Medical Imaging
MVA 2013-2014
http://www-sop.inria.fr/asclepios/cours/MVA-2013-2014/Module2/

X. Pennec
Statistics on Riemannian manifolds
and Lie groups

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Medical Imaging – Module 2
MVA 2013-2014
Fri Jan. 17: Introduction to Medical Image Acquisition and Analysis [HD]
Fri Jan. 24: Statistics on Riemannian manifolds and Lie groups [XP]
Fri Jan. 31: Diffusion tensor Computing [XP]
Fri Feb. 7: Computational Anatomy [XP]
Fri Feb. 28: Diffeomorphic Registration & Statistics on Deformation [XP]
Fri Mar. 7: Mechanical Modeling and simulation [HD]
Fri Mar. 14: Physiological Modeling [HD]
Fri Mar. 21: Exam [HD, XP]

Course overview

Introduction
  • Why do we need statistics on manifolds?

The geometrical tool

Application to registration
Per-operative registration of MR/US images

Variability of a registration algorithm

Computational Anatomy
Methods of computational anatomy

Remodelling of the right ventricle of the heart in tetralogy of Fallot

- Mean shape
- Shape variability
- Correlation with clinical variables
- Predicting remodeling effect

Shape of RV in 18 patients

Shapes: forms & deformations

"Shape space" embedding [Kendall]

- Shape = what remains from the object when we remove all transformations from a given group
  - Transformation (rigid, similarity, affine) = nuisance factor
  - Shape manifold = quotient of the Object manifold by the group action
- Quotient spaces are non-linear (e.g. $\mathbb{R}^n / \text{scaling} = S^n$)
- Kendall size & shape space: $\left(\mathbb{R}^n\right)_{S^n}$

Statistics on shape spaces?

Anatomical structures segmented in Brain Images

- Sulcal lines at the surface of the cortex
- Surface of deep brain structures
- Fiber tracts from DTI

How to measure the variability across subjects?
Generic framework to deal with all object types?
**Shapes: forms & deformations**

Measure of deformation [D’Arcy Thompson 1917, Grenander & Miller]
- Object = “random” deformation of a reference template
- Reference template = Mean (atlas)
- Shape variability encoded by the deformations

Statistics on diffeomorphism?
How to determine an unbiased (centered) atlas?

**Diffusion Tensor Imaging**

Covariance of the Brownian motion of water
- Architecture of axonal fibers

Very noisy data
- Tensor image processing
  - Robust estimation
  - Filtering, regularization
  - Interpolation / extrapolation
- Information extraction (fibers)

Symmetric positive definite matrices
- Convex operations are stable
  - mean, interpolation
- More complex operations are not
  - PDEs, gradient descent...

Intrinsic computing on Manifold-valued images?

**Medical Image Analysis**

Measures are geometric and noisy or variable
- Feature extracted from images
- Transformations in registrations or shape analysis
- Diffusion tensor imaging
- Shape population variability

Goal: develop
- Consistent statistical tools
- A consistent computing framework
**Basic probabilities and statistics**

**Measure:** random vector \( x \) of pdf \( p_x(z) \)

**Approximation:**
- Mean: \( \mu = \mathbb{E}(x) = \int x \cdot p_x(z) \, dz \)
- Covariance: \( \Sigma_{xx} = \mathbb{E}[(x - \mu)(x - \mu)^T] \)

**Propagation:**
\[
y = h(x) - \left( h'(\mu), \frac{\partial h}{\partial x} \Sigma_x \frac{\partial h}{\partial x} \right)
\]

**Noise model:** additive, Gaussian...

**Principal component analysis**

**Statistical distance:** Mahalanobis and \( \chi^2 \)

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**Some problems with geometric features**

**Extrinsic means of 3D rotations:** invariance w.r.t. the chart
\[
R = \frac{1}{n} \sum R_i \quad q = \frac{1}{n} \sum q_i \quad p = \frac{1}{n} \sum p_i
\]

**Noise on 3D rotations:** invariance w.r.t. the transformation group

\[
\Sigma = \Sigma + \sigma^2 \quad \Sigma' = \Sigma + \sigma^2
\]

