Résolution de quelques problèmes inverses pour Δ en 2 et 3D par des techniques d'approximation de fonctions ; applications à l'EEG.

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joint work with

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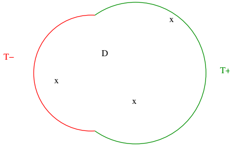
Overview

- EEG inverse problem, cortical mapping, source localization, spherical model
- Cauchy problems for Δ
  - Harmonic and analytic functions, Hardy spaces, approximation issues in the disk [or 2D (conformally equiv. to) circular (or annular) domains]
  - Also in 3D balls , using spherical harmonics (or spherical shells)
  - Few numerical examples
- Source recovery in balls (or ellipsoids)
  - Rational approximation on disks (2D slices)
  - Numerical examples
  - Conclusion

Overview



$\boxed{\nabla.(\sigma \nabla u) = \delta}$ in $D \subset \mathbb{R}^m$, $m = 2, 3$,

given conductivity $\sigma > 0$ (isotropic)

available data: u, $\partial_n u$ on $T_+ \subset \partial D$

σ cst or pcw $\longrightarrow$ $\boxed{\Delta u = \delta / \sigma}$

(i) $\delta = 0$

find (data on) T_-

Cauchy pb, EEG cortical map., Robin, bdy geometry

best harmonic / analytic approx.

BEP in Hardy spaces, 2 and 3D

(ii) $\delta \neq 0$

find (supp) δ

singularities, EEG source (MEG)

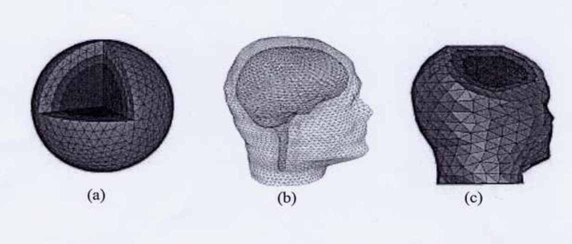
best rational / meromorphic approx.

2D quadrature domains

also, smooth variable σ for $m = 2$, $\delta = 0$ (Beltrami, tokamak)

EEG inverse problem

Models for the head $D = \cup_{i=1}^{3} D_i$, brain D_1, skull D_2, skin D_3



recover cerebral current (source term) δ inside the brain $D_1 \subset D \subset \mathbb{R}^3$ from electroencephalographic (EEG) measurements on part T_+ of scalp ∂D of a solution to

$$-\mathrm{div}\ (\sigma\ \mathrm{grad}\ u) = -\nabla \cdot (\sigma \nabla u) = \delta$$

(Maxwell, electrostatic [Feynman, F&al, H&al])

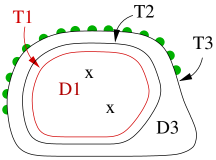
(IP): potential diff., current flux $(u, \partial_n u)_{|T_+} \rightarrow \delta$ supported in D_1

EEG - IP: which head!?

given piecewise constant conductivity: $\sigma_{|D_i} = \sigma_i$, $1 \leq i \leq 3$ usually;
D union of homogeneous layers D_i (scalp, skull, brain,...)

spherical model (a): ∂D_i spheres T_i

conductivity defaults = pointwise dipolar sources $\{C_k\}$ in brain
$D_1 = \mathbb{B}$, unit ball



$$\begin{cases} \Delta u = 0 \text{ in } D_3 \text{ and } D_2 \\ \Delta u = \delta = \sum_{k=1}^{L} m_k \cdot \nabla \delta_{C_k} \text{ in } D_1 \\ u\,,\ \sigma \partial_n u \text{ continuous on } T_i \\ u = \nu \text{ on } T_+ \subset T_3 \\ \partial_n u = \phi \text{ on } T_+ \end{cases}$$

(IP): from $(u, \partial_n u)_{|T_+}$, find moments $\{m_k\}$, sources $\{C_k\}$ in $\mathbb{B}$
(pointwise values of ν, while $\phi = 0$ on T)

EEG - IP : well posedness

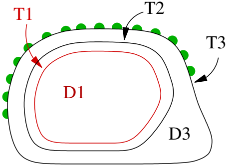
- existence of solution to direct problem $(\delta, \phi) \to u$ (unique up to additive cst) if compatibility condition:

$$\int_{\partial D} \phi \, ds = 0$$

- $\phi \in L^2(\partial D) \Rightarrow u \in$ Sobolev $W^{1,2}(D)$, Hölder $\bar{D} \setminus \{C_k\}$ [...]
- identifiability, uniqueness of solution to (IP) [HD-EB]
- stability/continuity: partial results, need for SC [Al,Ve]

EEG - IP

(i) (Cauchy-IP): get Cauchy data - current flux, potential diff. - from $T_+ \subset T_3$ to $T_1 = \mathbb{S}$: $(u, \partial_n u)_{|\mathbb{S}}$ (cortical mapping)



$$
\text{(C-IP)} \begin{cases} \Delta u = 0 & \text{in } D_3 \text{ and } D_2 \\ u = \nu & \text{on } T_+ \\ \partial_n u = \phi & \text{on } T_+ \\ u\,,\ \sigma \partial_n u & \text{continuous on } T_i \end{cases}
$$

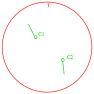
$$\nu, \phi \text{ on } T_+ \subset T_3 \longrightarrow u, \partial_n u \text{ on } T_- = T_2 \text{ ?}$$

$$u, \partial_n u \text{ on } T_+ = T_2 \longrightarrow u, \partial_n u \text{ on } T_- = T_1 \text{ ?}$$

using best analytic approximation on T_+ then ....

