On Smoothness Measures of Active Contours and Surfaces

Hervé Delingette
INRIA , Projet Epidaure
06902 Sophia-Antipolis, France
Email : Herve.Delingette@inria.fr

Abstract

In this paper, we propose to study different smoothness measures of planar contours or surfaces. We first define a smoothness measure as a functional that follows three types of invariance : invariance to changes of contour parameterization, invariance to contour rotations and translations and invariance to the contour sizes. We then introduce different smoothness measures that can be classified into local or global functionals but also that can be of geometric or algebraic nature. We finally discuss their implementation by observing the advantages and disadvantages of explicit and implicit contour representations.

1 Introduction

The notion of smoothness functionals is at the heart of several practical and theoretical problems in computer vision. For instance, it is naturally related to the problem of visual reconstruction [2] and regularization [1]. Through their first variation, these functionals can also be used for smoothing curves or surfaces in an intrinsic manner to obtain multiscale representations of these manifolds.

In this paper, we first define some mathematical criteria corresponding to our intuitive notion of curve or surface smoothness. We then propose several functionals that meet these criteria. Among them, most of those based on geometry have been already proposed in the litterature [13, 5]. Also, Gage [7] and Sapiro [14] have studied curve flows having remarkable invariance properties, but that do not correspond to the minimization of a functional.

In addition to the geometry-based smoothness measures, we show that functionals based on algebraic invariants can also be considered as smoothness measures. By extending the calculus of variation to more general functionals, we introduce new curve flows that have interesting invariance properties. Finally, we discuss the different possible implementations based on implicit or explicit representations.

2 Properties of Fonctionals

2.1 Definitions

In this section, we mostly concentrate on the evaluation of smoothness functionals defined on planar contours. To describe analytically each contour, we use an explicit contour representation : for each parameter value $u \in [0, u_0]$ we associate a point on the Euclidean plane $\mathcal{C}(u) = (x(u), y(u))^T \in \mathbb{R}^2$. The choice between implicit and explicit contour representation does not have any impact on the definition of smoothness functionals but only on their numerical computation (this is discussed in section 5.3). A contour may be closed or open. We write as $\mathbf{N}(u)$, $\mathbf{T}(u)$, the normal and tangent vector at point $\mathcal{C}(u)$ while $k(u)$ is the curvature at that point.

A functional $E(\mathcal{C}(u))$ is an application that associates a positive scalar value with a given contour : $E : \mathcal{C}(u) \in \mathcal{F} \mapsto E(\mathcal{C}) \in \mathbb{R}^+$. The set of admissible contour $\mathcal{F}$ depends on the nature of the functional. For a functional relying on the contour first derivatives, $\mathcal{F}$ is the set of continuously differentiable contours.

2.2 Invariance Properties

A first way of characterizing a functional is to look at its invariance properties. In particular two different invariance properties are of high importance :

- **Invariance under the application of a group of transformation**. Given a group of transformations $\mathcal{T}$ of the Euclidean plane $\mathbb{R}^2$, such as the group of rigid or similarity transformations, a functional is invariant under the application of $\mathcal{T}$ iff :

$$E(T(\mathcal{C})(u)) = E(\mathcal{C}(u)) \quad \forall T \in \mathcal{T} \tag{1}$$

- **Invariance under change of contour parameter**. A contour, in its explicit formulation is described by a parameter u. A functional is invariant to any change of parameterization iff :

$$E(\mathcal{C}(u)) = E(\mathcal{C}(u^\star)) \quad \forall u^\star = \phi(u) \tag{2}$$

In this case, the functional is called "intrinsic" since it only depends on the contour shape. For an intrinsic functional we use the following notation : $E(\mathcal{C}(u)) = E(\mathcal{C})$. Finally, only intrinsic functionals can be written with an implicit formulation of a contour.

2.3 First Variation of a functional

Under certain conditions, it is possible to compute the first variation of a functional that should be seen as the functional first derivative. Indeed, the first variation of a functional $\delta E(\mathcal{C}(u))$ is defined as :

$$\delta E(\mathcal{C}(u)) = \lim_{\|\epsilon(u)\|\mapsto 0} \frac{E(\mathcal{C}(u)+\epsilon(u)) - E(\mathcal{C}(u))}{\|\epsilon(u)\|}$$

The Euler-Lagrange equation provides the expression of the first variation when the functional consists of an integral along the contour. Also, if a functional is a function of two functionals, then the following chain rule applies :

$$\begin{aligned} E(\mathcal{C}) &= h(E_1(\mathcal{C}), E_2(\mathcal{C})) \\ \delta E(\mathcal{C}) &= \frac{\partial h}{\partial E_1}\delta E_1(\mathcal{C}) + \frac{\partial h}{\partial E_2}\delta E_2(\mathcal{C}) \end{aligned} \quad (3)$$

This relation allows us to easily calculate the first variation of complex functionals by decomposing it into simpler functionals.

