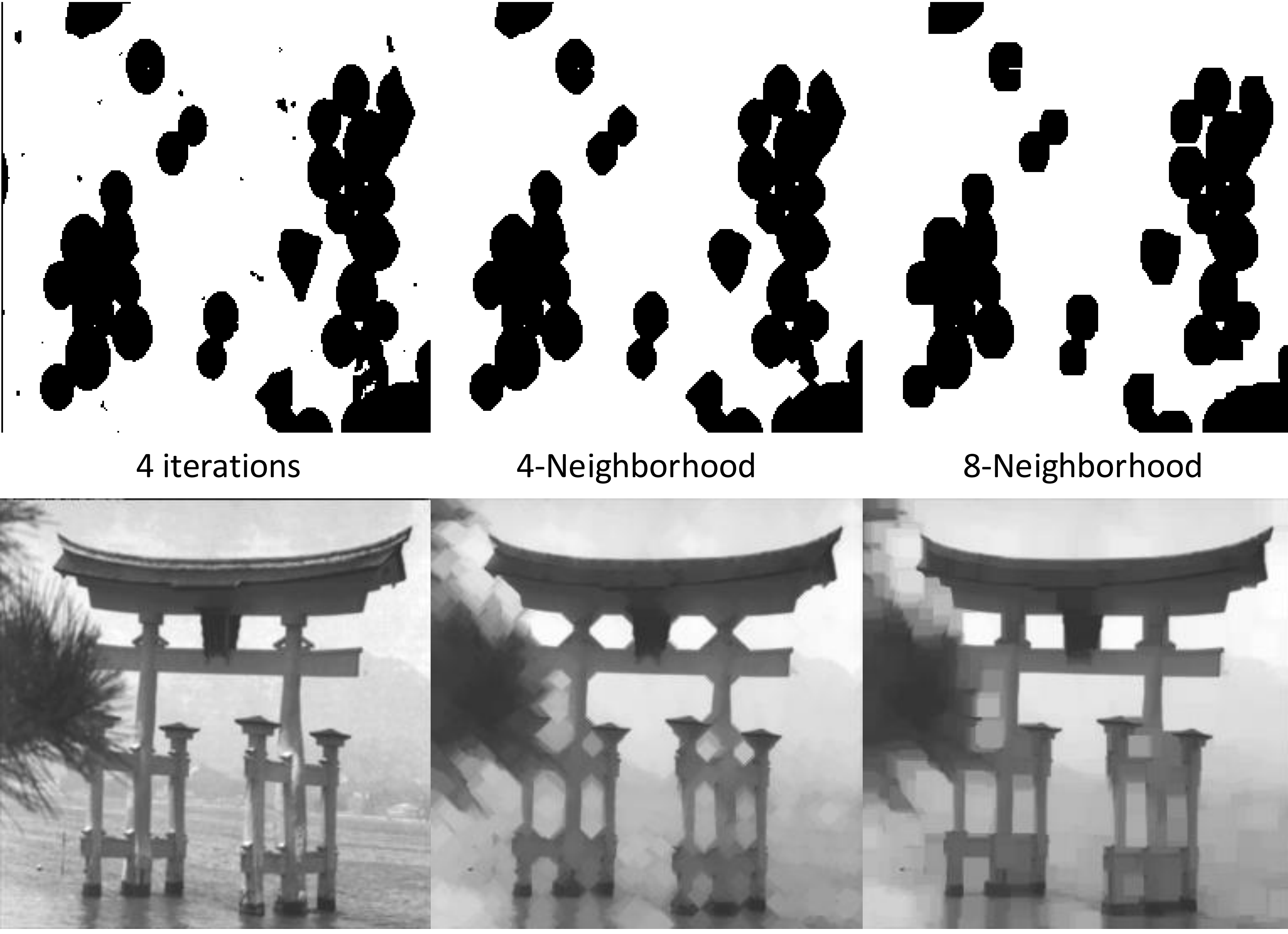
Opening

- X = Foreground Object
- B = Structuring Element
- Opening defined as $X \circ B = (X \ominus \breve{B}) \oplus B$