EEG - IP

(ii) (Sources-IP): from propagated data $u, \partial_n u$ on T_1, recover (dipolar pointwise) sources C_k in (brain) D_1:



$$\text{(S-IP)} \quad \Delta u = \delta = \sum_{k=1}^{L} m_k \cdot \nabla \, \delta_{C_k} \text{ in } D_1 = \mathbb{B}$$

$$u \,, \; \partial_n u \text{ on } T_+ = T_1 \to L, m_k \in \mathbb{R}^3, C_k \in D_1 \text{ ?}$$

using best rational approximation on 2D slices (disks)

Here:

- best analytic approximants and extremal problems

  $\longrightarrow$ constructive solutions to Cauchy inverse problems (C-IP), data propagation (from T_+ to T_-)

  $\longrightarrow$ also for Robin type coeff., from incomplete data, or geometrical boundary problems

- best rational / meromorphic approximants and behaviour of their poles

  $\longrightarrow$ constructive solutions to inverse sources problems (S-IP), localization of $\{C_k\}$

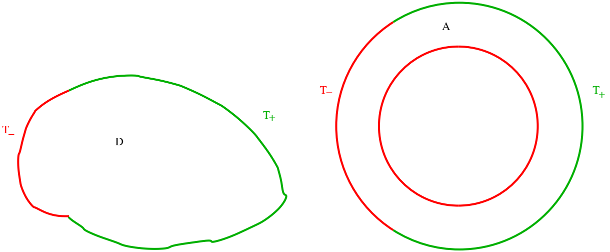
  $\longrightarrow$ also for cracks and others geometrical issues

Cauchy problems for Δ

$D \subset \mathbb{R}^m$, $m = 2, 3$, domain with smooth boundary $T = T_+ \cup T_-$
$T_\pm$ with disjoint interiors both of positive Lebesgue measure

From prescribed data ϕ (flux) and associated measurements ν (potential) on T_+, find u, $\partial_n u$ on T_-:

(C-IP) $$\begin{cases} \Delta u = 0 & \text{in} \quad D \\ u = \nu & \text{on} \quad T_+ \\ \partial_n u = \phi & \text{on} \quad T_+ \end{cases} \qquad \nu, \phi \text{ on } T_+ \to u, \partial_n u \text{ on } T_-?$$



Cauchy problems for Δ

(C-IP) $\quad \begin{cases} \Delta u = 0 & \text{in} \quad D \\ u = \nu & \text{on} \quad T_+ \\ \partial_n u = \phi & \text{on} \quad T_+ \end{cases}$ $\quad \nu, \phi$ on $T_+ \to u, \partial_n u$ on T_-?

(C-IP) admits unique solution u on T_- $\quad \phi \in L^2(T_+) \Rightarrow u \in W^{1,2}(T)$

but ill-posed: strongly discontinuous w.r.t. data

$\exists$ (many) resolution algorithms [Kozlov, MC&al., ...]

but need for numerical robustness w.r.t. errors
and stronger convergence results for non compatible data

Cauchy problems for Δ $\rightarrow$ approximation

(C-IP) $\begin{cases} \Delta u = 0 & \text{in} \quad D \\ u = \nu & \text{on} \quad T_+ \\ \partial_n u = \phi & \text{on} \quad T_+ \end{cases}$ ν, ϕ on $T_+ \rightarrow u, \partial_n u$ on T_-?

(C-IP) stated as best approximation issue on T_+ constrained on T_-

(regularization) in Hilbert classes of analytic functions in D

$\rightarrow$ Bounded Extremal Problems (BEP) in Hardy spaces [JuL&al]
of analytic / harmonic functions bounded in L^2 norm

well-posed (not interpolation)

constructive resolution algorithms for $m = 2, 3$

Harmonic/analytic functions 2D

Because $\Delta u = 0$ in $D \subset \mathbb{R}^2 \simeq \mathbb{C}$, the function $G = u + iv$ is analytic in D, for conjugate harmonic function v

Cauchy-Riemann equations on bdy $T \simeq \mathbb{T}$, for simply connected $D \simeq \mathbb{D}$ unit disk up to conformal map [Rudin, ...]

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial \theta}, \ \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial n}$$

Hence the function F given from boundary data:

$$F(e^{i\theta}) = \nu(e^{i\theta}) + i \int^{\theta} \phi(e^{i\tau})\, d\tau = u + i \int^{\theta} \partial_n u \,, \quad e^{i\theta} \in T_+$$

is the trace on T_+ of a function analytic in $\mathbb{D}$.

Recovery of harmonic/analytic fos in $\mathbb{D}$

- from (noisy) boundary data on $T_+ \subset \mathbb{T}$, get

$$F(e^{i\theta}) = \nu(e^{i\theta}) + i \int^{\theta} \phi(e^{i\tau})\, d\tau = u + i \int^{\theta} \frac{\partial u}{\partial n}, \quad e^{i\theta} \in T_+$$

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- recover analytic function G in $\mathbb{D}$ from its values $F = G_{|T_+}$ on $T_+ \subsetneq T \longrightarrow$ solution u to (C-IP): $\boxed{u = \mathrm{Re}\, G}$

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- Ill-posed boundary interpolation issue
$\Rightarrow$ best approximation in Hardy spaces H^2 of $\mathbb{D}$
(functions analytic in $\mathbb{D}$ bounded $L^2(\mathbb{T})$)

Hardy spaces $H^2(\mathbb{D})$

Functions analytic in $\mathbb{D}$ bounded in $L^2(r\mathbb{T})$, $r \leq 1$, or,
$L^2(\mathbb{T})$ functions with vanishing negative Fourier coeff.

$$H^2 = H^2(\mathbb{D}) = \{G(\zeta) = \sum_{p\geq 0} g_p\, \zeta^p \;,\; \sum_{p\geq 0} |g_p|^2 < \infty \;,\; \zeta \in \mathbb{D}\}$$

$$\to G(e^{i\theta}) = \sum_{p\geq 0} g_p\, e^{ip\theta} \in L^2(\mathbb{T})$$

Example $g \in H^2$: polynomials, or rationals without poles in $\overline{\mathbb{D}}$, exp or log with singularities out of $\overline{\mathbb{D}}$...

