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(a) (b) (c)

Figure 6 a) Initial contour **b)** Contour smoothed with the Gaussian filter **c)** Contour smoothed with the intrinsic polynomial filter of order 1.

3. Conclusion

In this paper, we have extended the framework of regularization by transforming an ill-posed problem into a well-posed differential equation $\sigma(v) + \lambda \cdot \delta D(v) = 0$ where σ is called a differential stabilizer. This equation corresponds to the Euler-Lagrange equation $\delta E(v) = \delta S(v) + \lambda \cdot \delta D(v) = 0$ obtained from the regularization theory. Considering the one dimension case, we have shown that a desirable property of the stabilizer is its circle-invariance and that no linear stabilizer had this property. The intrinsic polynomial stabilizers of order p leave circles invariant as well as curves of intrinsic equation $k = a_{2p+1}s^{2p+1} + a_{2p}s^{2p} + ... + a_0$ where s is the arc length.

We are now extending this work to three-dimensional curves as well as three-dimensional surfaces.For 3D curves, the circle-invariance becomes the circular-helix-invariance and for surfaces it becomes sphere-invariance.The formulation of a stabilizer is made difficult in the case of surfaces because unlike the three-dimensional curves that can be described with their curvature and torsion, surfaces are intrinsically defined by their two fundamental forms with three compatibility equations.

IPS tries to minimizes the variation curvature.

(a) (b) (c)

Figure 5 : **a)** Initial curve. **b)** Curve fit using thin plate stabilizer associated with its curvature profile. **c)** Curve fit using the intrinsic polynomial stabilizer associated with its curvature profile.

2.3.3. Smoothing Filter

Given a differential stabilizer σ, we can associate a smoothing filter Σ defined by: $\Sigma(v) = v + \alpha\sigma(v)$. where α is a constant. For the thin plate stabilizer the associated operator is a linear filter which kernel is the L_4 function of Schoenberg.[12] For the intrinsic polynomial stabilizer, the resulting filter is non linear and is therefore difficult to characterize. We give an example with the contour of the United States filtered with respectively a gaussian filter and intrinsic polynomial filter (Figure 6.a 6.b 6.c). We have applied iteratively the intrinsic polynomial filter about twenty times so that the amount of smoothing of the two filters are similar. But both were used with the same scale coefficient

The IP filter compares favorably with the Gaussian filter because the distortion is less important while releasing a comparably smooth contour.

2.3. Results

2.3.1. Clothoid Spline

In this section, we present some results concerning the problem of fitting a curve through a set of points (Pi) and with a number of end conditions. Formally, it is equivalent to solving:

$$\sigma(v) + \sum_{i=0}^{n} (v(u_i) - P_i) + (v'(u_0) - P'_0) + (v'(u_n) - P'_n) + ... = 0$$

If the number of end-conditions is correctly chosen, the solution is unique and is made of piecewise integral trajectories (curves for which $\sigma(v)=0$). For the stabilizer of first order, the ends conditions consist of two tangents and the second order stabilizer, they consist of two tangents and two curvatures. We find the trajectory solution by initializing the curve as a set of straight lines and by applying the stabilizer until the curve reaches its equilibrium (Figure 4.a). Figure 4.b is an example of trajectory solution:

(a) **(b)**

`Figure 4` : **a)** Deformation of a string from line to clothoid. **b)** A piecewise clothoid with its curvature profile.

The intrinsic nature of these curves and especially their ability to have continuous curvature profile make them suitable for generating trajectories for mobile robots[5].

