Equilibrium plasma shape recovery from magnetic measurements in tokamaks

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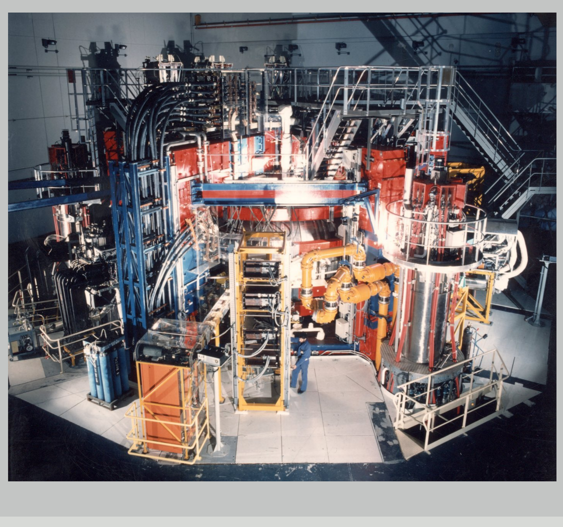
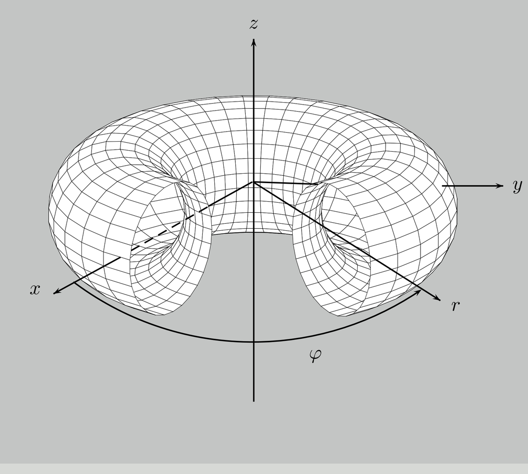
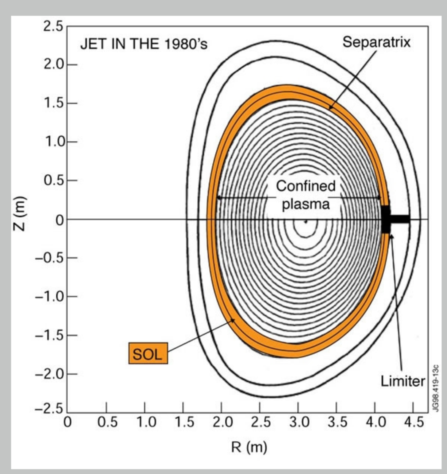
Introduction



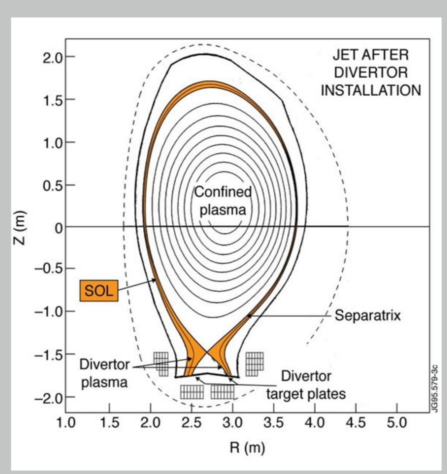
In order to control plasma in tokamaks, it is essential to determine its boundary from magnetic measurements in sufficiently small time. We propose an efficient algorithm to recover the plasma boundary in a computationally cheap way (hence suitable for real-time application) which allows to take into account data available from both direct measurements and pre-computation performed by heavy software packages.



1. Physical formulation



JET tokamaks



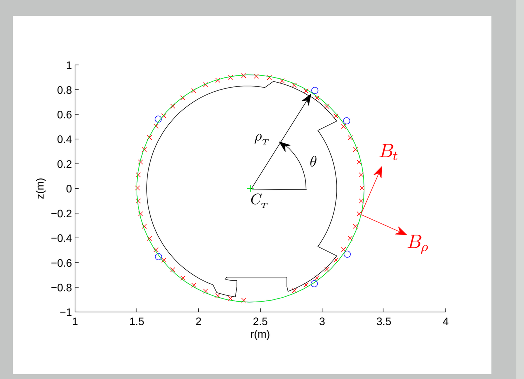
Equilibrium plasma equations:

$$\begin{cases} \nabla\cdot\mathbf{B} &= 0, \\ \nabla\times(\mathbf{B}/\mu) &= \mathbf{j}, \\ \nabla p &= \mathbf{j}\times\mathbf{B}. \end{cases}$$