---

**Statistical computing on manifolds**

**The geometric framework**
- (Geodesically complete) Riemannian manifolds

**The statistical and computational tools**
- Mean, Covariance, Parametric distributions / tests
- Interpolation, filtering, diffusion PDEs

**The application examples**
- Rigid body transformations (evaluation of registration performances)
- Tensors: Diffusion tensor imaging, Variability of brain sulci
- Statistics of deformations for non-linear registration
Riemannian geometry is a powerful structure to build consistent statistical computing algorithms

Shape spaces & directional statistics
- [Kendall StatSci 89, Small 96, Dryden & Mardia 98]

Numerical integration, dynamical systems & optimization
- [Helmke & Moore 1994, Hairer et al 2002]
- Matrix Lie groups [Owen BIT 2000, Mahony JGO 2002]
- Optimization on Matrix Manifolds [Absi, Mahony, Sepulchre, 2008]

Information geometry (statistical manifolds)

Statistics for image analysis
- Rigid body transformations [Pennec PhD96]
- General Riemannian manifolds [Pennec JMA98, NSIP99, JMIV06]
- PGA for M-Reps [Fletcher IPMI03, TMI04]
- Planar curves [Klassen & Srivastava PAMI 2003]

Definition of a manifold

Intuitive idea
- "A manifold is a topological space which is locally Euclidean"
- "A Manifold is a topological space for which the neighborhood of each point is homeomorphic to the euclidean space"
- Homeomorphism: F is bijective and F and F⁻¹ are continuous (no folding or tearing transformation)

Definition
- Given an integer \( n \geq 1 \) and given some \( k \) such that \( k \) is either an integer, with \( k \geq 1 \), or \( k = \infty \), a \( C^k \)-manifold of dimension \( n \) consists of a topological space, \( M \), together with an equivalence class, \( A \), of \( C^k \)-atlases on \( M \). Any atlas, \( A \), of \( A \) is called a differentiable structure of class \( C^k \) (and dimension \( n \))

Differentiable manifolds
- When \( k = \infty \), we say that \( M \) is a smooth manifold
- When \( k=1 \), \( M \) is a differential manifold
- When \( k=0 \), \( M \) is a topological manifold

Traditional atlas definition

Given: Manifold \( M \)
Construct: Atlas \( A \)

Chart
- Region \( U_i \) in \( M \) (open disk)
- Region \( c \) in \( \mathbb{R}^n \) (open disk)
- Function \( \alpha_i \) taking \( U_i \) to \( c \)
  - Inverse

Atlas is collection of charts
- Every point in \( M \) in at least one chart
- Overlap regions
- Transition functions: \( \psi_{ij} = \alpha_j \circ \alpha_i^{-1} \) smooth
Riemannian Manifolds: geometrical tools

Riemannian metric:
- Dot product on tangent space
- Speed, length of a curve
- Distance and geodesics (angle, great circles)

Exponential chart (Normal coord. syst.):
- Development in tangent space along geodesics
- Geodesics = straight lines
- Distance = Euclidean
- Star shape domain limited by the cut-locus
- Covers all the manifold if geodesically complete

Cut locus
Reinterpretation of Basic Operations

<table>
<thead>
<tr>
<th>Operation</th>
<th>Euclidean space</th>
<th>Riemannian manifold</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subtraction</td>
<td>$xy = y - x$</td>
<td>$xy = \log_y(x)$</td>
</tr>
<tr>
<td>Addition</td>
<td>$y = x + xy$</td>
<td>$y = \exp(x)$</td>
</tr>
<tr>
<td>Distance</td>
<td>$\text{dist}(x, y) = |y - x|$</td>
<td>$\text{dist}(x, y) = |y - x|$</td>
</tr>
<tr>
<td>Gradient descent</td>
<td>$x_{i+1} = x_i - \delta\nabla(x_i)$</td>
<td>$x_{i+1} = \exp_{x_i}(-\delta\nabla(x_i))$</td>
</tr>
</tbody>
</table>

Metric choice

Transformations (Lie group):
- Left (or right) invariant: $\text{dist}(g, h) = \text{dist}(f \circ g, f \circ h) = \|f^{-1} \circ g - h\|
- Practical computations: $\exp_{f}\left[\delta f\right] = f \circ \exp_{g}\left[\delta g\right] = f^{-1} \circ \exp_{1}\left[-\exp_{x}^{-1} \circ y\right]$
- No bi-invariant metric