Among all contours having given boundary conditions, it is important for characterizing a functional to look for the contours minimizing this functional. Depending on the nature of the functional, its natural boundary conditions may be posed in terms of end positions ($\mathcal{C}(0) = \mathbf{P}$ and $\mathcal{C}(u_0) = \mathbf{Q}$) but can also include the contour end first or second derivatives.

The first variation of a functional is useful to find the set of contours minimizing this functional. Indeed, a *necessary* condition for $\mathcal{C}^\star$ to be an extremum of $E(\mathcal{C}(u)$ is that $\delta E(\mathcal{C}^\star) = \mathbf{O}$. A *sufficient* condition for $\mathcal{C}^\star$ to be an extremum is not simply related to the sign of the second variation $\delta^2 E(\mathcal{C})$ but is based on more complex notions of the "extremum field" theory. However, we will abusively call *minimal contours* of a functional $E(\mathcal{C}(u))$, the contours verifying ;

$$\delta E(\mathcal{C}^\star) \equiv \mathbf{O} \quad (4)$$

This set of minimal contours provides important clues about the nature of the functionals since they correspond to the "smoothest" contours according to the smoothness metric given by the functional.

If a functional is convex then existence and unicity of the minimal contours is guaranteed.

2.4 Associated contour evolution

Also of interest for characterizing a functional, is the study of the contour evolution when minimizing this functional. In this case, an evolving contour $\mathcal{C}(u,t)$ depending the time parameter t is controlled through a partial differential equation defining its law of motion. Two laws of motion are mainly used :

- **Lagragian law of motion** : $\frac{\partial \mathcal{C}(u,t)}{\partial t} = \delta E(\mathcal{C}(u,t))$ which corresponds to a gradient descent of the functional.
- **Newtonian law of motion** : $\frac{\partial^2 \mathcal{C}(u,t)}{\partial t^2} = -\gamma \frac{\partial \mathcal{C}(u,t)}{\partial t} + \delta E(\mathcal{C}(u,t))$ which corresponds to the equation of a mechanical spline minimizing its potential energy in a viscous medium (γ is the damping factor).

For an analysis of this *pde*, it is easier to consider a Lagrangian law of motion since it only depends on the nature of the functional (and not on the value of the damping factor γ). Several authors including Kimia *et al.* [10] have studied the general contour evolution associated with this *pde*. A first result is that the evolution of the contour shape only depends on the normal component $\beta(u,t)$ of $\delta E(\mathcal{C}(u,t))$. Furthermore, they provide simple *pde*'s that give the evolution of most geometric entities of the contour as a function of $\beta(u,t)$. For instance, the contour length $\mathcal{L}(t)$ evolves as :

$$\frac{\partial \mathcal{L}}{\partial t} = \int_{\mathcal{C}} k(u)\beta(u,t)du$$

2.5 Smoothness Functionals

We define a smoothness functional as a functional that measures the geometric regularity of a shape. Since several definitions of smoothness are possible, we provide a list of criteria that should be met in order to state that a functional is a smoothness functional. With these criteria, we try to capture the intuitive perception of shape smoothness :

1. **Invariance with respect to parameterization**. A contour smoothness measure should be intrinsic and not depend on a given parameterization.
2. **Invariance with respect to similarity transformations**. The position, orientation and size of a contour should not affect its smoothness measure.
3. **Circles and lines should belong to the set of curves minimizing smoothness functionals**.
4. **Dependance on inner-scale**. The smoothness measure should be a scale-related value. A contour may be rough at a small scale and smooth at a large scale. Without any scale dependence, it is assumed that the smoothness is evaluated at the finest scale.

2.6 Quadratic Functionals

The most widely used functionals in computer vision are quadratic functionals that can be written as :

$$E(\mathcal{C}(u)) = \int_0^{u_0} \|F(u) \star C(u)\|^2 du$$

where $F(u) \star \mathcal{C}(u)$ designates the convolution of $\mathcal{C}(u)$ with function $F(u)$. In particular, Tikhonov stabilisers corresponding to the Sobolev norms have been extensively used for the regularization of active contours or active surfaces to solve ill-posed problems such as image segmentation or surface reconstruction [1].

Their main advantage is to lead to linear systems of equations that can be solved in closed form. Also they have been extensively studied in the approximation theory. Unfortunately, none of these functionals follow the 4 above criteria since they are not invariant with scale changes :$E(\sigma\mathcal{C}(u)) = \sigma^2 E(\mathcal{C}(u))$. More importantly they are not independent to changes of parametrisation (can be shown using the Parseval theorem) which makes them ill-suited for measuring shape characteristics of contours or surfaces. However, Tikhonov stabilisers are invariant with the application of any rotations and translations on contours or surfaces [3].

2.7 Classification of Smoothness Functionals

We propose to classify smoothness functionals according to two criteria : geometric *vs* algebraic and local *vs* global.