Working under axisymmetric assumption (i.e. $\partial_\phi = 0$), we are interested in the poloidal component of magnetic field $\mathbf{B_p} = (B_r, B_z)$. Introducing the magnetic flux $u(r,z) := \frac{1}{2\pi}\int_{D_r}\mathbf{B}\cdot\mathbf{ds} = \int_0^r B_z\rho d\rho$, we obtain $\mathbf{B_p} = \frac{1}{r}[\nabla u\times\mathbf{e}_\phi]$. Thus, $\mathbf{B}\cdot\nabla u = 0$, $\mathbf{B}\cdot\nabla p = 0$, $\mathbf{j}\cdot\nabla p = 0$, allowing to claim that flux is constant on isobaric surfaces and hence to define plasma boundary as the outermost closed level line of u in the vacuum chamber. Outside the plasma ($\mathbf{j} = \mathbf{0}$, $p = 0$):

$$\operatorname{div}\left(\frac{1}{r}\nabla u\right) = 0$$

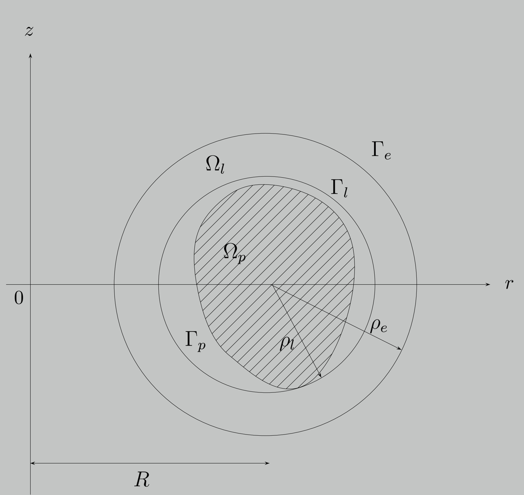
(Grad-Shafranov equation in vacuum).



Poloidal section of the tokamak Tore Supra (CEA-IRFM, Cadarache, France). The limiter is at the bottom. Red crosses and blue bubbles represent points with available data: tangential component of poloidal magnetic field $\mathbf{B_t}$ and magnetic flux u, respectively.

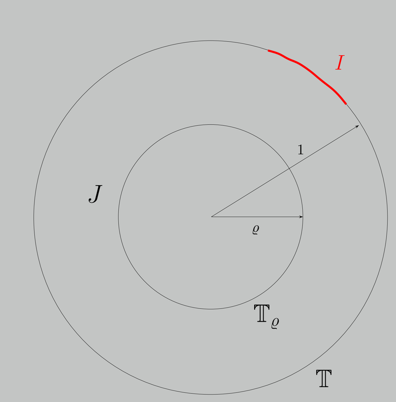
2. Mathematical formulation and approach

Consider the limiter geometry and choose domain Ω as the set confined by the plasma boundary Γ_p and the exterior circle Γ_e, where measurements are available.



Then, given $I\subset\partial\Omega$, $g, h\in L^2_{\mathbb{R}}(I)$, we want to find u on J by solving the overdetermined inverse problem for the "conductivity" equation

$$\begin{cases} \operatorname{div}\left(\frac{1}{x}\nabla u\right) &= 0 \quad \text{in } \Omega, \\ u|_I &= g, \\ \partial_n u|_I &= h. \end{cases}$$



To attack the problem, a complex analysis approach is taken: instead of $\mathbb{R}^2$, we are going to work in $\mathbb{C}$. For notational convenience, rename the coordinates (r,z) into (x,y) and let $z := x+iy$ denote a complex variable. Assume $u = \operatorname{Re} f$ with $f(z,\bar{z}) = u(x,y) + iv(x,y)$ satisfying the conjugate Beltrami equation

$$\partial_{\bar{z}} f = \nu\overline{\partial_z f},$$

where $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$, $\nu := \frac{x-1}{x+1}$. Then, as it can be shown, u and v are intertwined by

$$\begin{cases} \partial_x u &= x\partial_y v, \\ \partial_y u &= -x\partial_x v, \end{cases}$$

resembling the Cauchy-Riemann equations, whereas f in this case is termed as generalized analytic function (formally, if one replaces $\frac{1}{r} =: x$ in the "conductivity" equation with 1, u is harmonic and f is analytic). Also, similarly to u, v satisfies $\operatorname{div}(x\nabla v) = 0$ in Ω and, moreover, the Neumann boundary conditions for u naturally translate into Dirichlet data for v.