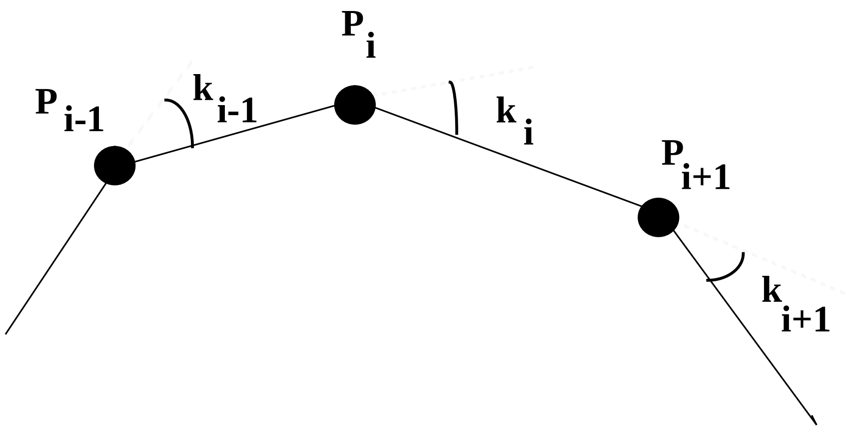
2.3.2. Curve Fitting

We now present results concerning curve $\{P_i\}$fitting which is equivalent to solving:

$$\sigma(v) + \lambda\left(\sum_{i=0}^{n} (v(u_i) - P_i)\right) = 0$$

The curve we have chosen is a step edge combined with two circles of opposite curvature (Figure 5.a) and we have applied both thin plate and intrinsic polynomial stabilizer of order 1. We have used a small value of λ so that the smoothing term is predominant and we have normalized the two stabilizers so that the results could be compared. The results (Figure 5.b 5.c) show that a smoother edge is obtained with the thin plate stabilizer because it tries to minimizes the curvature while the

2.2. Discretization



`Figure 3` : Definition of the discretized curvature k_i

We now present the formulation of the intrinsic polynomial stabilizers for a discrete curve. Let a discrete curve be defined by a set of points $\{P_i = (x_i, y_i)\}$ $(i=0,n)$, then we can define the set $\{\kappa_i\}$ $(i=0,n)$ where κ_i is the angle between the two segments $[P_i, P_{i+1}]$, $[P_i, P_{i-1}]$.

Because i is the parameter of the curve, κ_i corresponds to $\frac{d\phi}{du}$, the vector $P_{i-1}P_{i+1}$ corresponds to $2,\frac{ds}{du}\cdot\vec{t}$. and $\vec{z}\times P_{i-1}P_{i+1} = P_{i-1}P_{i+1}^{\perp}$ corresponds to $2,\frac{ds}{du}\cdot\vec{n}$

To derive the discrete formulation of the IPS we use the relation: $\frac{d^2s}{du^2}\cdot\vec{t} = \frac{d^2v}{du^2} - \left(\frac{ds}{du}\cdot\frac{d\phi}{du}\right)\vec{n}$. The expressions of the stabilizers are:

:

$$\text{Zero order : } \sigma(P_i) = \frac{P_{i-1}P_{i+1}}{2} - P_i - \left(\tan\left(\frac{k_i}{2}\right)\cdot\frac{P_{i-1}P_{i+1}^{\perp}}{2}\right)$$

$$\text{First order : } \sigma(P_i) = \frac{P_{i-1}P_{i+1}}{2} - P_i - \left(\tan\left(\frac{\sum_{u=-u_0}^{u_0} k_{i+u}}{2\cdot(2u_0+1)}\right)\cdot\frac{P_{i-1}P_{i+1}^{\perp}}{2}\right)$$

Second order :

$$\sigma(P_i) = \frac{P_{i-1}P_{i+1}}{2} - P_i - \left(\tan\left(\frac{k_{i-1}+k_i+k_{i+1}}{6} - \frac{\sum_{u=-u_0}^{u_0}(k_{i+u-1} - 2k_{i+u} + k_{i+u+1})}{6\cdot(2u_0+1)}\right)\cdot\frac{P_{i-1}P_{i+1}^{\perp}}{2}\right)$$

$$\frac{d\phi}{du}(u) = \frac{\left(\int_{-u_0}^{u_0} \frac{d\phi}{dt}(u+t)dt\right)}{2 \cdot u_0}$$

which is equivalent to

$$k(u) = \frac{\left(\int_{-u_0}^{u_0} k(u+t)dt\right)}{2 \cdot u_0}$$

because $k = \frac{d\varphi}{ds} = \frac{du}{ds} \cdot \frac{d\varphi}{du} = a \cdot \frac{d\varphi}{du}$. To be a solution of this functional equation *k(u)* has to be a linear function of *u* and therefore of the arc length *s*. The IT are curves in which curvature is a linear function of arc-length. These curves are called *Cornu's Spirals* or *Clothoids* (Figure 2).