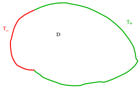
Also , bdd analytic fcs in $\mathbb{C} \setminus \overline{\mathbb{D}}$:

$$\overline{H}_0^2 = H_0^2(\mathbb{C} \setminus \overline{\mathbb{D}}) = \{G(\zeta) = \sum_{p<0} g_p\, \zeta^p \;,\; \sum_{p<0} |g_p|^2 < \infty\} \;\; \zeta \in \mathbb{C} \setminus \mathbb{D}$$

Hardy spaces, notations, properties

disk $\mathbb{D} \subset \mathbb{R}^2 \simeq \mathbb{C}$, $\partial\mathbb{D} = \mathbb{T}$
$L^2 = L^2(\mathbb{T}) = H^2 \oplus \overline{H}^2_0$

(or conformally equiv. simply connected D, or annular domain)



$\mathbb{T} = T_+ \cup T_-$, $L^2_\pm = L^2(T_\pm)$, norm/inner product $\| \ \|_\pm$, $< \ , \ >_\pm$

uniqueness on subsets T_+ of $\mathbb{T}$ of positive measure:
if $G \in H^2$ and $G_{|_{T_+}} = 0$, then $G \equiv 0$

if $T_- = \mathbb{T} \setminus T_+$ of positive measure, $H^2_{|_{T_+}}$ dense in L^2_+; however, if $F \in L^2_+$ and $G_n \to F$ in L^2_+, then either $F \in H^2_{|_{T_+}}$, or $\|G_n\|_- \to \infty$

Hardy spaces in 3D

ball $\mathbb{B}$ (or 3D spherical shell), $\partial\mathbb{B} = \mathbb{S} = T_+ \cup T_-$

$H^2(\mathbb{B})$: $G = \nabla u$ gradients of functions harmonic in $\mathbb{B}$ (analytic), G bounded in $L^2(\mathbb{S})$ norm,

$$G = (\frac{\partial u}{\partial n}, \nabla_{\mathbb{S}} u)$$

Cauchy-Riemann equations for analytic functions, Riesz systems $\nabla \, . \, G = \nabla \times G = 0$ [SteinWeiss]

$$\mathcal{L}^2 = \mathcal{L}^2_{\nabla}(\mathbb{S}) = \left\{(f, \nabla_{\mathbb{S}}\phi) : \, f \in L^2(\mathbb{S}), \phi \in W^{1,2}(\mathbb{S})\right\},$$

with normalization $\int_{\mathbb{S}} f \, d\sigma = \int_{\mathbb{S}} \phi \, d\sigma = 0,$

$L^2_{\pm} = \mathcal{L}^2_{|T_{\pm}}$, norm/inner product $\| \ \|_{\pm}$, $< \ , \ >_{\pm}$

Still: $\mathcal{L}^2 = H^2(\mathbb{B}) \oplus \overline{H}^2_0(\mathbb{B})$ $\overline{H}^2_0(\mathbb{B})$:... outside $\mathbb{B}$...

and uniqueness on T_+, density of $H^2_{|T_+}$ in L^2_+ (but unbdd on T_-...)

From (C-IP) to (BEP)

Back to inverse problem (C-IP) in (2D) and (3D) situations: extension issue of finding $\boxed{G \in H^2,\ G_{|_{T_+}} = F}$ from (noisy) boundary data on T_+:

$$2\text{D: } F = u + i \int^{\theta} \partial_n u\,, \qquad 3\text{D: } F = (\phi, \nabla_{\mathbb{S}} \nu) = (\frac{\partial u}{\partial n}, \nabla_{\mathbb{S}} u)$$

One can fit arbitrarily closely to noisy data F on T_+ ($F \notin H^2_{|_{T_+}}$)
But with unstable behaviour elsewhere, on T_-

Related to ill-posedness of Cauchy type or interpolation issues

Add a T_- norm constraint on the H^2 function G:
well-posed best constrained approximation issues

Bounded extremal problems

Given $F \in L^2_+$, $M \geq 0$, find $G_* \in H^2$, $\|G_*\|_- \leq M$

$$\text{(BEP)} \qquad \|F - G_*\|_+ = \inf\{\|F - G\|_+ : G \in H^2, \|G\|_- \leq M\}$$

admits unique solution G_* [JuL&al]

Further, if $F \notin \{G \in H^2, \|G\|_- < M\}_{|T_+}$, then $\|G_*\|_- = M$

Proof: best approximation projection onto closed cvx subsets of Hilbert spaces

(BEP) also in Sobolev norm $W^{k,2}$, in Banach spaces H^p/L^p, or in $H^p(\mathbb{A})$ for the annulus, with other constraints (mixed: L^2 / L^∞, or on Re / Im parts), criteria (in Re / Im part) [Apics&al]

Bounded extremal problems

$$(\text{BEP}): \min_{G \in H^2} \left(\|F - G\|_+^2 + \lambda \|G\|_-^2 \right)$$

$\pi \perp$ projection L^2 or $\mathcal{L}^2 \to H^2$, $\chi_\pm$ characteristic function of $T_\pm$

Toeplitz operator $\mathcal{T}$ on H^2 defined by $\mathcal{T}_{k,j} = \mathcal{T}_{k-j}$

$$< \mathcal{T} G, \Gamma > = < G, \Gamma >_- = \int_{T_-} G \cdot \Gamma \quad \text{or} \quad \mathcal{T} G = \pi (\chi_- G) \quad \in H^2$$

Construct the solution, solve variational equation:

$$< (I + (\lambda - 1) \mathcal{T}) G_*, \Gamma > = < F, \Gamma >_+ = < \chi_+ F, \Gamma > , \quad \text{for all } \Gamma \in H^2$$

for (unique) value $\lambda > 0$ (Lagrange param.): $\|G_*\|_- = M$

Toeplitz operator $\mathcal{T}$ in 2D

Computations, Toeplitz matrices on bases of L^2 and H^2

(2D) $D = \mathbb{D}$, Fourier basis of $L^2(\mathbb{T})$,

$$\pi\chi_+ F(e^{i\theta}) = \sum_{k=0}^{\infty} \hat{F}_k \, e^{ik\theta}$$

$T_+ = (e^{-i\theta_0}, e^{i\theta_0})$; Cauchy formula: $\mathcal{T} = (\mathcal{T}_{k,j})_{k,j\geq 0}$

$$\mathcal{T}_{k,m} = \begin{cases} 1 - \dfrac{\theta_0}{\pi} & k = j \\ -\dfrac{\sin(k-j)\theta_0}{(k-j)\pi} & k \neq j \end{cases}$$

for $D = \mathbb{A}$, add $r\mathbb{T}$ and Fourier coefficients $\hat{F}_k$, $k < 0$