Homogeneous manifolds: $\text{dist}(x, y) = \text{dist}(x \ast g, y \ast g)$
- Invariance wrt the isotropy group
- Practical computations: $\exp\left[\Delta x\right] = \exp_{f}^{-1} \circ \Delta x \ast f \circ \exp_{x}^{-1} \circ y \ast f^{-1}$

General Riemannian manifolds:
- $\exp$ and $\log$ through numerical optimization / integration

Example on 3D rotations

Space of rotations $SO(3)$:
- Manifold: $\mathbb{R}^3 \setminus \{0\}$ and $\det(R) = 1$
- Lie group:
  - Composition: $R_1 \circ R_2 = R_1 \circ R_2$
  - Inversion: $R^{-1} = R$

Tangent space:
- At Identity (skew symmetric matrices)
- At any point by left or right translation

Metrics on $SO(3)$:
- Left / right invariant metrics
- Induced by the ambient space: bi-invariance
Example on 3D rotations

Group exponential
- One parameter subgroups
- Matrix exponential and Rodrigue’s formula

Exponential map for the bi-invariant metric
- Geodesic starting at identity = one parameter subgroups
- Geodesic everywhere by left (or right) translation

More details in the memo on rotations on the web

Space of rotations SO(3):
- Manifold: \( R \cdot R = \text{Id} \) and \( \det(R) = +1 \)
- Lie group:
  - Composition: \( R_1 \circ R_2 = R_1 \cdot R_2 \)
  - Inversion: \( R^{-1} = R^T \)

Tangent space
- At Identity (skew symmetric matrices)
- At any point by left or right translation

Metrics on SO(3)
- Left / right invariant metrics
- Induced by the ambient space: bi-invariance

Group exponential
- One parameter subgroups = bi-invariant Geodesic starting at Id
  - Matrix exponential and Rodrigue’s formula
- Geodesic everywhere by left (or right) translation

Statistics on Riemannian manifolds

The geometric framework

Simple statistical tools
- Mean, Covariance, Parametric distributions / tests
- Statistics on the spine

Application examples
- Evaluation of registration performances
- Rigid body transformations
Basic probabilities and statistics

Measure: random vector $x$ of pdf $p_z(x)$

Approximation:
- Mean: $\mathbb{E}(x) = \int x p_z(x) dx$
- Covariance: $\text{Cov}(x) = \int (x - \mathbb{E}(x)) (x - \mathbb{E}(x))^T p_z(x) dx$

Propagation:
$y = h(x) - \left( \mathbb{E}(x), \frac{\partial h}{\partial x} + \sum_i \frac{\partial h}{\partial x_i} \right)$

Noise model: additive, Gaussian...

Principal component analysis

Statistical distance: Mahalanobis and $\chi^2$.

Statistical tools on Riemannian manifolds

Metric -> Volume form (measure) $dM(x)$

Probability density functions
$\forall x, P(x \in X) = \int p_z(x) dM(x)$

Expectation of a function $\phi$ from $M$ into $\mathbb{R}$:

- Definition: $\mathbb{E}(\phi) = \int \phi(y)p_z(y) dM(y)$
- Variance: $\sigma^2 = \mathbb{E}(\text{dist}(y, z)^2) = \int \text{dist}(y, z)^2 p_z(z) dM(z)$
- Information (neg. entropy): $\text{Entropy}(\phi) = \mathbb{E}(\text{log}(p_z(x)))$

Fréchet expectation (1944)

Minimizing the variance:
$E[x] = \arg\min_{y \in M} \mathbb{E}([\text{dist}(y, x)])$

Existence
- Finite variance at one point

Characterization as an exponential barycenter (P(C)=0)
$\text{grad}([\text{dist}(y, x)]) = 0 \Rightarrow E[x] = \int x p_z(x) dM(z) = 0$

Uniqueness $K$archer 77 / Kendall 86 / Afsari 10 / Le 10
- Unique Karcher mean (thus Fréchet) if distribution has support in a regular geodesic ball with radius $r < r^* = \frac{1}{2} \min(\text{inj}(M), n/\sqrt{K})$ (k upper bound on sectional curvatures on $M$)