- By applying the same reasoning, it can be shown that the IT of the stabilizer of order n are curves in which curvature is a polynomial function of degree 2n+1 with respect to arc length (Figure 2).

Thus, the criterion of smoothness used here is related to the derivatives of the curvature while for the thin plate the criterion of smoothness was related to the square value of the curvature.

a) b)

Figure 2: **a)** Clothoid curve of equation $k = s$. **b)** Curve of equation $k=s^3- s$

It is easy to verify that these stabilizers are invariant with rotation and translation. Their integral trajectories are all expressed in terms of intrinsic equation *k=f(s)* and they are circle-invariant because circles correspond to curves with constant curvature.

The intrinsic polynomial stabilizers are scale sensitive because they depend on the constant u_0: $\sigma(v)=\sigma(v,u_0)$. u_0 is a scale parameter which has the same meaning that the Gaussian's standard deviation: it defines the scale at which the smoothing is performed. If the curve has a finite length then the ratio $2,(u_0/(u_{max}-u_{min}))$ indicates the relative scale of the stabilizer with respect to the length of the curve. This scale dependance allows scale-space filtering[17] that is to build a qualitative description of the signal over scale.

$$\text{Zero order : } \sigma(v) = \frac{d^2 s}{du^2}\vec{t}$$

$$\text{First order : } \sigma(v) = \frac{d^2 s}{du^2}\vec{t} + \frac{ds}{du} \cdot \left(\frac{d\phi}{du}(u) - \frac{\left(\int_{-u_0}^{u_0} \frac{d\phi}{dt}(u+t)dt \right)}{2 \cdot u_0} \right)\vec{n}$$

$$\text{Second order : } \sigma v = \frac{d^2 s}{du^2}\vec{t} + \frac{-1}{3} \cdot \frac{ds}{du} \cdot \left(\frac{d^3 \phi}{du^3} - \frac{\int_{-u_0}^{u_0} \frac{d^3 \phi}{du^3}(u+t)dt}{2 \cdot u_0} \right)\vec{n}$$

$$\text{Order n : } \sigma(v) = \frac{d^2 s}{du^2}\vec{t} + \frac{(-1)^{n+1}}{\binom{2n+1}{n}} \cdot \frac{ds}{du} \cdot \left(\frac{d^{(2n+1)} \phi}{du^{2n+1}} - \frac{\left(\int_{-u_0}^{u_0} \frac{d^{(2n+1)} \phi}{du^{2n+1}}(u+t)dt \right)}{2 \cdot u_0} \right)\vec{n}$$

where s is the arc length $\frac{ds}{du} = \sqrt{\frac{dx}{du}^2 + \frac{dy}{du}^2}$; $\vec{t}$ and $\vec{n}$ are the tangent and the normal vector; u_0 is a constant and where $\phi(u)$ is the polar angle of the tangent.

It should be noticed that if *f'(x)* is the derivative of the function *f* then the mean value of *f'(u)* for $u \in [x + x_0, x - x_0]$ is:

$$\frac{f(x + x_0) - f(x - x_0)}{2 \cdot x_0} = \int_{-x_0}^{x_0} f'(x+t)dt$$

such that the stabilizer of first order for example can also be written on the form:

$$\sigma(v) = \frac{d^2 s}{du^2}\vec{t} + \frac{ds}{du} \cdot \left(\frac{d\phi}{du}(u) - \frac{(\phi(u + u_0) - \phi(u - u_0))}{2 \cdot u_0} \right)\vec{n}$$

The integral trajectories are:

- For *n=0*, the IT verify $u = as+b$, which means that the parameter u is linearly proportional to the arc-length s. Therefore all plane curves that are continuous, $C^0$are integral trajectories because for all C^0 curves, a normal parametrization can be defined. If a curve is discretized in a set of points, then $u = as+b$ means that all nodes are equidistant.