Toeplitz operator $\mathcal{T}$ in 3D

Computations, Toeplitz matrices on bases of $\mathcal{L}^2$ and H^2

(3D) $D = \mathbb{B}$, basis of spherical harmonics

$$\pi \chi_+ F(\sigma) = \nabla \sum_{k=0}^{\infty} p_k(\sigma)$$

$$< \mathcal{T} \nabla p_k, \nabla p_j > = < \nabla p_k, \nabla p_j >_-$$

$$= j(j+k+1) \left(\int_{T_-} p_k p_j + \int_{\partial T_-} p_k \partial_3 \bar{p}_j \right)$$

for $D = \mathbb{S} \setminus r\mathbb{S}$, add Kelvin transforms $\mathcal{K}[p_k]$

Bounded extremal problems

Convergent and robust algorithms in 2D and 3D

- compute an adequate L^2 extension $\chi_+ F$ of F to the whole T
  from pointwise data, approx. interpolate (splines, ...)

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- compute (iteratively) $\boxed{G = (I + (\lambda - 1)\,\mathcal{T})^{-1}\,\pi\,\chi_+ F}$

varying $\lambda > 0$ (dichotomy) until $\|G\|_- = M$: $\boxed{G_*}$

Bounded extremal problems

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  varying $\lambda > 0$ (dichotomy) until $\|G\|_- = M$: $\boxed{G_*}$
- approximation of L^2_+ functions: (robust interpolation for $H^2_{|T_+}$)
  compromize between $\|G_*\|_- = M$ and error $\|F - G_*\|_+$

(Cauchy-IP)

From $G*$ in $\mathbb{D}$ and on T_-, get u:

$$\text{2D: } u \simeq \text{Re } G_* , \qquad \frac{\partial u}{\partial n} \simeq \frac{\partial \text{Im } G_*}{\partial \theta}$$

or

$$\text{3D: from } G_* \simeq \left(\frac{\partial u}{\partial n}, \nabla_{\mathbb{S}} u\right) \quad \left(\text{in fact, algo.} \to u\,,\ \frac{\partial u}{\partial n}\right)$$

Numerical computations, example (EEG)

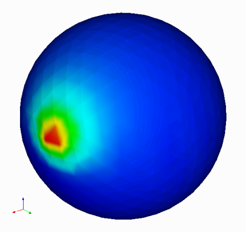
3-sphere model, radii $\rho_i = .87, .92, 1$, conductivities $\sigma_i = 1, 1/30, 1$
one dipolar source at $C = (.7, .2, .1)$ (BEM), [MC&al]

ν numerically generated on $T_+ = S_3$ from $u(X) \simeq \dfrac{< p, X - C >}{\|X - C\|^3}$

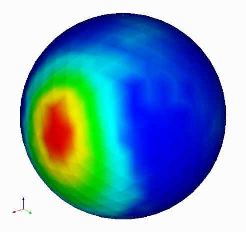
(BEP) solved with $T_- = S_2$, then with $T_+ = S_2$ and $T_- = S_1$

hence (IP), $G_* \simeq \nabla u$, cortical potential u on S_1:

explicit data



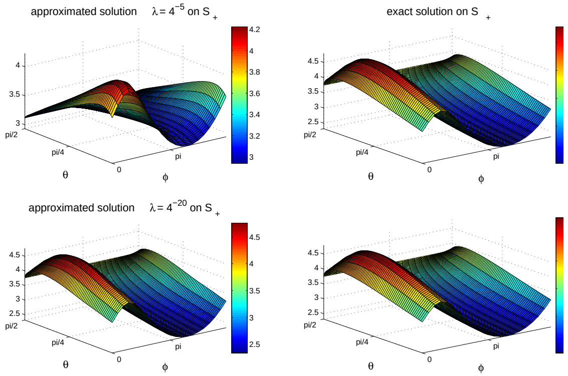
(BEP) solution



Numerical computations, example

$f = \nabla u$, $u(X) = \sum_{k=1}^{3} \frac{1}{\|X - C_k\|}$ monopolar sources

$D = $ ball $\mathbb{B}$, $T_+ =$ upper $1/2$ sphere $\mathbb{S} \cap \{x_3 > 0\}$



error $\|F - G_*\|_+ \simeq .07$ for $\lambda = 4^{-20}$ still too many coeffs...

More about harmonic/analytic mD functions

- Cauchy-Riemann equations for analytic functions [SteinWeiss]
  $G = (G_1, \cdots, G_m)$ in $D \subset \mathbb{R}^m$ Riesz systems

$$\begin{cases} \partial_j G_i = \partial_i G_j & \text{or } \nabla \times G = 0 \quad \Rightarrow G = \nabla g \text{ or } G_i = \partial_i g \\ \displaystyle\sum_{j=1}^{m} \partial_j G_j = 0 & \text{or } \nabla . G = 0 \quad \Rightarrow g \text{ harmonic} \end{cases}$$

More about harmonic/analytic 3D functions

In $\mathbb{B} \subset \mathbb{R}^3$, analytic functions $\boxed{G = \nabla g}$ for g harmonic in $\mathbb{B}$

Hardy spaces $H^2(\mathbb{B})$: [SteinWeiss]
$G = (G_1, G_2, G_3)$ analytic in D, with $G_i = \partial_i g$ bounded in $L^2(T)$