3. Analysis of the problem: functional spaces

Without loss of generality (thanks to conformal mapping), we consider the annular domain $\Omega = \mathbb{A} := \mathbb{D}\backslash\bar{\mathbb{D}}_\rho$ (between the unit disk and the disk of radius $0<\rho<1$).

Suitable spaces for analysis of the problem are the so-called generalized Hardy classes on an annulus $H^2_\nu(\mathbb{A})$ which properties mainly stem from $H^2_\nu(\mathbb{D})$, the Hardy space of generalized analytic functions on the unit disk (and hence satisfying the conjugate Beltrami equation on $\mathbb{D}$) such that

$$\|f\|_{H^2_\nu(\mathbb{D})} = \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta\right)^{1/2} < \infty.$$

Main properties of $H^2_\nu(\mathbb{D})$:

- $\|f\|_{H^2_\nu(\mathbb{D})} = \|f\|_{L^2(\mathbb{T})}$, where $\mathbb{T} := \partial\mathbb{D}$ is the unit circle.
- $H^2_\nu(\mathbb{D})$ is Banach space (as a closed subspace of the complete Lebesgue space $L^2(\mathbb{T})$).
- Traces (non-tangential limits) $\operatorname{tr} f$ are well-defined almost everywhere on $\mathbb{T}$.
- For $I\subset\mathbb{T}$ such that $|\mathbb{T}\backslash I|>0$, $\operatorname{tr} H^2_\nu\big|_I$ is dense in $L^2(I)$, meaning we can approximate a given $L^2(I)$ by a sequence of $\operatorname{tr} H^2_\nu\big|_I$.
- For $|I|>0$, $\operatorname{tr} f|_I = 0$ implies $f\equiv 0$ on $\mathbb{D}$.
- Maximum modulus principle holds for $f\in H^2_\nu(\mathbb{D})$.
- Blaschke factorization of isolated zeros (important to incorporate point measurements inside the chamber).

Solution to our problem would be an extrapolation of purely Dirichlet (for both u, v and hence f) $L^2(I)$ boundary data to $H^2_\nu(\Omega)$.

4. Reformulation of the problem and solution

However, the available data u, $\partial_n u$ are prone to measurement/pre-computation errors, and thus are generally incompatible to be exactly the trace of a $H^2_\nu(\Omega)$-function. Unfortunately, the problem is sensitive (unstable) to such perturbations being ill-posed due to the following result:

- Given $F\in L^2(I)$, $|I|>0$, let $\{f_n\}_{n=1}^\infty$ be a sequence of $H^2_\nu(\Omega)$-functions such that $\lim_{n\to\infty}\|F - f_n\|_{L^2(I)} = 0$. Then, for $J := \partial\Omega\backslash I$, $\|f_n\|_{L^2(J)}\to\infty$ as $n\to\infty$, unless F is the trace of a $H^2_\nu(\Omega)$-function.

Therefore, to remedy the situation, an issue of constrained approximation in form of the following bounded extremal problem should be considered: given $F\in L^2(I)$, $M>0$, find $f\in H^2_\nu(\Omega)$ such that the norm $\|\operatorname{tr} f - F\|_{L^2(I)}$ is minimal subject to the constraint $\|\operatorname{tr} f\|_{L^2(J)}\le M$.

Let $\mathcal{H}_\nu : L^2_{\mathbb{R}}(\partial\Omega)\to L^2_{\mathbb{R}}(\partial\Omega)$ be the Hilbert conjugation operator and $P_\nu : L^2_{\mathbb{R}}(\partial\Omega)\to H^2_\nu(\Omega)$ be the Riesz projector defined, respectively, as

$$\operatorname{Re}\operatorname{tr} f\mapsto\operatorname{Im}\operatorname{tr} f \quad\text{and}\quad \operatorname{Re}\operatorname{tr} f\mapsto\frac{1}{2}\left[(I + i\mathcal{H}_\nu)\operatorname{Re}\operatorname{tr} f + \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)d\theta\right].$$

Then, solution $f\in H^2_\nu(\Omega)$ to the bounded extremal problem is given by

$$f = (I + \lambda P_\nu\chi_J)^{-1} P_\nu(\chi_I F),$$

where χ_J denotes the characteristic function of the set J, and the parameter $\lambda\in(-1,\infty)$ has to be chosen such that the constraint on J is saturated.