- For *n=1*, the IT verify $u = as+b$ and

1.3. The large deflection thin plate stabilizer

The thin plate energy makes the assumption that a curve is smooth if its derivatives of order two are bounded and small. This explains that its IT are parametrization dependant, because the derivatives are considered with respect to the current parametrization which may not be a normal parametrization. To overcome this, Blake[2,3] proposed to use the following energy:

$$E(v) = \int k^2 ds = \int \frac{(x_u y_{uu} - y_u x_{uu})^2}{(x_u^2 + y_u^2)^{5/2}} du$$

This energy is the total elastic energy stored in a thin beam and is also called the large deflection thin plate energy. Indeed, the thin plate energy is only a linearization of this energy for small deflections. The differential stabilizer derived from *E(v)* is obtained from the Euler-Lagrange equation and is:

$$\delta E(v) = \sigma(v) = \frac{d}{du}\left[k^2 \vec{t} + 2,\left(\frac{dk}{ds}\vec{n}\right)\right] = \left(\frac{ds}{du}\right) \cdot \left(k^3 + 2\frac{d^2k}{ds^2}\right)\vec{n}$$

where s is the arc-length of the curve *v(u)* and where $\vec{t}$ and $\vec{n}$ are respectively the tangent and the normal of the curve. The integral trajectories of these DS are called mechanical splines or curves of least energy[9] verify the intrinsic equation:

$$k^3 + 2\frac{d^2k}{ds^2} = 0$$

which can be set in the form[9]:

$$k^2 = \mu \cdot \cos(\varphi - \varphi_0)$$

where μ and ϕ_0 are two constants and where ϕ is the polar angle of the tangent $k = \frac{d\varphi}{ds}$. Therefore, the large deflection thin plate is independent of the parametrization but is not circle invariant because the circles are not integral trajectories. This result shows that this DS will exhibit shrinking effect too, if applied for curve fitting.

2. The Intrinsic Polynomial stabilizer

2.1. Definition

We propose a set of differential stabilizers, the Intrinsic Polynomial Stabilizer (IPS), that verify the following properties :

- They are invariant with respect to translation and rotation.
- Their IT are invariant with respect to parametrization.
- They are circle invariant.
- They are scale sensitive.

The expression of these stabilizers are :

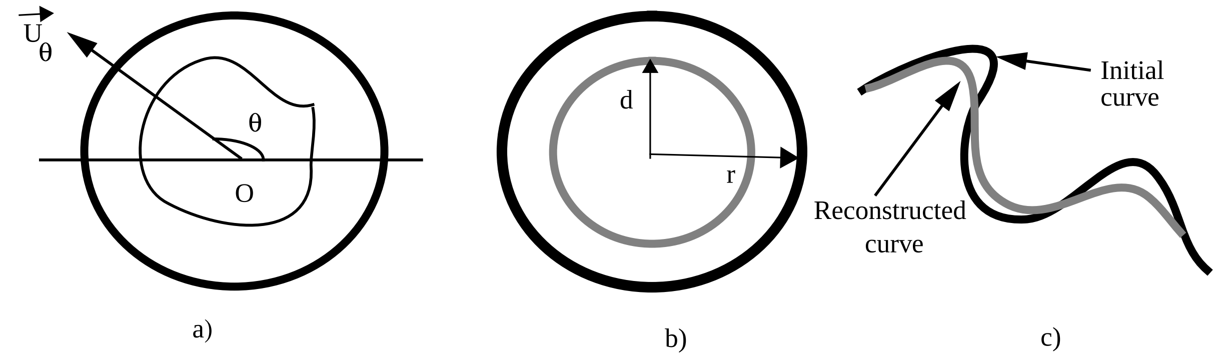


Figure 1: **a)** Definition of u_q **b)**Solution of the fitting problem is a circle of radius d **c)** Shrinking effect for tl solution of the fitting problem.