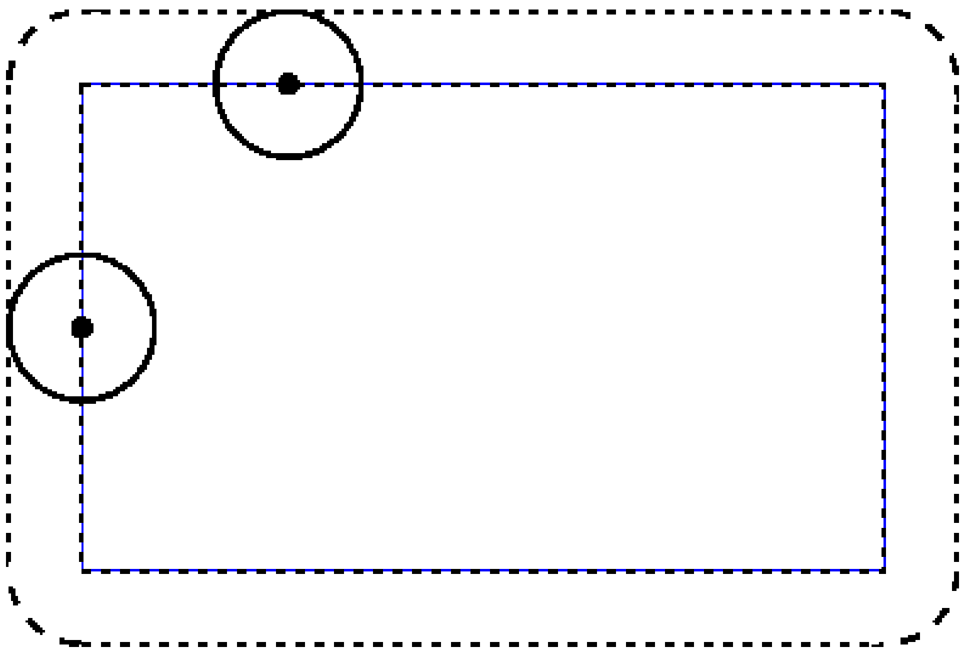


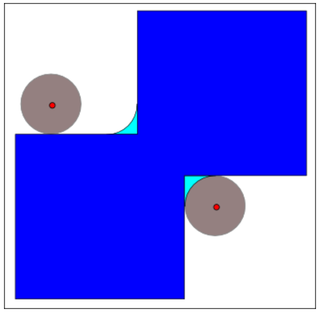
Opening of a set is the dilated of the eroded set (if B is symmetric)


4 iterations
4-Neighborhood
8-Neighborhood

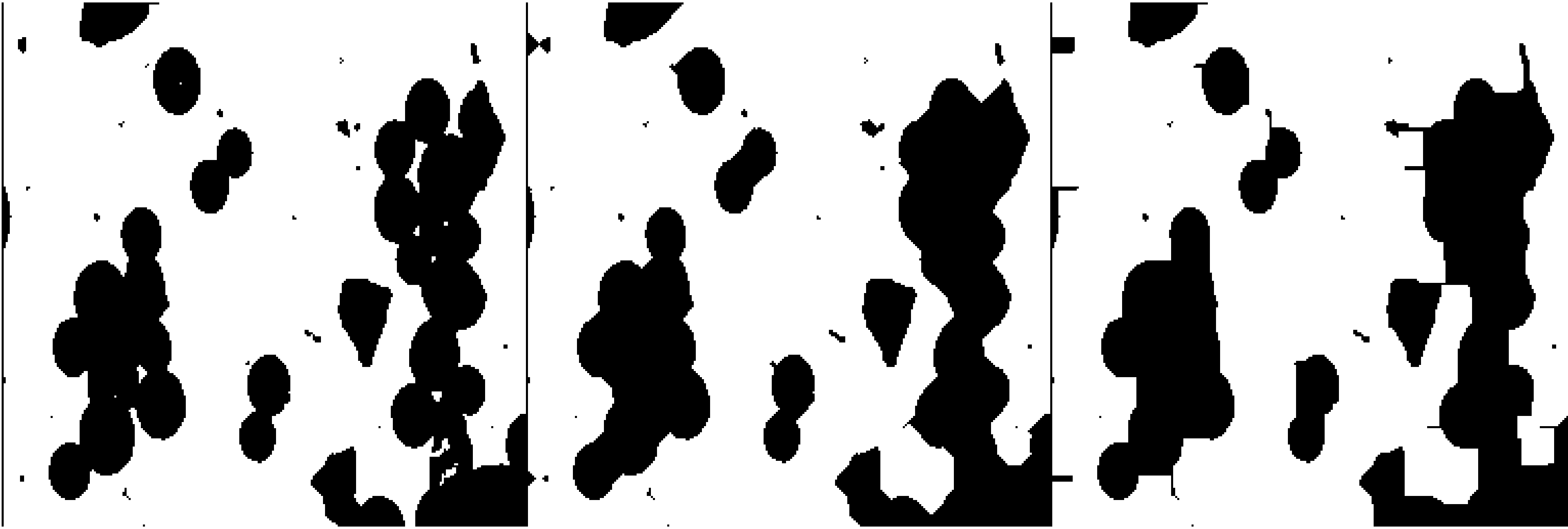
Closure

- Closure defined as $X \cdot B = (X \oplus \breve{B}) \ominus B$





Closure of a set is the eroded of the dilated set (if B is symmetric)



4 itérations | 4-élément structurant | 8-élément structurant

Link with Partial Differential Equation

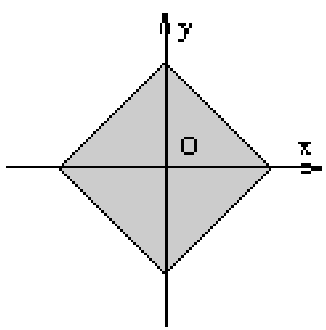
- Erosion

$$\frac{\partial I}{\partial t} = -\|\nabla I\|_{\mathrm{p}}$$

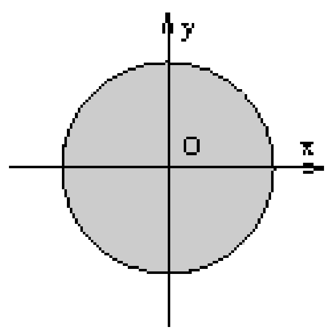
- Dilation

$$\frac{\partial I}{\partial t} = \|\nabla I\|_{\mathrm{p}}$$

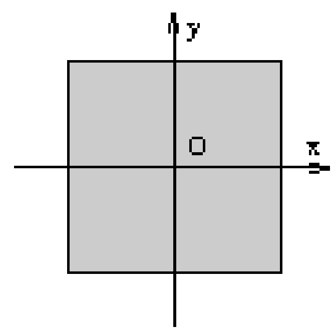
Choice of p-norm



P=1



P=2



P=∞

P=1

Partial differential Equations for Morphological Operators, F. Guichard, P. Maragos, J-M. Morel, Chap. In book Tribute to Georges Matheron, Springer, 2005

Distance Maps

- Allow
  - implementation of dilation / erosion of large size (with thresholding)
  - Compute easily the skeletonization of a binary shape (as regional maxima of distance function)