Geometric functionals only depends on intrinsic geometric entities. For a planar contour, it implies that it only depends on arc length s, curvature and the derivatives of curvature $\frac{dk}{ds}$. On the other hand, algebraic functionals can be expressed as :

$$\begin{aligned} E(\mathcal{C}) &= \int_{x_0}^{x_1} P(x,y)dx + \int_{y_0}^{y_1} Q(x,y)dy \\ &= \int_0^{\mathcal{L}} \begin{pmatrix} -Q(x,y) \\ P(x,y) \end{pmatrix} \cdot \mathbf{N} ds \end{aligned}$$

The latter equation has a simple mechanical interpretation : if $\begin{pmatrix} -Q(x,y) \\ P(x,y) \end{pmatrix}$ is a force field defined on the Euclidean plane, then $E(\mathcal{C})$ corresponds to the work done by the force as a point mass moves along the contour.

Furthermore, if the contour is closed, and Ω is the domain enclosed by the contour then the Stoke's theorem states that :

$$\begin{aligned} E(\mathcal{C}) &= \int_{x_0}^{x_1} P(x,y)dx + \int_{y_0}^{y_1} Q(x,y)dy \\ &= \int\int_{\Omega} \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy \end{aligned}$$

For the second criterion, we define local smoothness functionals, a functional for which its smoothness is the sum of the smoothness of its parts. If $\mathcal{C}_1$ is the part of $\mathcal{C}$ for $u \in]0, u^\star[$ and $\mathcal{C}_1$ is the part of $\mathcal{C}$ for $u \in]u^\star, u_0[$ then we have :

$$E(\mathcal{C}) = E(\mathcal{C}_1) + E(\mathcal{C}_2)$$

With these functionals, the first variation $\delta E(\mathcal{C}(u))$ only depends on the characteristics of the contour around point $\mathcal{C}(u)$ and not of parameters related to the whole contour. Therefore, with local smoothness functionals each contour part contributes independently to the functional value whereas there is a coupling occuring between each contour part in the evaluation of a global functional.

3 Geometric Smoothness Functionals

3.1 Contour Length

The contour length is computed as :

$$E(\mathcal{C}) = \mathcal{L} = \int_0^{u_0} \frac{ds}{du} \quad du = \int_0^{u_0} \sqrt{(\frac{dx}{du})^2 + (\frac{dy}{du})^2} \quad du$$

and its first variation is $\delta E = -\frac{ds}{du} k\mathbf{N}$ which implies that the minimal curves are straight lines. The associated *pde* is the well studied *mean curvature motion* [8] which exhibits remarkable properties. Clearly, this functional is dependent on the contour size which explains why all closed contours shrink to a point under the mean curvature motion. Therefore, the contour length cannot be considered as an adequate smoothness measure.

As an alternative, it is possible to consider, instead of the Euclidean contour arc length, the contour arc length which is invariant with respect to similarity transformations [14] : kds. Since the curvature k is the variation of polar angle $k = \frac{d\phi}{ds}$, then the " similarity contour length" of a closed contour is :

$$E(\mathcal{C}) = 2\pi n(\mathcal{C})$$

where $n(\mathcal{C})$ is the number of turns of the contour. This measure is linked with the contour embedding in the plane but does not characterise its shape. Therefore, it cannot be used also as a smoothness measure.

3.2 Bending Energy

The bending energy is :

$$E(\mathcal{C}) = \mathcal{M} = \int_0^{\mathcal{L}} k^2 ds$$

Its first variation is :

$$\delta E(\mathcal{C}) = \frac{ds}{du}\left(k^3(u) + 2\frac{d^2k}{ds^2}\right)\mathbf{N}$$

Polden [13] have studied the associated *pde* :

$$\frac{\partial \mathcal{C}}{\partial t} = \left(k^3(u) + 2\frac{d^2k}{ds^2} \right) \mathbf{N}$$

The curves minimizing the sum of their square curvature, also called *mechanical splines*, or *M-curves* have been studied by many authors including Horn[9] and Su et al[15].

Surprisingly, the bending energy is not scale invariant, and Horn[9] first noticed that circles do not minimize this functional. Therefore, the bending energy cannot be considered as a smoothness measure of planar contours.

To achieve scale invariance, Bruckstein[4] *et al.*have increased the bending energy for longer curves, by defining the energy below :

$$E(\mathcal{C}) = \mathcal{L}\mathcal{M} = \int_0^{\mathcal{L}} ds \quad \int_0^{\mathcal{L}} k^2(s)ds \qquad (5)$$

The first variation of this *normalized bending energy* can be computed with equation 3 as :

$$\delta E(\mathcal{C}) = \frac{ds}{du}\left(\mathcal{L}k^3 + 2\mathcal{L}\frac{d^2k}{ds^2} - \mathcal{M}k \right)$$

Clearly lines and circles are among the contours that minimize this functional. For a straight line, the smoothness measure is zero whereas it is equal to $4\pi^2$ for a circle. Therefore, this functional is a smoothness measure that follows the first three criteria. To obtain a scale dependent smoothness measure, we propose to use the functional below :

$$E(\mathcal{C}) = \mathcal{L}\int_0^{\mathcal{L}} \frac{\|\mathcal{C}(s-dl) - 2\mathcal{C}(s) + \mathcal{C}(s+dl)\|^2}{16dl^4} ds \quad (6)$$

When dl decreases towards zero, then the functional of equation 6 converges towards equation 5. Also, it is important to note that scale-dependent version of the *normalized bending energy* can be evaluated on a curve that may not be twice differentiable which makes it a more powerful measure of smoothness.