For spherical domains [Axler&al, DautrayLions]

$$\begin{cases} H^2(\mathbb{B}) : \quad g(X) = \sum_{k \geq 0} p_k(X) \\ \\ (p_k) \text{ homogeneous harmonic polynomials degree } k : \\ X \cdot \nabla p_k(X) = k p_k = \partial_n p_k \text{ on } \mathbb{S} \\ \\ \sum_{k \geq 0} k(2k+1) \| p_k \|^2_{L^2(\mathbb{S})} < \infty \end{cases}$$

Spherical harmonics

Spherical harmonics $\mathcal{H}_k$:
traces on $\mathbb{S}$ of homogeneous harmonic polynomials degree k

$$\boxed{L^2(\mathbb{S}) = \oplus_{k\geq 0}\mathcal{H}_k}$$

$$\begin{cases} p_k(X) = p_k(r,\sigma) = r^k \sum_{m=-k}^{k} \gamma_k^m Y_k^m(\sigma) \\ Y_k^m(\sigma) = Y_k^m(\theta,\varphi) = P_k^{|m|}(\cos\theta)\, e^{im\varphi} \text{ in } \mathbb{C} \end{cases}$$

$P_k^m(t)$ 1st kind Legendre functions $\quad (1-t^2)P'' - 2tP' + (k(k+1) - \frac{m^2}{1-z^2})P = 0$

Example: $\quad p_5(X) = 63x_1^5 - 70x_1^3 + 15x_1,\ p_{5|_{\mathbb{S}}} \in \mathcal{H}_5$, on $\mathbb{S}$, $p_5(X) = -40x_1^3x_3^2 + 30x_1x_2^2x_3^2 + 15x_1x_3^4$

$$\sum_{m=-k}^{k} Y_k^m(\theta,\varphi)\, Y_k^m(\theta',\varphi') = c_k P_k^0(\cos\psi)$$

ψ spherical distance , $\cos\psi = \cos\theta' \cos\theta + \sin\theta' \sin\theta\, cos(\varphi' - \varphi)$

Spherical harmonics

Bases of spherical harmonics and Fourier coefficients

In 2D: $\mathcal{H}_k$ spanned by $\{e^{\pm ik\theta}\}$ $\longrightarrow L^2(\mathbb{T})$
(complex, or $\{\cos(k\theta), \sin(k\theta)\}$ real)

Spherical harmonics

For spherical domains $\mathbb{B}$, $\mathbb{R}^3 \setminus \mathbb{B}$ [Axler&al, DautrayLions]

$$\begin{cases} \overline{H}^2_0(\mathbb{B}): \quad g = \sum_{k\geq 0} \mathcal{K}[p_k] \\ \mathcal{K} \text{ Kelvin transform :} \\ \mathcal{K}[p_k](X) = \frac{1}{|X|} p_k\left(\frac{X}{|X|^2}\right) = \frac{p_k(X)}{|X|^{2k+1}} \end{cases}$$

$\mathcal{K}[p_k](r,\sigma) = r^{-(k+1)} \sum_{m=0}^{k} \gamma_k^m Y_k^m(\sigma)$

$\mathcal{K}[g] = g$ on $\mathbb{S}$; g harmonic in $\mathbb{B} \Rightarrow \mathcal{K}[g]$ harmonic in $\mathbb{R}^3 \setminus \mathbb{B}$; $\mathcal{K}[\mathcal{K}[g]] = g$

Inversion $X \mapsto X/|X|^2$ conformal

(S-IP) 3D sources recovery $\mathbb{B} \to \cup_p \mathbb{D}_p$

$$\Delta u = \delta = \sum_{k=1}^{L} m_k \cdot \nabla \, \delta_{C_k} \text{ in } \mathbb{B}$$

(S-IP) ν, ϕ on $\mathbb{S} = \partial\mathbb{B} \to m_k \in \mathbb{R}^3, C_k \in \mathbb{B}$?

$\mathbb{B} \subset \mathbb{R}^3$, convolution with fundamental solution
(Newton or Green potential), $X \in \mathbb{B} \setminus \{C_k\}$

$$u(X) = H(X) + \sum_{k=1}^{L} \frac{< m_k \, , X - C_k >}{4\pi \, \|X - C_k\|^3} = H(X) + f(X)$$

H harmonic in $\mathbb{B}$: $f = P_- u$ (spherical harmonics)

$X_p = (x, y, z_p) \simeq$ complex var. $\xi = x + i\, y$,
$\xi \in$ disk $\mathbb{D}_p = (\{z = z_p\} \cap \mathbb{B}) \subset \mathbb{R}^2 \simeq \mathbb{C}$

$f(X_p) = \tilde{f}_p(\xi)$, $f_p(\xi) = P_- \tilde{f}_p(\xi)$ (Fourier coeffs < 0 index)

... $\mathbb{B} \to \cup_p \mathbb{D}_p$, 2D statements

fix p, $\mathbb{D}_p$:

$$f_p(\xi) = \sum_{k=1}^{m} \frac{P_{k,p}(\xi)}{(\xi - \xi_{k,p})^{3/2}}$$

f_p^2 has poles and branchpoints $\{\xi_{k,p}\} \in \mathbb{D}_p$, $P^2_{k,p}$ analytic in $\mathbb{D}_p$, $\mathbb{T}_p = \partial \mathbb{D}_p$

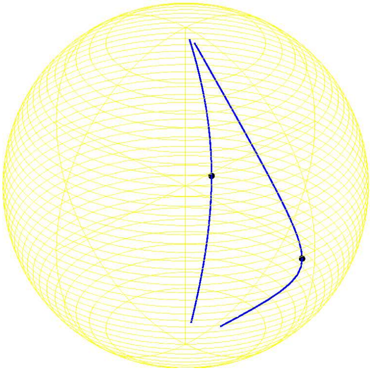
$C_k = (x_k, y_k, z_k)$, affix $\xi_k = x_k + iy_k$; assume $\xi_k \neq 0$