Because for a circle with normal parametrization $v_{uuuu} = k^3 \cdot \vec{n} = \frac{1}{r^3} \cdot \vec{n}$; the solution of this equation is a circle of radius *d* which verifies the following equation:.

$$\beta + \lambda \cdot d^3(d - r) = 0$$

Therefore the solution is not the initial circle but a circle of smaller radius if β and λ are positive.This counter-intuitive result is due to the fact that a circle is not an integral trajectory of the thin plate and therefore the solution is a trade-off between smoothness and accuracy. If we would have replaced the circle with a curve (that is not a cubic spline) the result would have looked like Figure 1.c, where the solution of the fitting problem would have systematically under-estimated the curvature of the initial curve. This shrinking effect of the stabilizer can be explained by the fact that v_{uuuu} is directed toward the center of curvature and therefore pushes the points in that direction. The more smoothing there is, the more distortion the solution will exhibit. In practice, if the curve or surface to be reconstructed are poorly curved, then the distortion will be 'acceptable'.

Because all planar curves are locally equivalent to a circle-- the osculatory circle--a necessary condition for a stabilizer not to exhibit a shrinking effect is that all circles are integral trajectories of the stabilizer. If we call Γ the set of plane circles, then we propose the following definition.:

$$(\sigma \text{ is Circle Invariant }) \Leftrightarrow (\forall c \in \Gamma, \sigma(c)=0) \Leftrightarrow (\sigma(\Gamma)=0)$$

If σ is a linear operator that is rotation invariant then it is easy to show that there are two linear operators *H* and G such that

$$\sigma(v) = \sigma\left(\begin{bmatrix} x(u) \\ y(u) \end{bmatrix}\right) = \begin{bmatrix} H(x) + G(y) \\ H(y) - G(x) \end{bmatrix}$$

If σ is circle-invariant then *H* and G must verify *H(cos(ku))* = *H(sin(ku))* =0 $\forall k \in \Re$. Because the sines and cosines functions are the eigenfunctions for a linear operator, *H*() is therefore the null operator[7]. So the only linear and rotation invariant stabilizer is the null operator. which explains that linear filtering such as Gaussian smoothing, shrinks or distorts the data. Several methods have been proposed to overcome these undesirable effects of linear smoothing: Lowe[10] has proposed an algorithm to compensate for the shrinkage of Gaussian smoothing while Zhou[18] proposed to fit cubic splines locally. Horn and Weldon[8] used the extended circular image representation of closed curves in order to perform linear smoothing.

Thus, another property that is desirable for a one-dimensional differential stabilizer is its circle invariance.

$$\sigma(v) = -\frac{d}{dx}(\alpha \cdot v_x) + \frac{d^2}{dx^2}(\beta \cdot v_{xx})$$

$$\sigma(v) = -\left[\frac{\partial}{\partial \mathbf{x}}(\alpha \cdot \mathbf{v_x}) + \frac{\partial}{\partial \mathbf{y}}(\alpha \cdot \mathbf{v_y})\right] + \left[\frac{\partial^2}{\partial x^2}(\beta \cdot \mathbf{v}_{xx}) + 2 \cdot \frac{\partial^2}{\partial xy}(\beta \cdot v_{xy}) + \frac{\partial^2}{\partial y^2}(\beta \cdot \mathbf{v}_{yy})\right]$$

We can associate with a Differential Stabilizer, a set of *Integral Trajector*ies (IT), such that $(v \in C_\sigma) \Leftrightarrow (\sigma(v)=0)$. These trajectories correspond to the curves or surfaces of maximum smoothness. For curve or surface fitting, the solutions of the differential equations are pieces of Integral Trajectories. The IT associated with the Tikhonov's stabilizer are obtained by solving the equation $\sigma(v)=0$ that is linear if the $w_m(v)$ are continuous. If $w_m(x)=\delta_p(x)$where δ is the Kronecker symbol, then the IT are polynomial of degree $2p-1$. For the membrane and thin plate spline, the IT is a cubic spline if $\alpha=0$ and $\beta=cste$. It, is a line if $\beta=0$ and $\alpha=cste$, and it is a spline under tension if α and β are piecewise constant.