Finally, it should be noticed that this smoothness measure is global since its first variation depends on the shape of the whole contour through the two parameters $\mathcal{L}$ and $\mathcal{M}$.

3.3 Bending Energy of 3D Surfaces

We now consider smoothness measures defined on 2-manifolds of $\mathbb{R}^3$. We consider a surface $\mathcal{S}(u,v)$ as a mapping from its parametric domain $\Omega \in \mathbb{R}^2$ into $\mathbb{R}^3$. The corresponding bending energy defined on tridimensional surfaces is :

$$E(\mathcal{S}) = \int\int_\Omega H^2(u,v)dA \qquad (7)$$

where :

- H is the mean curvature of the surface equal to the average sum of the two principal curvatures.
- dA designates the elementary surface element equal to $\sqrt{LN-M^2}dudv$ where $L = \|\frac{\partial \mathcal{S}}{\partial u}\|^2$, $N = \|\frac{\partial \mathcal{S}}{\partial v}\|^2$ and $M = \frac{\partial \mathcal{S}}{\partial u} \cdot \frac{\partial \mathcal{S}}{\partial v}$

In fact, this total energy is closely related to the total curvature of the surface :

$$E(\mathcal{S}) \quad = \quad \int\int_\Omega \frac{k_1^2(u,v) + k_2^2(u,v)}{4} dA$$

where $k_1(u,v)$ and $k_2(u,v)$ are the two principal curvatures. Since $k_1^2 + k_2^2 = 4(H^2 - 4K)$ and $\int\int_\Omega K(u,v)dA$ is a topological invariant of the surface, the total curvature is equivalent to the using the bending energy.

Unlike the bending energy defined on contours, this functional defined on tridimensional surfaces is clearly invariant with respect to the surface size. Furthermore, its first variation is computed as :

$$\delta E(\mathcal{S}) = \sqrt{LN-M^2}\left(\Delta^\star H + 2H^3 - 2HK\right)\mathbf{N}$$

where $\Delta^\star H$ is the Laplace-Beltrami operator applied on the surface mean curvature. It appears that all planes and spheres (for which $H = 1/r$ and $K = 1/r^2$) minimizes this bending energy. The bending energy is equal to zero for planes and 4π for spheres. Therefore, the surface bending energy of is an appropriate smoothness measure.

To obtain a scale-dependent smoothness measure, it is necessary to evaluate the mean-curvature at different scales. We propose to use the following property of the mean curvature : $H(u,v)\mathbf{N} = \Delta^\star \mathcal{S}(u,v)$. This intrinsic Laplacian operator can be approximated using a circular neighborhood around point $\mathcal{S}(u,v)$. If $\mathcal{N}(u,v,dl)$ is the tridimensional curve corresponding to the set of points on $\mathcal{S}$ located at the geodesic distance dl from point $\mathcal{S}(u,v)$ (see figure 1), then we can approximate the mean curvature $H(u,v,dl)$ at scale dl with the following relation :

$$H(u,v,dl) = \frac{1}{dl^2}\left\| \mathcal{S}(u,v) - \frac{\int_{\mathcal{N}(u,v,dl)} \mathcal{S}(m,n)ds}{\int_{\mathcal{N}(u,v,dl)} ds} \right\|$$

Therefore, the mean curvature is proportional to the distance between $\mathcal{S}(u,v)$ and the centroid of the curve $\mathcal{N}(u,v,dl)$. When dl converges towards zero, $H(u,v,dl)$ converges towards the continuous mean curvature $H(u,v)$. The scale-dependent version of the bending energy can be written as :

$$E(\mathcal{S}) = \int\int_\Omega \left\| \frac{\mathcal{S}(u,v)}{dl^4} - \frac{\int_{\mathcal{N}(u,v,dl)} \mathcal{S}(m,n)ds}{dl^4 \int_{\mathcal{N}(u,v,dl)} ds} \right\|^2 dA \qquad (8)$$

Again this functional provides a way to measure smoothness on surfaces without requiring to have twice differentiability everywhere on the surface.

