Région
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Solutions of the conjugate Beltrami equation in generalized Hardy spaces and applications to inverse problems in Tokamaks

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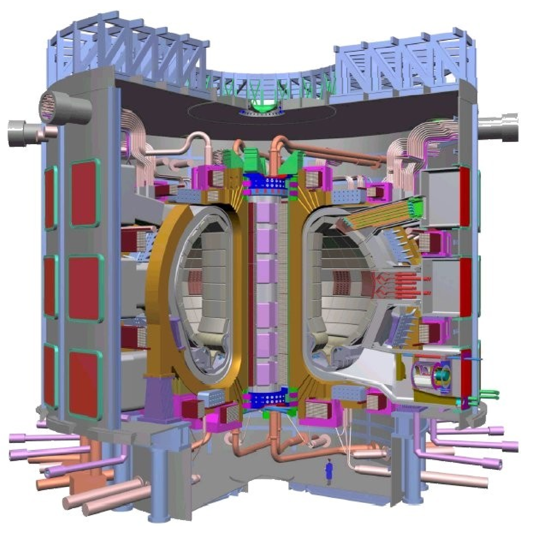
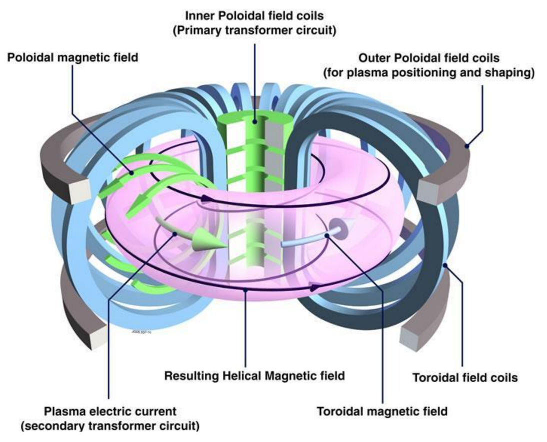
Inverse Problems in Tokamaks:





Model of plasma equilibrium:

Physical problem of plasma confinment in a Tokamak (thermonuclear fusion)

Maxwell equations + axisymmetric assumption $\Rightarrow$ Grad-Shafranov equation in poloidal plane sections ($\varphi = constant$) : $-\,div\left(\frac{1}{x}\nabla u\right) = j_T$ ($= 0$ in the vacuum)

where (x, y, φ), $u(x, y)$ and j_T denote respectively the cylindrical coordinates, the poloidal component of the magnetic flux and the toroidal component of the current density vector.

Motivation:

The above issue is a practical important motivation for considering the following so-called conductivity equation in a planar annular domain Ω

$$div(\sigma\nabla u) = 0,\ \sigma \text{ real-valued and Lipschitz-continuous, } 0 < c < \sigma < C \text{ in } \Omega \subset \mathbb{R}^2$$

From such questions, several inverse boundary value problems may be considered

- given Dirichlet data u on the boundary $\partial\Omega$, recover u in Ω,
- given overdetermined Cauchy data u and the normal derivative $\partial_n u$ on a strict subset $I \in \partial\Omega$, recover u in Ω (afterward Cauchy data on $J = \partial\Omega\backslash I$)

1. Conjugate Beltrami equation:

When Ω is simply connected ($\Omega \sim \mathbb{D}$ the unit disc) and when $z = x + iy$ denotes the complex variable, the conductivity equation is linked to the conjugate Beltrami equation by a system of real second order elliptic equations

$f(z, \overline{z}) = u(x, y) + i\,v(x, y)$, u and v in $W^{1,p}_{\mathbb{R}}(\Omega)$

solves $\overline{\partial} f = \nu\overline{\partial f}$ a.e in Ω $\iff$ u and v satisfy $\begin{cases} div(\sigma\nabla u) = 0 \text{ in } \Omega \\ div(\frac{1}{\sigma}\nabla v) = 0 \text{ in } \Omega \end{cases}$ with $\sigma = \dfrac{1-\nu}{1+\nu}$

with $\nu \in W^{1,\infty}_{\mathbb{R}}(\Omega)$ and $\|\nu\|_{L^\infty(\Omega)} \leq \kappa,\ \kappa \in (0, 1)$

The study of the solutions of the conjugate Beltrami equation is a generalization of the harmonic case $\overline{\partial} f = 0$ (when $\sigma = 1$, last system reduces to the Cauchy-Riemann equations).