Other central primitives:
$E^*[x] = \arg\min_{y \in M} \left( \mathbb{E}[\text{dist}(y, x)]^2 \right)^{1/2}$
Other definitions of the mean

Doss [1949] / Herer [1988]: \( E^x = \{ y \in M / \text{dist}(y, \tau) \leq E[\text{dist}(y, x)] \} \)

Convex barycenters (Emery / Arnaudon)

\( E^x = \{ y \in M / \alpha(y) \leq E[\alpha(x)] \} \) for \( \alpha \) convex on the support of \( x \)

- Convex functions in compact spaces are constant

Emery 1991:

- if the support of \( x \) is included in a strongly convex open set:
  - Exponential barycenters ⊂ Convex Barycenters

Picard 1994: Connector (→) Connection (→) metric

- Difference between barycenters is \( O(\epsilon) \)

A gradient descent (Gauss-Newton) algorithm

Vector space

\[
\begin{align*}
  f(x + v) &= f(x) + V f' x + \frac{1}{2} v^T H_f v \\
  x_{n+1} &= x_n + v & \text{with} & & v = -H_f^{-1} \nabla f
\end{align*}
\]

Manifold

\[
\begin{align*}
  f(\exp(x, v)) &= f(x) + \nabla f(\exp(x, v)) + \frac{1}{2} v^T H_f(\exp(x, v)) \\
  \nabla f(\exp(x, v)) &= -2 E[(y-x)^2] + \frac{2}{n} \sum_i x_i \\
  H_f &= 2Id
\end{align*}
\]

- Geodesic marching
  \( x_{n+1} = \exp_x(v) \quad \text{with} \quad v = E[y - x] \)

Statistical tools: Moments

Frechet / Karcher mean minimize the variance

\[
E[x] = \arg \min_{x \in M} \left\{ E[\text{dist}(y, x)] \right\} \Rightarrow \quad E\left[\frac{1}{n} \sum_{i=1}^n \text{dist}(x, y_i)^2 \right] = \frac{1}{n} \int_{\mathcal{M}} \text{dist}(x, p(z))^2 d\mathcal{M}(z) = 0 \quad \text{[P(\mathcal{C}) = 0]}
\]

- Geodesic marching
  \( x_{n+1} = \exp_x(v) \quad \text{with} \quad v = E[y - x] \)

- Covariance et higher moments
  \[
  \Sigma_{xx} = E\left[\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}) (y_i - \bar{y}) \right] = \int_M \left[ \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right] p_x(z) d\mathcal{M}(z)
  \]

[ Pennec, JMI'06, RR-5093, NSIP'99 ]
Example with 3D rotations

Principal chart: rotation vector : \( r = \theta n \)

Distance: \( \text{dist}(R_1, R_2) = \| r_{\theta 1} - r_{\theta 2} \| \)

Frechet mean:

\( \bar{R} = \arg \min_{R \in H} \sum_i \text{dist}(R, R_i) \)

Centered chart:

mean = barycenter

Distributions for parametric tests

Uniform density:
- maximal entropy knowing \( X \)
  \[ p_x(z) = \text{Ind}_x(z) / \text{Vol}(X) \]

Generalization of the Gaussian density:
- Stochastic heat kernel \( p(x,y,t) \) [complex time dependency]
- Wrapped Gaussian [Infinite series difficult to compute]
- Maximal entropy knowing the mean and the covariance

\[ N(y) = k \exp \left( \frac{y^T G_y y}{2} \right) \]

\[ k = (2\pi)^{-\frac{n}{2}} \text{vol}(\Sigma)^{-\frac{1}{2}} \text{det}(\Sigma)^{-\frac{1}{2}} \]

Mahalanobis D2 distance / test:
- Any distribution:
  \[ E[\mu^2(x)] = r \]
- Gaussian:
  \[ \mu^2(x) \propto \chi^2 + O(\sigma^2) + O(\sigma / r) \]

[ Pennec, JMIV06, NSIP’99 ]

Gaussian on the circle

Exponential chart: \( x = r\theta \in [-\pi r; \pi r] \)

Gaussian: truncated standard Gaussian

\( r \to \infty: \) standard Gaussian (Ricci curvature \( \to 0 \))

\( y \to 0: \) uniform pdf with \( \sigma^2 = (x y)^2 / 3 \)