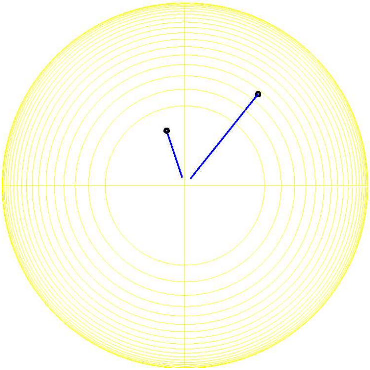
- $(\xi_{k,p})_p \ // \ \xi_k$
- $|\xi_{k,p}|$ maximum at p^* such that $z_{p^*} = z_k$ where $\xi_{k,p^*} = \xi_k$

(S-IP): from $(f_p)_p$ on $(\mathbb{T}_p)_p$, find L, $(\xi_{k,p})_p$, $1 \leq k \leq L$ and sort them out to get $(\xi_k, z_k) = C_k$, m_k

...$\longrightarrow \mathbb{D}_p$, 2D

For 2 dipoles $\{C_1\ ,\ \ C_2\} \subset \mathbb{B}$, $\{\xi_{k,p}\}_p$:





$\mathbb{B} \to \cup_p \mathbb{D}_p$: compute $\xi_{k,p}$ wrt ξ_k, z_k, z_p

at $X = (x, y, z_p)$ and with $\xi = x + i\, y$,

$$\|X - C_k\|^3 = Q_{p,k}(\xi)^{3/2}$$

where, if $h_{p,k} = z_p - z_k$,

$$Q_{p,k}(\xi) = |\xi - \xi_k|^2 + h_{p,k}^2 = -\frac{\xi_k}{\xi}(\xi - \xi_{p,k}^+)(\xi - \xi_{p,k}^-)$$

has 1 root in $\mathbb{D}_p$ at $\xi_{p,k}^-$: pole and branchpoint to f_p^2
with $r_p^2 = 1 - z_p^2$,

$$\xi_{p,k}^{\pm} = \frac{\xi_k}{2|\xi_k|^2}\left\{|\xi_k|^2 + r_p^2 + h_{p,k}^2 \pm \sqrt{(|\xi_k| + r_p)^2(|\xi_k| - r_p)^2 + h_{p,k}^2}\right\}$$

2D (S-IP)$_p$

(S-IP)$_p$: given $f_{p|_{\mathbb{T}}}$, find L singularities $\xi_{k,p} \subset \mathbb{D}_p$, functions (moments) $P_{k,p}$, such that

$$f_p(\xi) = \sum_{k=1}^{L} \frac{P_{k,p}(\xi)}{(\xi - \xi_{k,p})^{3/2}}$$

fix p: $\xi \longrightarrow f_p^2(r_p\, \xi)$

continous in $\mathcal{C}_\varepsilon = \{z \in \overline{\mathbb{D}}\,;\ 1 - \varepsilon < |z| \leq 1\}$, analytic in $\overset{\circ}{\mathcal{C}}_\varepsilon$,
can be analytically extended in $\mathbb{D}$ except for finit. many
(poles or) branchpoints

$\Rightarrow$ poles of its best L^2 (or L^∞...) meromorphic or rational approximants r_n "cv" to singularities $\{\xi_{k,p}\}_k$ [Bar.&al]

Poles of $r_n \rightarrow$ singularities $\{\xi_{k,p}\}_k$

$$f_p^2(\xi) = \sum_{k=1}^{L} \frac{\Pi_{k,p}(\xi)}{(\xi - \xi_{k,p})^3} + \sum_{k=1}^{L} \frac{\Pi_{j,k,p}(\xi)}{(\xi - \xi_{k,p})^{3/2}\,(\xi - \xi_{j,p})^{3/2}}$$

n poles of r_n in $\mathbb{D}_p$ discretize / approximate singularities $\{\xi_{k,p}\}_k$ of f_p in $\mathbb{D}_p$

accumulate to branchpoints $\{\xi_{k,p}\}_k$ on curve joining them of minimal Green capacity [Bar.&al]

($m = 2$: hyp. geodesic arc between $\{\xi_{1,p}, \xi_{2,p}\}$)

Meromorphics and rationals in Hardy classes

Hardy class of $\mathbb{D}$, hilbertian case:

$$H^2 = H^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D}\,,\ \sup_{r<1} \|f\|_{L^2(\mathbb{T}_r)} < \infty\}$$

and meromorphics:

$$H^2_n = H^2_n(\mathbb{D}) = \{\frac{h}{q_n}\,,\ h \in H^2\,,\ q_n(z) = \prod_{j=1}^{n}(z - \eta_j)\,,\ \eta_j \in \mathbb{D}\}$$

$$= H^2 + \text{ rational with less than } n \text{ poles all in } \mathbb{D}$$

Best L^2 rational approximation

$f_{p|_{\mathbb{T}}} \to r_n \in H^2_n$, best $L^2(\mathbb{T})$ meromorphic approximant deg. $\leq$ n

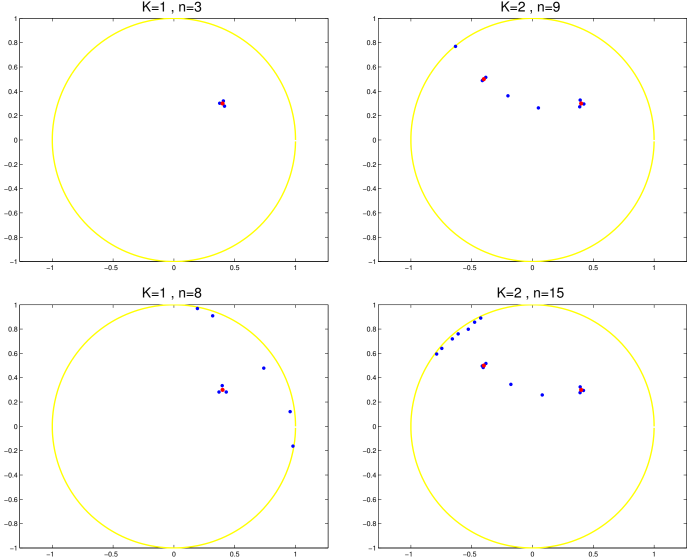
$$\min_{h/q \in H^2_n} \| f_p^2 - \frac{h}{q} \|_{L^2(\mathbb{T})} = \Psi_n \to r_n = \frac{h_n}{q_n}$$

$$f_p^2 \simeq r_n \text{ and poles } (0 \text{ of } q_n) \to \text{singularities of } f_p$$

existence and constructive results, parametrization, gradient algorithms (local minima...) [Miaou-Apics]

(also H^∞_n, [Nehari, AAK, Hankel op.] and H^l_n, $2 < l < \infty$, [LB+FS], matrices,...)