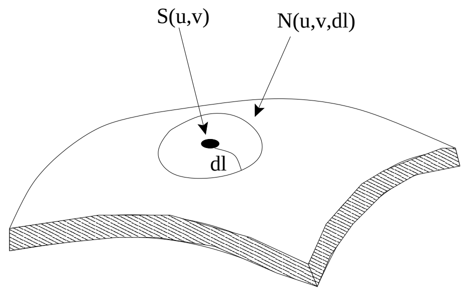

Figure 1: Definition of the curve $\mathcal{N}(u,v,dl)$ located at the geodesic distance dl from point $\mathcal{S}(u,v)$

3.4 Extension to other dimensions

In [5] Chen *et al.*have studied differential problems defined on hypersurfaces, i.e. on n-manifolds of $\mathbb{R}^{n+1}$. In particular, they have studied the following functionals :

$$E(\mathcal{S}) = \mathcal{H}_p = \int_\Omega \|H\|^p dA$$

where $H = \sum k_i$ is the manifold mean curvature (k_i is a principal curvature). The first variation when $p \neq 1$ and $n \neq 1$ is :

$$\delta E(\mathcal{S}) = \sqrt{|g_{\alpha\beta}|}(p\Delta^\star H^{p-1} + n^2(p-1)H^{p+1} \\ -2pH^{p-1}R)\mathbf{N}$$

where $|g_{\alpha\beta}|$ is the determinant of the metric tensor, $\Delta^\star$ is the generalized Laplace-Beltrami operator and R is the scalar curvature defined as $R = \sum_{i,j} k_i k_j$

It first appears that $\mathcal{H}_p$ is scale invariant only if $p = n$. Furthermore, it can be shown can hyperspheres and hyperplanes verify $\delta E(\mathcal{S}) = \mathbf{O}$ if $p = n$. Therefore, we can claim that $\int_\Omega \|H\|^n dA$ is a smoothness measure for n-manifolds with $n > 1$.

4 Algebraic Smoothness Functionals

4.1 Enclosed Area

The first algebraic functional that we can associate with a planar contour is the signed area $\mathcal{A}$ enclosed that a curve $\mathcal{C}$:

$$E(\mathcal{C}) = \mathcal{A}(\mathcal{C}) = \frac{1}{2}\int_\mathcal{C} (xdy - ydx)$$

When a contour is open with end points $\mathbf{p}_0$ and $\mathbf{p}_1$ then $\mathcal{A}(\mathcal{C})$ is equal to the sum of the corresponding closed contour area when a straight line is drawn between $\mathbf{p}_0$ and $\mathbf{p}_1$ and the area of triangle $(\mathbf{O}\mathbf{p}_1\mathbf{p}_0)$ ($\mathbf{O}$ is the reference frame origin).

The functional $\mathcal{A}(\mathcal{C})$ is not a smoothness measure since it is a signed functional and since it is obviously not size invariant. Its first variation simply writes as :

$$\delta E(\mathcal{C}) = \frac{ds}{du}\mathbf{N}$$

4.2 Isoperimetric Ratio

We first propose to the use the isoperimetric ratio as a smoothness measure :

$$E(\mathcal{C}) = \frac{4\pi|\mathcal{A}(\mathcal{C})|}{\mathcal{L}^2} \tag{9}$$

In fact, in many textbooks, the isoperimetric ratio is often written as the inverse of equation 9. Indeed, for computation stability, we prefer to divide by the contour length which is greater than zero for non-degenerate curves rather than dividing by the enclosed area which may be zero even for non-degenerate curves. Therefore, unlike previous functionals, this isoperimetric ratio must be maximised and not minimized. Jacob Steiner in the middle pf the XIXth century, has proved that from all planar curves, circles maximise the isoperimetric ratio among all closed curves :

$$4\pi\mathcal{A}(\mathcal{C}) \leq \mathcal{L}^2$$

This can be proved also by looking at the first variation of this functional :

$$\delta E(\mathcal{C}) = \frac{ds}{du}\frac{1}{\mathcal{L}^2}\left(1 - \frac{2\mathcal{A}k}{\mathcal{L}}\right)\mathbf{N}$$

For closed planar curves, therefore, the isoperimetric ratio is a smoothness measure since it is invariant to the application of rotations, translations and scale changes.

However, for open curves, two problems arise. First, the isoperimetric ratio depends on the position of the curve with respect to the origin of the reference frame. Therefore, it is no longer invariant to translation. To solve this problem, we propose to compute the enclosed area $\mathcal{A}(\mathcal{C})$ of an open contour as the enclosed area of the closed contour build by linking the two end points with a straight line (see figure 2).

Second, straight lines minimize the isoperimetric ratio whereas circles maximize it. Therefore, the opposite value of the isoperimetric ratio should be considered for open curves.

The isoperimetric ratio easily generalizes itself to tridimensional surfaces :

$$E(\mathcal{S}) = \frac{36\pi\mathcal{V}(\mathcal{S})^2}{\mathrm{Area}(\mathcal{S})^3}$$

where $\mathcal{V}(\mathcal{S})$ is the volume enclosed by the surface defined as an algebraic invariant ($\mathcal{V}(\mathcal{S}) = \frac{1}{3}\int\int_\Omega (xdydz + ydydz + zdxdy)$) and $\mathrm{A}rea(\mathcal{S})$ is the area of the surface $\mathcal{S}$.