(compact manifolds)

\( y \to \infty: \) Dirac
PCA vs PGA

PCA
- Find the subspace that best explains the variance
- \( \rightarrow \) Maximize the squared distance to the mean
- Generative model: Gaussian

PGA (Fletcher, Joshi, Lu, Pizer, MMBIA 2004)
- Find a low dimensional sub-manifold generated by geodesic subspaces that best explain measurements
- \( \rightarrow \) Minimize the squared Riemannian distance from the measurements to that sub-manifold (no closed form)
- Generative model:
  - Implicit uniform distribution within the subspace
  - Gaussian distribution in the vertical space

Different models in curved spaces (no Pythagorean theorem)

Computing on manifolds: a summary

The Riemannian metric easily gives
- Intrinsic measure and probability density functions
- Expectation of a function from \( M \) into \( \mathbb{R} \) (variance, entropy)

Integral or sum in \( M \): minimize an intrinsic functional
- Fréchet / Karcher mean: minimize the variance
- Filtering, convolution: weighted means
- Gaussian distribution: maximize the conditional entropy

The exponential chart corrects for the curvature at the reference point
- Gradient descent: geodesic walking
- Covariance and higher order moments
- Laplace Beltrami for free

Statistical Analysis of the Scoliotic Spine

Database
- 307 Scoliotic patients from the Montreal’s Sainte-Justine Hospital
- 3D Geometry from multi-planar X-rays

Mean
- Main translation variability is axial (growth?)
- Main rotation var. around anterior-posterior axis

PCA of the Covariance
- 4 first variation modes have clinical meaning
Statistical Analysis of the Scoliotic Spine

• Mode 1: King's class I or III
• Mode 2: King's class I, II, III
• Mode 3: King's class IV + V
• Mode 4: King's class V (+II)

Statistics on Riemannian manifolds

Simple statistical tools
- The geometric framework
- Mean, Covariance, Parametric distributions / tests
- Statistics on the spine

Application examples in rigid registration
- Evaluation of registration performances
- Error propagation on rigid body transformations
- Consistency loops and bronze standard

Performance evaluation and validation

Synthetic data (simulation):
- Available ground truth
- Difficult to identify and model all sources of variability

Real data in a controlled environment (Phantom):
- Possible gold standard
- Performances evaluation in specific conditions
  - Difficult to test all clinical conditions
  - May hide a bias

Image database representative of the clinical application
- Usually no ground truth
- Should span all sources of variability
Performance evaluation without Gold Standard

Registration or consistency loops
- Holden et al. TMI 19(2), 2000

Cross-comparison of criterions

Ground truth as a hidden variable (EM like algorithms)
- Warfield, MICCAI 2002, [Staple, segmentation]
- Nicolau, IS4TM 2003

Error prediction

Uncertainty of feature-based registration

Matches estimation (landmarks)
- Alignment
- Geometric hashing
- ICP

Least square registration
\[ C(T, Z) = \sum_i y_i T * x_i \]

- Propagation of the errors from the data to the optimal transformation at the first order (implicit function theorem):
\[ \Sigma_{\epsilon_T} = \sigma^2 I_d \Rightarrow \Sigma_{\epsilon_T} = \sigma^2 H^{-1} \text{ with } H = \frac{\partial^2 C(T, Z)}{\partial T^2} \]

Error propagation

Optimum of a criterion = implicit function of data
\[ \hat{T} = \arg \min_T (C(X, T)) \quad \Phi(X, T) = \left[ \frac{\partial C}{\partial T}(X, T) \right]^T = 0 \]

First order Taylor expansion
\[ \Phi(X + \delta X, T + \delta T) = \Phi(X, T) + \frac{\partial \Phi}{\partial X} \delta X + \frac{\partial \Phi}{\partial T} \delta T + O(\delta X^2, \delta T^2) \]
\[ \delta T = -\left( \frac{\partial \Phi}{\partial T} \right)^{-1} \frac{\partial \Phi}{\partial X} \delta X + O(\delta X^2) \]

Data = random variable
\[ \Sigma_{\epsilon_T} = \mathbb{E}(\delta T \cdot \delta T^\top) = H^{-1} \left( \frac{\partial \Phi}{\partial X} \right) \Sigma_{\epsilon_X} \left( \frac{\partial \Phi}{\partial X} \right)^\top \cdot H^{-1} \]
**Registration of CT images of a dry skull**