Several properties are desirable for a differential stabilizer in order to render feasible and computable solutions. These properties are:

- Invariance with respect to translation and rotation. This property is essential because it allows the modelling of a surface or curve independently of the frame from which the data was obtained. Few existing methods[4,16,3] guaranties this invariance because they usually leads to non-linear and non-convex minimization.

- Stability. A special case of stability is convexity where uniqueness of the solution and convergence are guaranteed at the same time.

- Invariance of the IT with respect to parametrization.. In another words, the IT should be described in terms of intrinsic parameters of curve or surface. For example, the IT of the thin plate, a cubic spline s(u), cannot be described in terms of curvature as a function of arc length (intrinsic equations).This is especially useful in computer vision where most of the problems are formulated in terms of real geometric entities such as normal, tangent, curvature, Gaussian curvature.

In the next sections, we will restrain the problem to one-dimensional stabilizers used for spline fitting. We will also show that another property is desirable for a stabilizer in order to perform correctly and that none of previous stabilizers have this property.

1.2. The shrinking effect

We now want to exhibit a bias inherent to the thin plate stabilizer and more generally to every linear differential stabilizer. Let C be a circle of radius r and center O in the plane. We set the following problem: find the curve $v(u) = \left[x(u)\ y(u)\right]^t$ such that its distance to the circle C is minimum with the constraint of the thin plate stabilizer. It is equivalent of solving:

$$\beta \cdot v_{uuuu} + \lambda[(\sqrt{x^2 + y^2} - r)]\vec{u}_\Theta = 0$$

with $\vec{u}_\Theta$ being the unit vector of polar angle $\Theta = \operatorname{atan}\left(\frac{y}{x}\right)$ (Figure 1).

where the $w_m(x)$ are nonnegative functions. This class of stabilizer were extended to multivariate functions[11, 6] and later were generalized by Terzopoulos[13,14] in order to handle discontinuities of different order. In computer vision, stabilizers are mostly of zero and first order in order to solve spline or surface fitting and they correspond then respectively to the deflection energy of a membrane and the bending energy of a thin plate. If we denote v_x to be the derivative of v with respect to x, (v is either a scalar or a vector), their expressions are:

Spline case

$$S(v) = \int_{\Re} (\alpha(x) \cdot v_x^2(x))dx + \int_{\Re} (\beta(x) \cdot v_{xx}^2(x))dx$$

Surface case

$$S(v) = \int\int_{\Re^2} (\alpha(x, y) \cdot [v_x^2(x, y) + v_y^2(x, y)])dxdy + \int\int_{\Re^2} (\beta(x, y) \cdot [v_{xx}^2(x, y) + 2v_{xy}^2(x, y) + v_{yy}^2(x, y)])dxdy$$

A *necessary* condition for v to minimize $E(v)$ is that v verifies: $\delta E(v) = \delta S(v) + \lambda \cdot \delta D(v) = 0$ where $\delta E(v)$ corresponds to the gradient (or force in mechanics) of the energy $E(v)$. In general, $E(v)$ is formulated as a variational principle and therefore $\delta E(v)$ is obtained via the Euler-Lagrange differential equations. Most of the time, the solution of the inverse problem is numerically approximated by solving the differential equation $\delta E(v)=0$ using methods such as Jacobi or Gauss-Seidel relaxation or non-convex methods such as GNC[2] or simulated annealing if there are several local minima.

It is therefore natural to extend the framework of the regularization theory by replacing the necessary condition $\delta S(v) + \lambda \cdot \delta D(v) = 0$ by the more general condition $\sigma(v) + \lambda \cdot \delta D(v) = 0$. The following definitions are set:

- *Stabilization* is the transformation of the ill-posed problem $Av = f$ into the well-posed differential equation $\sigma(v) + \lambda \cdot \delta D(v) = 0$
- $\sigma(v)$ is an operator of F^{np} into F^{np} (F^{np} is the space of function of $\Re^n$ into $\Re^p$). $\sigma()$ is called a *Differential Stabilizer* (DS).
- $\delta D(v)$ is the differential of $D(v) = \|Av - f\|_2^2$.