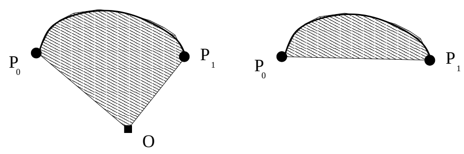

Figure 2: (Left) Geometrical definition of $\mathcal{A}(\mathcal{C})$ on open contours; (Right) Translation invariant definition of $\mathcal{A}(\mathcal{C})$

Similarly, spheres are known to maximize the isoperimetric ratio of closed surfaces. The associated first variation is :

$$\delta E(\mathcal{S}) = \sqrt{LN - M^2}\frac{36\pi\mathcal{V}(\mathcal{S})}{\mathrm{Area}(\mathcal{S})^3}\left(1 - 3H\frac{\mathcal{V}(\mathcal{S})}{\mathrm{Area}(\mathcal{S})}\right)\mathbf{N}$$

For open surfaces (surfaces with end contours such as cylinders), to obtain a translation invariant functional it is necessary to close each end contour by a surface. The natural choice is to close each end contour with a minimal surface, surface minimizing the area of all spanned surfaces. However, in practice, defining these surfaces is known to be a non-trivial task, even if efficient iterative algorithms have been proposed [12] to find triangulated approximations of these surfaces. Therefore, we can consider that this isoperimetric ratio is not suited for the smoothness measure of open surfaces.

4.3 Moment Invariants

Moment invariants have been widely used in computer vision for object recognition tasks. In particular, several authors have reported systematic methods to build moment invariants for any given transformation group [11]. In this paper, we only present smoothness measures based on the basic geometric moments, even if more sophisticated moments such as Zernike moments could be also used. The geometric moments are defined on any type of manifolds, curves, surfaces or volumes. If $\mathcal{D}$ is the parameterization space of the manifold, then the geometric moments on $\mathbb{R}^2$ are defined as :

$$m_{pq} = \int_{\mathcal{D}} x^p y^q dxdy$$

If we write $\overline{x} = \frac{m_{10}}{m_{00}}$ and $\overline{y} = \frac{m_{01}}{m_{00}}$ as the centroid of the manifold, then the centered geometric moments μ_{pq} are :

$$\mu_{pq} = \int_{\mathcal{D}} (x - \overline{x})^p (y - \overline{y})^q dxdy$$

The first two moments that are invariant with respect to rotations, translations and scales are :

$$\varphi_1^\star = \frac{\mu_{20} + \mu_{02}}{m_{00}^2} = \frac{m_{00}(m_{20} + m_{02}) - m_{01}^2 - m_{10}^2}{m_{00}^3}$$

$$\varphi_2^\star = \frac{(\mu_{20} - \mu_{02})^2 + 4\mu_{11}}{m_{00}^2}$$

An infinite number of moment invariants involving geometric moments of higher order can also be derived. In this paper, we only study functionals related to $\varphi_1^\star$ defined on contour $\mathcal{C}$ or the area enclosed by $\mathcal{C}$ if the contour is closed.

4.4 Contour moment invariants

In this section, we consider the different geometric moments of a two-dimensional contours :

$$m_{pq}(\mathcal{C}) = \int_{\mathcal{C}} x^p(s) y^q(s) ds$$

Thus m_{00} is the contour length $\mathcal{L}$ and $\overline{\mathcal{C}} = (\frac{m_{10}}{m_{00}}, \frac{m_{01}}{m_{00}})^T$ is the contour centroid. The first variation of the geometric moments is :

$$\delta m_{pq}(\mathcal{C}) = \frac{ds}{du}\begin{pmatrix} px^{p-1}y^q \\ qx^p y^{q-1}\end{pmatrix} - \frac{d}{du}(x^p y^q \mathbf{T})$$

The first variation of $\varphi_1^\star$ is then computed from the previous equation :

$$\delta\varphi_1^\star(\mathcal{C}) = \frac{1}{\mathcal{L}^2}\frac{ds}{du}(2(\mathcal{C}(u) - \overline{\mathcal{C}})\cdot\mathbf{N} - k(\mathcal{C}(u) - \overline{\mathcal{C}})^2 - k\mathcal{L}\phi_1)\mathbf{N}$$

For a straight line $k = 0$ and $(\mathcal{C}(u) - \overline{\mathcal{C}})\cdot\mathbf{N} = 0$ which implies that $\delta\varphi_1^\star = 0$. For a circle also, $(\mathcal{C}(u) - \overline{\mathcal{C}}) = r\mathbf{N}$ and $\phi_1 = r$ which also implies that $\delta\varphi_1^\star = 0$. Therefore, $\varphi_1^\star(\mathcal{C})$ can be considered as a valid smoothness measure that can be computed on closed and open contours.