550 matched frames among 2000

Typical object accuracy: 0.04 mm

Typical corner accuracy: 0.10 mm

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**Registration of MR T1 images of the head**

860 matched frames among 3600

Typical object accuracy: 0.06 mm

Typical corner accuracy: 0.125 mm

---

**Validation of the error prediction**

Comparing two transformations and their Covariance matrix:

\[ \mu^1(T_1, T_2) = \chi^2 \]

Mean: 6, Var: 12

KS test

Brigham and Women’s Multiple sclerosis database

- 24 3D acquisitions over one year per patient
- T2 weighted MR, 2 different echo times, voxels 1x1x3 mm
- Predicted object accuracy: 0.06 mm.

[X. Pennec et al., Int. J. Comp. Vis. 25(3) 1997, MICCAI 1998]
**Validation of the error prediction**

Comparing two transformations and their Covariance matrix:

\[ \mu^1(T_1, T_2) \approx \chi^2 \]

Mean: 6, Var: 12
KS test

Intra-echo: \( \mu^1 \approx 6 \), KS test OK
Inter-echo: \( \mu^1 > 50 \), KS test failed, Bias!

Bias estimation: (chemical shift, susceptibility effects)
- \( \sigma_{\text{bias}} = 0.06 \) deg (not significantly different from the identity)
- \( \sigma_{\text{bias}} = 0.2 \) mm (significantly different from the identity)

Inter-echo with bias corrected: \( \mu^1 \approx 6 \), KS test OK

[X. Pennec et al., Int. J. Comp. Vis. 25(3) 1997, MICCAI 1998]

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**Accuracy Evaluation (Consistency)**

\[ \sigma^2_{\text{bias}} = 2\sigma^2_{\text{LUT,0.2}} + \sigma^2_{\text{LUT}} + \sigma^2_{\text{LUT}} \]

---

**Validation using Bronze Standard**

Best explanation of the observations (ML):
- LSQ criterion
- Robust Fréchet mean
- Robust initialization and Newton gradient descent
- Grid scheduling for efficiency

Result \( T_{\text{LSQ}}, \sigma_{\text{bias}}, \sigma_{\text{bias}} \)

Results on per-operative patient images

Data (per-operative US)
- 2 pre-op MR (0.9 x 0.9 x 1.1 mm)
- 3 per-op US (0.63 and 0.95 mm)
- 3 loops

Robustness and precision

<table>
<thead>
<tr>
<th>Method</th>
<th>var rot (deg)</th>
<th>var trans (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MI</td>
<td>0.53</td>
<td>0.25</td>
</tr>
<tr>
<td>CR</td>
<td>0.66</td>
<td>0.17</td>
</tr>
<tr>
<td>BCR</td>
<td>0.25</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Consistency of BCR

<table>
<thead>
<tr>
<th>Method</th>
<th>var rot (deg)</th>
<th>var trans (mm)</th>
<th>var test (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple MR</td>
<td>0.06</td>
<td>0.06</td>
<td>0.10</td>
</tr>
<tr>
<td>Loop</td>
<td>2.22</td>
<td>0.62</td>
<td>2.33</td>
</tr>
<tr>
<td>M/US</td>
<td>1.57</td>
<td>0.58</td>
<td>1.65</td>
</tr>
</tbody>
</table>

(Roche et al., TMI 20(10), 2001

Mosaicing of Confocal Microscopic in Vivo Video Sequences.

Cellvizio: Fibered confocal fluorescence imaging

Mosaic image creation

- Interpolation / approximation with irregular sampling

Common coordinate system
- Multiple rigid registration
- Refine with non rigid

[ T. Vercauteren et al., MICCAI 2005, T.1, p.753-760 ]

Mosaicing of Confocal Microscopic in Vivo Video Sequences.
**Conclusion**

Statistics on geometric objects (e.g. transformations)
- Use a representation centered at the current estimation
- Developing along geodesics captures first order non-linearity due to curvature

Evaluation of registration performances
- Synthetic data
- Phantom (real images)
- Methods without gold-standard (Bronze standard)