2. Definition of generalized Hardy spaces ($1 < p < \infty$):

Dirichlet problem:

Find u defined on $\mathbb{D}$ with prescribed trace on $\mathbb{T}$ $\iff$ Find f defined on $\mathbb{D}$ with prescribed Ref on $\mathbb{T}$

This problem was solved in $W^{1,p}(\mathbb{D})$ only for $W^{1-1/p,p}_{\mathbb{R}}(\mathbb{T})$-boundary data. With $L^p(\mathbb{T})$-boundary data, f does no longer belong to $W^{1,p}(\mathbb{D})$ but to $H^p_\nu(\mathbb{D})$.

Definition of $H^p_\nu(\mathbb{D})$:

$$H^p_\nu(\mathbb{D}) = \text{measurable functions } f \text{ on } \mathbb{D} \text{ such that } \sup_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} := \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} < +\infty$$

and solving (CB) in the sense of distributions in $\mathbb{D}$

3. Properties of $H^p_\nu(\mathbb{D})$:

- Equipped with the norm $\|.\|_{H^p_\nu(\mathbb{D})}$, $H^p_\nu(\mathbb{D})$ is a Banach space,
- If $\nu = 0$, $H^p_0(\mathbb{D}) = H^p(\mathbb{D})$, the classical Hardy space of holomorphic funtions on $\mathbb{D}$ such that $\sup_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} < \infty$,
- Each $f \in H^p_\nu(\mathbb{D})$ has a non-tangential limit trf a.e on $\mathbb{T}$ called the trace of f,
- $trH^p_\nu(\mathbb{D})$ is a closed subspace of $L^p(\mathbb{T})$,
- On $I \subset \mathbb{T}, |I| > 0, (trf)_{|_I} = 0 \Longrightarrow f \equiv 0$ in $\mathbb{D}$,

Moreover, the Hilbert transform $\mathcal{H}$ and the Riesz-projection $\mathcal{P}$ are naturally extended to $\mathcal{H}_\nu$ and $\mathcal{P}_\nu$ with

$$\mathcal{H}_\nu : \quad L^p_{\mathbb{R}}(\mathbb{T}) \to L^p_{\mathbb{R}}(\mathbb{T}) \qquad Re(trf) = u_{|_\mathbb{T}} \mapsto Im(trf) = v_{|_\mathbb{T}}$$

and

$$\mathcal{P}_\nu : L^p_{\mathbb{R}}(\mathbb{T}) \to L^p_{\mathbb{R}}(\mathbb{T}) \qquad g \mapsto \frac{1}{2}(I + i\mathcal{H}_\nu)g + \frac{1}{4\pi}\int_\mathbb{T} g\,ds$$

Density result:

$$\forall\, I \subset \mathbb{T} \text{ such that } |\mathbb{T}\backslash I| > 0,\ trH^p_\nu(\mathbb{D})_{|_I} \text{ is dense in } L^p(I)$$

Solution of the Dirichlet problem:

$$\forall\,\varphi \in L^p_{\mathbb{R}}(\mathbb{T}), \exists!\, f = u + iv \in H^p_\nu(\mathbb{D}) \text{ with } \int_\mathbb{T} v = 0 \text{ such that,a.e on } \mathbb{T},\ Re(trf) = \varphi$$

4. The Cauchy problem:

Instability of the Cauchy solution:

data $\phi_{|_I}, \varphi_{|_I} \in L^p_{\mathbb{R}}(I)$ and $f_{|_I} = \phi_{|_I} + i\varphi_{|_I}$

- if $\phi_{|_I}$ and $\varphi_{|_I}$ satisfy generalized Cauchy-Riemann equations $\rightarrow$ $f_{|_I} \in trH^p_\nu(\mathbb{D})_{|_I}$ $\rightarrow$ $\exists!\, g_0 \in H^p_\nu(\mathbb{D})$, $\|g_0 - f_{|_I}\|_{L^p(I)} = \min_{g \in H^p_\nu(\mathbb{D})} \|g - f_{|_I}\|_{L^p(I)}$
- if $\phi_{|_I}$ and $\varphi_{|_I}$ don't satisfy generalized Cauchy-Riemann equations $\rightarrow$ $f_{|_I} \notin trH^p_\nu(\mathbb{D})_{|_I}$ $\rightarrow$ $\exists\, g_n \in H^p_\nu(\mathbb{D})$, $\|g_n - f_{|_I}\|_{L^p(I)} \to 0$ and $\lim_{n\to\infty} \|g_n\|_{L^p(J)} = \infty$