Regularization appears now as a special case of stabilization because every solution of a regularized problem verifies: $\sigma(v) + \lambda \cdot \delta D(v) = 0$ with $\sigma(v) = \delta S(v)$. In theory, solving $\delta E(v) = \delta S(v) + \lambda \cdot \delta D(v) = 0$ is not equivalent to minimizing $E(v)$ (it is only a *necessary* condition). In practice, it is equivalent because the solution for which $E(v)$ is maximum is unstable and therefore can never be obtained numerically. Stabilization is a generalization of regularization because for every $\sigma \in F^{np}$, it is not always possible to have an operator $S(v)$ such that $\sigma(v) = \delta S(v)$, meaning that the solution of the differential equation does not minimize an energy function $E(v)$. A justification of this approach is to draw a parallel with the equations of mechanics. The laws of mechanics are based on the minimization of the Lagrangian $L = T - U$, T being the kinetic energy and U the potential energy of the system. The Euler-Lagrange equations corresponding to the Lagrangian are the equations of the mechanics $m\vec{\Gamma} - \vec{F} = 0$. But some forces in mechanics are not derived from a potential such as viscous or friction forces, so that it is not always possible to set the problem in term of minimization of energy but only in terms of forces.

We can derive the differential stabilizer associated with the Tikhonov's stabilizer.

$$\sigma(v) = \sum_{m=0}^{p} \left((-1)^m \cdot \frac{d^m}{dx^m} \left(w_m(x) \left(\frac{d^m}{dx^m} v(x) \right) \right) \right)$$

For the membrane and the thin plate in one dimension and in two dimension, we have respectively:

Energy functions for regularization algorithms[1]

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Abstract

Energy functions used for regularization algorithms measure how smooth a curve or surface is. In order, to render acceptable solutions, these energies have to verify certain properties such as invariance with Euclidean transformations or invariance with parametrization. In this paper, we first extend the notion of smoothness energy to the notion of differential stabilizer. If we make an analogy with mechanics, smoothness energy corresponds to potential energy while differential stabilizers correspond to forces.We then show that to avoid the systematic underestimation of curvature for planar curve fitting, it is necessary that circles be the curves of maximum smoothness. We finally propose a set of stabilizers that meet this condition as well as invariance with rotation and parametrization.

1. Differential Stabilizer

1.1. Definition

Regularization techniques are widely used for inverse problem solving in computer vision such as surface reconstruction, edge detection or optical flow estimation. Formally, regularization transforms an ill-posed inverse problem into a well-posed minimization problem by constraining the solution to belong to a set of smooth functions.

More precisely, let v be a function of $\Re^n$ into $\Re^p$, $v \in F^{np}$, A be an operator of F^{np} into F^{nq} and f be a function of $\Re^n$ into $\Re^q$, $f \in F^{nq}$ Then the inverse problem $Av = f$ is transformed into the minimization of *E(v)*[1]*:*

$$E(v) = S(v) + \lambda \cdot D(v) = \|Pv\|_1^2 + \lambda\|Av - f\|_2^2$$

where *P* is an operator, λ is a real number, $\|.\|_1$ and $\|.\|_2$ are two seminorms. The term S(v) is the smoothness energy or stabilizer and measures how smooth the solution is: the smaller *S(v)* the smoother the shape. *D(v)* measures how well the solution matches the data. The coefficient λ quantifies the trade-off between smoothness and accuracy.

Because smoothness is a characteristic of a shape that cannot be intrinsically quantified, several types of stabilizers have been proposed. The most widely used are based on the Tikhonov's stabilizers[15] which are closely related to the spline theory. The Tikhonov's stabilizers of order p are defined by:

$$S(v) = \sum_{m=0}^{p}\left(\int_{-\infty}^{\infty} w_m(x)\left(\frac{d^m}{dx^m}v(x)\right)^2 dx\right)$$

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