4.5 Enclosed area moment invariants

By using Stoke's theorem, we can transform an integral over the area enclosed by a contour into an integral over a contour :

$$\int_C (P(x,y)dx + Q(x,y)dy) = \int\int_A (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$

Therefore, the moments m_{pq} on the enclosed area can be computed as :

$$m_{pq}(\mathcal{C}) = \frac{1}{2}\int_{\mathcal{C}}(-x^p\frac{y^{q+1}}{q+1}dx + \frac{x^{p+1}}{p+1}y^q dy)$$

The associated first variation takes the simple form :

$$\delta m_{pq}(C) = -\frac{ds}{du}x^p y^q \mathbf{N}$$

and the first variation of $\varphi_1^\star(\mathcal{C})$ is then :

$$\delta\varphi_1^\star(\mathcal{C}) = -\frac{ds}{du}\frac{1}{\mathcal{A}^2}\left((\mathcal{C}(u)-\overline{\mathcal{C}})^2 - 2\mathcal{A}^2\phi_1^\star\right)\mathbf{N}$$

We have verified that $\delta\varphi_1^\star(\mathcal{C}) = \mathbf{O}$ for all circles. However, this functional cannot be used to evaluate the smoothness of open contours (artificially closed by drawing a line between their two end points) since straight lines do not zero $\delta\varphi_1^\star(\mathcal{C})$.

5 Discussion

5.1 Comparison between the different smoothness measures

We have found four different classes of smoothness measures associated to the bending energy, the isoperimetric ratio and the moments computed on the manifold itself or its enclosed volume. We have first noticed that two of these functionals can only be computed on closed contours (see table 1). Also these two functionals are only valid for contours of $\mathbb{R}^2$ whereas the other two smoothness measures can be computed for contours of any codimension d ($d \geq 1$). Despite their restricted domain of application, the *isoperimetric ratio* and the *enclosed area moment invariants* have the advantage of being sensitive to the presence of contour self-intersections. Therefore, when smoothing a contour with the *pde* $\frac{\partial\mathcal{C}(u,t)}{\partial t} = \delta E(\mathcal{C}(u,t))$, these two functional should be more likely to avoid the creation of self-intersections than the other two functionals.

	Closed Contours	Opened Contours
Normalized Bending Energy	$\star$	$\star$
Isoperimetric Ratio	$\star$	
Contour Moment Invariants	$\star$	$\star$
Area Moment Invariants	$\star$	

Table 1: Properties of different contour smoothness measures.

For contours, the four smoothness measures are global in the sense that they cannot be broken-up into the smoothness measures of its different parts. However, we have seen that for surfaces of higher dimensions, it is always possible to find a local geometric smoothness measure based on surface integrals of mean curvature. We believe that local functionals correspond more closely to the intuitive notion of smoothness than global functionals. For instance when smoothing a surface, it is natural to consider that the contour evolution at a point does not depend on the contour shape far away from this point. When using algebraic measures, each contour point is smoothed based on the position of the contour centroid which completely depends on the global contour shape.

Finally, we have showed that it is possible to define scale-dependent geometric smoothness functionals. These functionals have the advantage of not requiring highly differentiable contours and surfaces.

5.2 Intrinsic Smoothing Filters

In this paper, we have systematically derived the first variation of all functionals since it can be used to smooth contours or surfaces through the associated *pde* $\frac{\partial\mathcal{C}(u,t)}{\partial t} = \delta E(\mathcal{C}(u,t))$. Also the first variation is used to find the minimal curves or surfaces of that functional. In fact, for many applications in computer vision (image enhancement, image segmentation,...), only the first variation of the functional is of practical use.

Therefore, it is also important to study curve or surface *pde*'s that do not correspond to the minimization of a global functional but that have also some remarkable invariance properties. From a general differential equation $\frac{\partial\mathcal{C}(u,t)}{\partial t} = \beta(\mathcal{C})\mathbf{N}$, the curve evolution is invariant to rotations and translations if $\beta(\mathcal{C})$ is itself invariant to these transformations. The invariance to changes of scale is verified if $\beta(\sigma\mathcal{C}) = \sigma\beta(\mathcal{C})$. Also, it is possible to verify if lines and circles are optimal curves if $\beta(\mathcal{C}) = 0$. Therefore, we can extend the notion of smoothness measures to the notion of *intrinsic smoothing filters* by requiring that these filters follow the criteria defined in section 2.2.

Obviously there are many possibilities for defining intrinsic smoothing filters. For instance, Delingette [6] has proposed to use the second derivative of curvature as the governing equation for smoothing contours :

$$\frac{\partial\mathcal{C}(u,t)}{\partial t} = \frac{d^2k}{ds^2}\mathbf{N}$$

This local filter has been also extended to include a scale factor :

$$\frac{\partial\mathcal{C}(u,t)}{\partial t} = \frac{1}{dl^2}\left(\frac{1}{2dl}\int_{s-dl}^{s+dl} k(u)du - kds\right)\mathbf{N}$$

Also in [7, 14] Gage and Sapiro have introduced two global intrinsic smoothing filters that have the property to keep the enclosed area constant during their evolution :

$$\frac{\partial\mathcal{C}(u,t)}{\partial t} = \left(k - \frac{2\pi}{\mathcal{L}}\right)\mathbf{N}$$
$$\frac{\partial\mathcal{C}(u,t)}{\partial t} = \left(k - \frac{\pi(\mathcal{C}\cdot\mathbf{N})}{\mathcal{A}}\right)\mathbf{N}$$

These non-local flows cannot be applied to open contours.