Bounded extremal problem :

By constraining Ref on J, Cauchy problem admits a unique solution. That solution is the best approximation of $\phi_{|_I}$ and $\varphi_{|_I}$. In case $p = 2$ and for $M > 0$, it is given by

$$g_0(\lambda) = (I + \lambda T)^{-1}\,\mathcal{P}_\nu(f_{|_I} \vee (\lambda + 1)h) \quad \text{with} \quad \|Ref - h\|_{L^2_{\mathbb{R}}(J)} \leq M$$

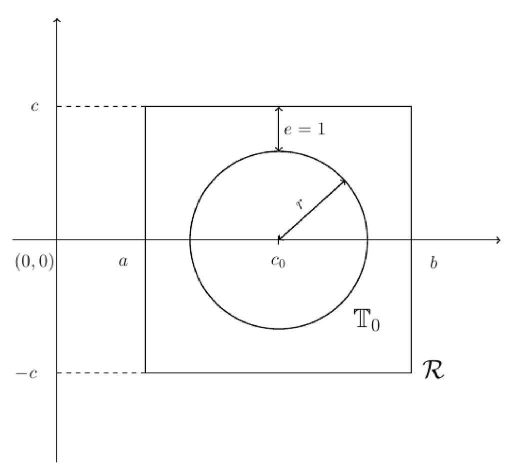
where $T(f) = \mathcal{P}_\nu(\chi_J f)$ is a Toeplitz operator and $\lambda \in (-1, +\infty)\,/\,\|Ref - h\|_{L^2_{\mathbb{R}}(J)} = M$

Bases of solutions in $trH^2_\nu(\mathbb{D}_0)$ and numerical results:

Solutions on a rectangle $\mathcal{R} \supset \mathbb{D}_0$:

$\mathbb{D}_0 = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + y^2 < R^2\}$

with $x_0 > 0$ and $R < |x_0|$



On $\mathcal{R}$, explicit solutions are, for every $N \in \mathbb{N}$, of Bessel-exponential type

$$b_1(x, y) = \sum_{n=1}^{N} xJ_1(\lambda_n x)[\alpha_n e^{\lambda_n y} + \beta_n e^{-\lambda_n y}] + \sum_{n=1}^{N} x[\gamma_n I_1(\mu_n x) + \delta_n K_1(\mu_n x)]\sin(\mu_n(y + c)) + a_0 x^2 + b_0 y + c_0$$

$$b_2(x, y) = \sum_{n=1}^{N} J_0(\lambda_n x)[\alpha_n e^{\lambda_n y} - \beta_n e^{-\lambda_n y}] + \sum_{n=1}^{N} [\delta_n K_0(\mu_n x) - \gamma_n I_0(\mu_n x)]\cos(\mu_n(y + c)) - b_0 \ln x + 2a_0 y + d_0$$

where $\mu_n = \dfrac{n\pi}{4c}$ and $(\lambda_n)_{n\geq 1}$ denotes zeros of $J_0(\lambda_n b) = 0$.

Density result: Define as $\mathcal{B}_1$ and $\mathcal{B}_2$ the families of solutions b_1 and b_2 given above, then

$$\mathcal{B}_{1_{|_{\mathbb{T}_0}}} \text{ and } \mathcal{B}_{2_{|_{\mathbb{T}_0}}} \text{ are } L^2 \text{ dense in } L^2_{\mathbb{R}}(\mathbb{T}_0) = Re(trH^2_\nu(\mathbb{D}_0))$$

Constructive aspects of solutions to the bounded extremal problem:

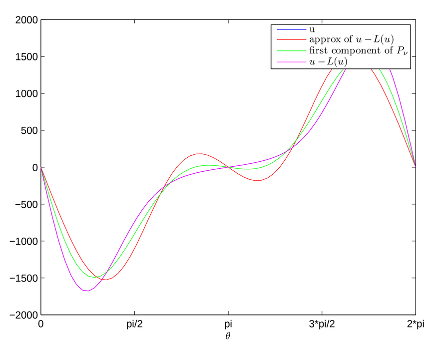
$u_{|_{\mathbb{T}_0}} \in L^2_{\mathbb{R}}(\mathbb{T}_0)$, $v_{|_{\mathbb{T}_0}} \in L^2_{\mathbb{R}}(\mathbb{T}_0)$ $\xrightarrow{L^2\text{ - density}}$ $b_1 \in \mathcal{B}_1, \|u - b_1\|_{L^2_{\mathbb{R}}(\mathbb{T}_0)} \longrightarrow 0$, $b_2 \in \mathcal{B}_2, \|v - b_2\|_{L^2_{\mathbb{R}}(\mathbb{T}_0)} \longrightarrow 0$ $\xrightarrow{\text{continuity of } \mathcal{H}_\nu}$ $\|\mathcal{H}_\nu(u) - \mathcal{H}_\nu(b_1)\|_{L^2_{\mathbb{R}}(\mathbb{T}_0)} \longrightarrow 0$, $\|\mathcal{H}_\nu(v) - \mathcal{H}_\nu(b_2)\|_{L^2_{\mathbb{R}}(\mathbb{T}_0)} \longrightarrow 0$ $\rightarrow$ $\mathcal{P}_\nu(b_1, b_2)$ and g_0

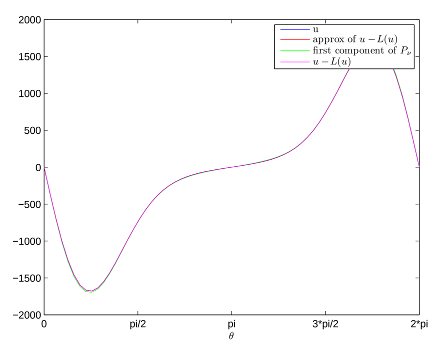
Numerical approximations:

Approximations are computed with $(c_0, r, e, N_1, N_2) = (5, 2, 1, 1, 5)$ and $f_{|_{\mathbb{T}_0}} = u_{|_{\mathbb{T}_0}} + iv_{|_{\mathbb{T}_0}} \in trH^2_\nu(\mathbb{D}_0)$.

- $u_{|_{\mathbb{T}_0}} = (x^2y^3 - \frac{3}{4}x^4y)_{|_{\mathbb{T}_0}}$

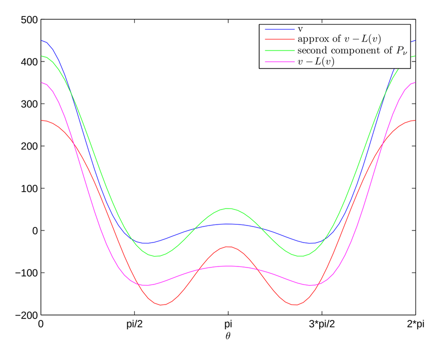
  Error is about 10^{-2} for $N = 5$

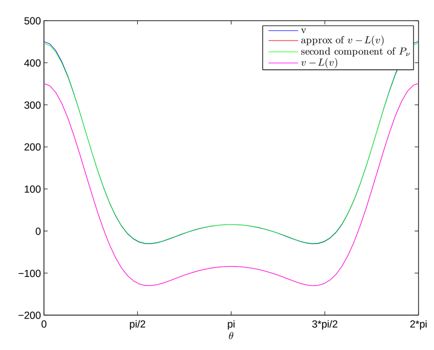




- $v_{|_{\mathbb{T}_0}} = (\frac{y^4}{2} - \frac{3}{2}x^2y^2 + \frac{3}{16}x^4)_{|_{\mathbb{T}_0}}$

  Error is about 10^{-3} for $N = 5$





Perspectives:

- In an annular domain $\mathbb{A} = \mathbb{D}\backslash\rho\mathbb{D}$? Use of the topological decomposition $H^p_\nu(\mathbb{A}) = H^p_{\nu_i}(\mathbb{D}) \oplus H^p_{\nu_e}(\mathbb{C}\backslash\rho\mathbb{D})$, where $\nu_i \in W^{1,\infty}(\mathbb{D}), \nu_e \in W^{1,\infty}(\mathbb{C}\backslash\rho\mathbb{D})$ such that $\nu_{i_{|\mathbb{A}}} = \nu_{e_{|\mathbb{A}}} = \nu$.

References:

[1]: K. Astala, L. Päivärinta, Calderón's inverse conductivity problem in the plane, *Ann. of Math.* (2) 163 (2006), no. 1, 265–299.
[2]: L. Baratchart, J. Leblond, S. Rigat, E. Russ, Hardy spaces for the conjugate Beltrami equation, submitted.
[3]: Y. Fischer, J. Leblond, Solutions to conjugate Beltrami equations equations and approximation in generalized Hardy spaces, *Advances in Pure and Applied Mathematics*, to appear.