5. Computational algorithm and improvements

Computational aspects:

- Toroidal harmonics constitute complete family of solutions to the "conductivity" equation $\{u_n(\tau,\eta)\}_{n=0}^\infty$ found by passing to the bipolar coordinates using change of variable: $x = a\frac{\sinh\tau}{\cosh\tau - \cos\eta}$, $y = a\frac{\sin\eta}{\cosh\tau - \cos\eta}$, where a is the y-distance to the center of the poloidal section. Similar holds for v.
- Computation of the solution f to the bounded extremal problem hinges on finding inverse of a Toeplitz operator. However, the operator's matrix in a truncated basis has a structure and can be efficiently inverted.

Practical algorithm:

- 1. Formally choose $\Gamma_p = \Gamma_l$, the boundary of the vacuum chamber including the limiter.
- 2. Perform expansion $u = \sum_{n=0}^N a_n u_n$ with coefficients a_n to be found from the boundary data on Γ_e. Choosing number of toroidal harmonics N sufficiently high to get desired accuracy, this approximates the solution that still remains valid outside of Ω all the way to the actual (yet unknown) plasma boundary.
- 3. Evaluate maximum of u on Γ_l. A level line corresponding to this value should be now taken as the plasma boundary Γ_p.
- 4. Similarly to u, construct an expansion approximating v.
- 5. Having defined the boundary of Ω, we solve the bounded extremal problem to find f.
- 6. Find maximum of $\operatorname{Re} f$ on Γ_l to obtain a level line improving the preliminarily estimated plasma boundary.

Further improvements:

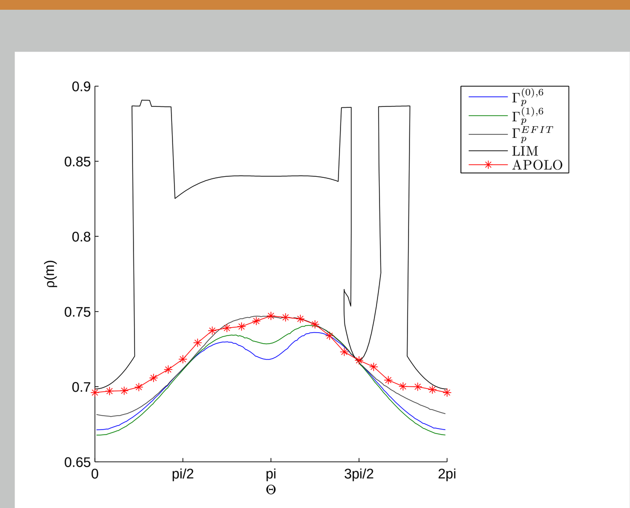
- Consider domain with $\partial\Omega$ having a cusp, in order to take into account X-point geometry of the new JET and ITER tokamaks.
- Extension of the result to incorporate additional measurements available from inside the chamber.

In particular, in case of pointwise data, generalizing the obtained result for a simply connected domain, we expect the solution to the bounded extremal problem to be given by

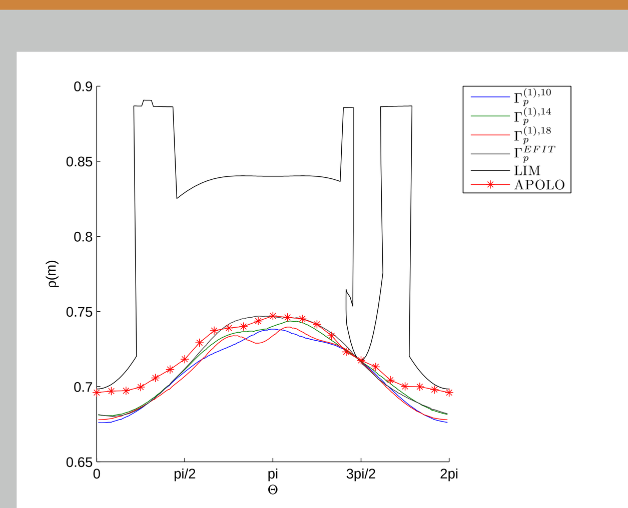
$$f = (I + \lambda P_\nu\chi_J)^{-1} P_\nu\left[\bar{b}(F - \psi)\chi_I - (1+\lambda)\bar{b}\psi\chi_J\right],$$

where b is the finite Blaschke product vanishing at each z_j and $\psi(z)$ is a regular interpolant such that $\psi(z_j) = y_j$, for given $\{z_j\}_{j=1}^M$, $\{y_j\}_{j=1}^M$.

6. Numerical illustrations



Preliminary and improved plasma boundary obtained with short series expansion.



Plasma boundary recovered with longer expansion over toroidal harmonics.

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