5.3 Implementation issues

In this section, we shortly discuss the implementation issues of the *pde*'s associated to the smoothness measures that have been presented in ths paper. In particular, each contour or surface can be represented using an implicit or explicit representation.

Implicit framework cannot easily represent open contours (or surfaces) since the zero level-sets are closed contours. Furthermore, it is well-suited for representing hypersurfaces but not manifolds of codimension greater than 1 (even if recent work has been done in this direction). Finally, despite numerous optimization schemes, it requires more computations than Lagrangian-based approaches since it requires the update of a $n+1$-manifold for computing the evolution of a n-manifold.

However, Eulerian methods have also several advantages. First of all, they do not require any explicit parameterizations of shapes. The second advantage of Eulerian methods is to naturally handle topology changes whereas the detection and processing of self-intersections on explicit surfaces has only been reported in the case of deformable contours [6] at a cost of greater implementation complexity.

For the implementation of the *pde*'s associated to these smoothness measures, two additional elements should be taken into account. First of all, many of these functionals are global and therefore require that some global contour parameters are evaluated. Extracting these parameters (such as contour length or enclosed volume) on implicit representations often requires an explicit extraction of these contours, through an isocontour extraction method, but also requires in some cases to find a suitable parameterization.

The second specificity is that the *pde* $\frac{\partial \mathcal{C}(u,t)}{\partial t} = \delta E(\mathcal{C}(u,t))$ corresponds to the gradient descent of a functional and therefore it can be implemented using the finite element method rather than the finite differences method. The advantage of the finite element method for instance using the Galerkin approximation is that it only requires to discretise the functional rather than its first variation.

6 Conclusion

In this paper, we have introduced four classes of smoothness functionals either based on geometric or algebraic invariants. In general, geometric smoothness measures lead to local functionals (except for planar contours). Using a local functional gives more intuitive results when smoothing shapes with the associated flow.

To further compare these measures it is necessary to implement them for planar contours and tridimensional surfaces. As a representation, we propose to use non-parametric surfaces meshes (polygons and triangulations). If algebraic functionals can be easily discretized, it is not the case of geometric functionals. To do so, we plan to expand the approach proposed by Pinkall *et al.* [12] to curvature integrals.

References

[1] M. Bertero, Poggio T., and V. Torre. Ill-posed problems in early vision. Technical Report A.I. Memo 924, M.I.T., A.I. Laboratory, May 1987.

[2] A. Blake and A. Zisserman. *Visual Reconstruction*. MIT Press, 1987.

[3] M. Brady and B.K.P. Horn. Rotationally symmetric operators for surface interpolation. *Computer Vision, Graphics, and Image Processing*, pages 70–94, 1983.

[4] A.N Bruckstein, A.M.and Netravali. On mimimal trajectories. *Computer Vision, Graphics, and Image Processing*, 1990.

[5] B.Y. Chen. On a variational problem of hypersurfaces. *Journal of London Mathematical Society,*, 2(6):321–325, 1973.

[6] H. Delingette and J. Montagnat. New algorithms for controlling active contours shape and topology. In D. Vernon, editor, *European Conference on Computer Vision (ECCV'2000)*, number 1843 in LNCS, pages 381–395, Dublin, Ireland, June 2000. Springer.

[7] M. Gage. On an area-preserving evolution equation for plane curves. *Contemporary Math Series*, 51:51–62, 1985.

[8] M. A. Grayson. The heat equation shrinks embedded plane curves to round points. *Journal of Diffential Geometry*, 28:285–314, 1987.

[9] B.K.P. Horn. The curve of least energy. *ACM Transactions on Mathematical Software*, 1983.

[10] B. Kimia, A. Tannenbaum, and S. Zucker. On the evolution of curves via a function of curvature i. the classical case. *Journal of Mathematical Analysis and Applications*, 163:438–458, 1992.

[11] Alexander G. Mamistvalov. n-dimensional moment invariants and conceptual mathematical theory of recognition n-dimensional solids. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 20(8), August 1998.

[12] U. Pinkall and K. Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental Mathematics*, 2(1):15–36, 1993.

[13] A. Polden. *Curves and Surfaces of Least Total Curvature And Fourth-Order Flows*. PhD thesis, Mathematisches Institut Universität Tübingen, Tübingen, Germany, September 1996.

[14] G. Sapiro and A. Tannenbaum. Area and length preserving geometric invariant scale-spaces. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 17(1):67–71, January 1995.

[15] BU-Qing Su and Liu Dingy-Yuan. *Computational Geometry- Curve and Surface Modeling*. Academic Press, Inc., 1989.