3D sources on 2D slice

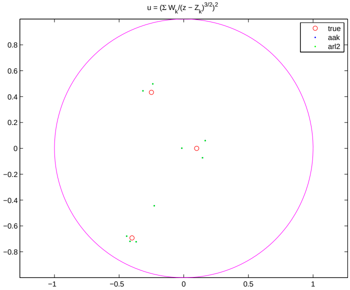


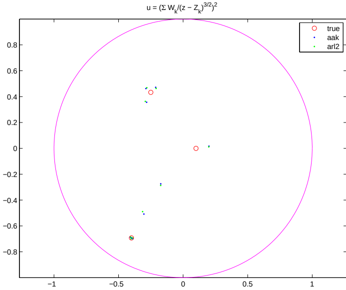
1 source (ARL2)

2 sources (from explicit 3D data)

Behaviour of poles on 2D slice

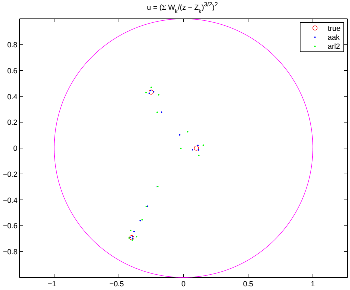
$n = 9$

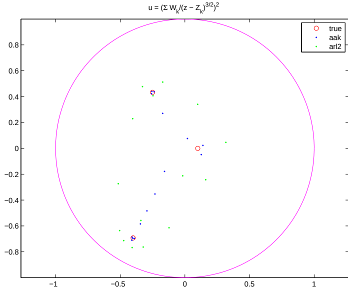




Behaviour of poles on 2D slice

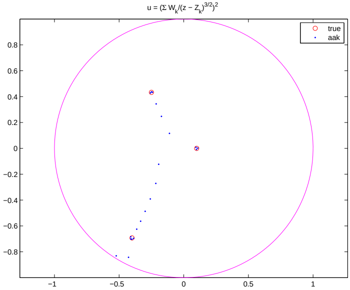
$n = 15$

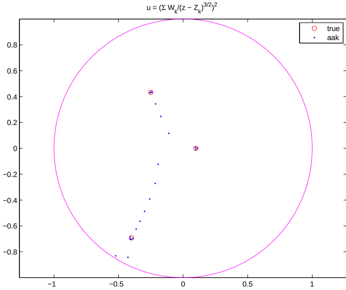




Behaviour of poles on 2D slice

AAK, $n = 21$





Algorithms... $\to \cup_p \mathbb{D}_p \to \mathbb{B}$

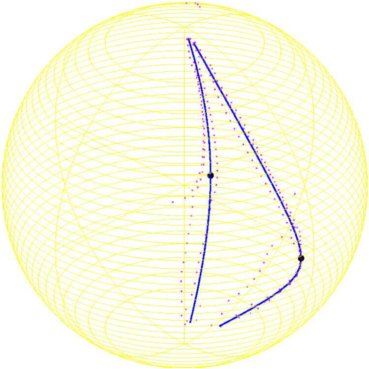
- $g = u_{|\mathbb{S}}$, $\phi = \partial_n u_{|\mathbb{S}} \to f_{|\mathbb{S}}$
- for each p, $-P \leq p \leq P$, $2D$:
  - $\to f_p^2(r_p\,\xi)$ on $\mathbb{T}$
  - best meromorphic approximation on $\mathbb{T}$,
    (ARL2) [Apics], Matlab: iterate a gradient algorithm from $n = 0$ to $\Psi_n \simeq 0$: $n \geq m \to r_n$ poles accumulate to $\{\xi_{k,p}\}_k$

(or AAK; also Endymion, C++)

- sort out aligned $(\xi_{k,p})_p$, then $//\ \xi_k$
- for each k, find p^* such that $|\xi_{k,p^*}| = \max_p |\xi_{k,p}|$:
  $\xi_{k,p^*} = \xi_k \ \to\ x_k, y_k$ and $z_{p^*} = z_k$

$$\longrightarrow\ C_k$$

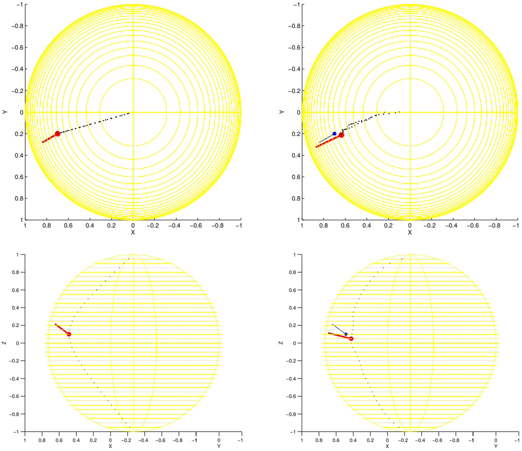
2 dipoles, $n = 6$



Numerical computations, example (EEG)

True source C • localized by best L^2 rational approximation •

explicit data | (BEP) solution

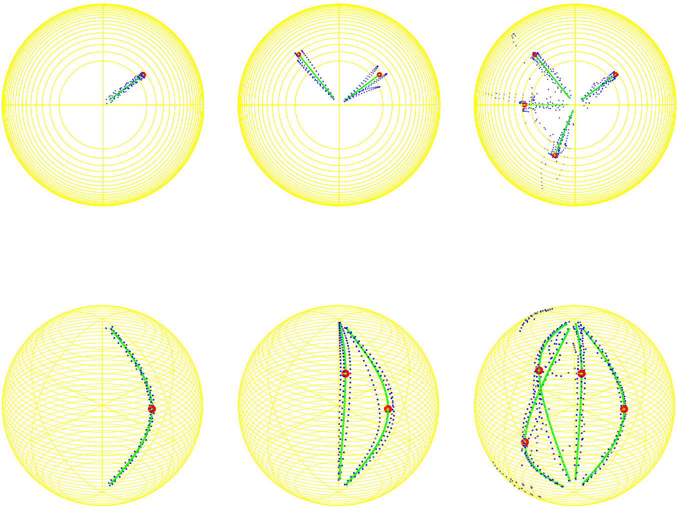


error on C: 10^{-4} | 10^{-2}

Numerical computations, example (EEG)

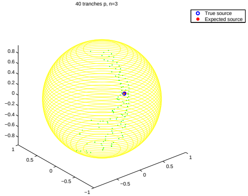
Several sources ● localized by best L^2 rational approximation ●

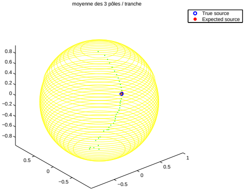
(explicit data)

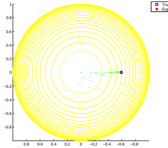


degree: $n = 3$ 6 12

1 dipole, numerical data (Odyssée), $n = 3$

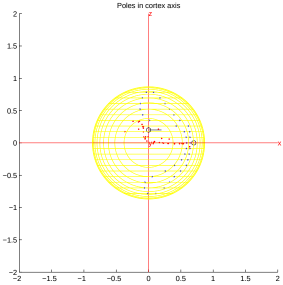


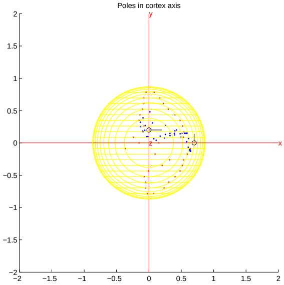




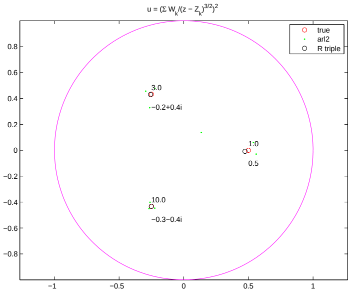
2 dipoles, numerical data (Odyssée), $n = 2$

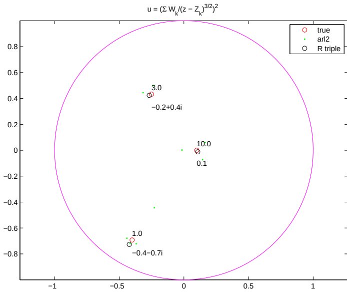
sections $p \perp 0y, 0z$





Triple poles degree $3n$, $n = 3$





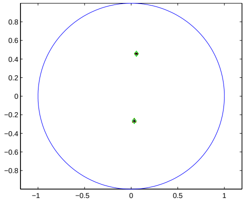
More realistic geometries

Ellipsoids (ellips. harmonics)

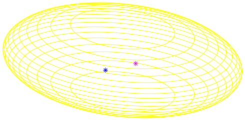
[JuL&al]

1 source

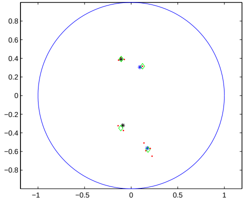
6 poles



2 sources



12 poles



Comments, conclusion

Under study / to be done:

- EEG: pre-/post-treatments + (BEM) + approx. on sections
  → FindSources3D software [RB, Apics+Odyssée]

  (Sphere2Circle, orient. spherical harmo., moments computation, several sections)

- → experimental EEG data (electrodes)?
- add MEG model and data (also martian rocks [W])
- other geometries (3D: 1/2-ellipsoid+1/2-sphere? 2D: quadrature domains)
- 3D (BEP) from partial data (computational issues / spherical harmo.)
- approx. / multiple poles and multipolar expansions [Bail.&al]
  distributed sources (small supports [Vo])
- variable conductivity, Beltrami equation (for plasma confinment in tokamak)
- inverse problem of conductivity recovery (EIT)
- geodesy... and inverse pbs for gravitational potential

+ various elliptic inverse pbs / related approximation, geometrical IP for corrosion detection or plasma recov.

(unknown boundary part, Bernoulli), 3D / quaternionic approximation?

References

Atfeh, Baratchart, Leblond, Partington. "Bounded extremal and Cauchy-Laplace problems on 3D spherical domains", subm.

Baratchart, Leblond, Marmorat. "Sources identification in a 3D ball from best meromorphic approximation on 2D slices", Elec. Trans. Num. Anal., 2006.

Baratchart, Ben Abda, Ben Hassen, Leblond. "Recovery of pointwise sources or small inclusions in 2D domains and rational approximation", Inverse Problems, 2005.

Baratchart, Mandréa, Saff, Wielonsky. "2D inverse problems for the laplacian: a meromorphic approximation approach", J. Maths Pures Appl., 2006.

Clerc, Leblond, Atfeh, Baratchart, Marmorat, Papadopoulo, Partington. "The Cauchy problem applied to cortical imaging: comparison of a boundary element method and a bounded extremal problem", *Proc. Brain Topography*, Springer Sci.& Bus. Media, 2005.

...

Clerc, Leblond, Marmorat, Baratchart, Papadopoulo. EEG source localization by best approximation of functions. *Proc. Human Brain Mapping*, 2006.

Leblond, Paduret, Rigat, Zghal. “Sources localisation in ellipsoids by best meromorphic approximation in planar sections”, subm.

Baratchart, Leblond, Rigat, Russ. “Beltrami equation and generalized analytic functions, and extremal problems”, in preparation.

and many other references....

[Dautray-Lions, ABR, Feynman, F&al, H&al, HD-EB, Kozlov, Alessandrini, Vessela, Hammari, Dassios, Baillet, Vogelius, Rudin, Stein-Weiss, ...]