Identification de l'équilibre du plasma dans un Tokamak en temps réel

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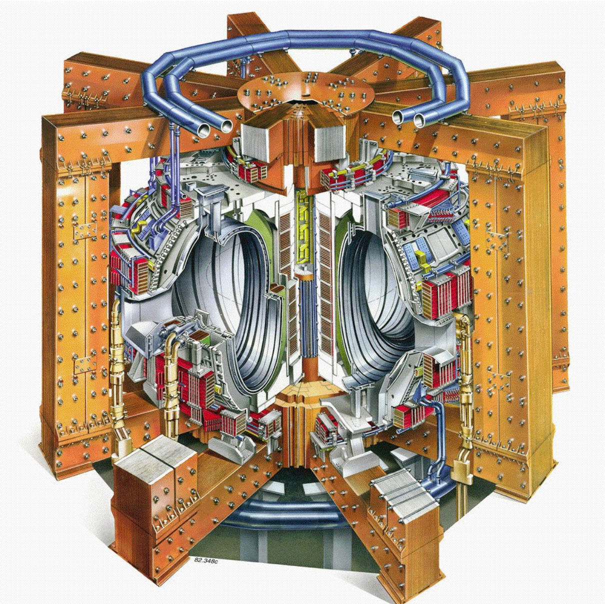
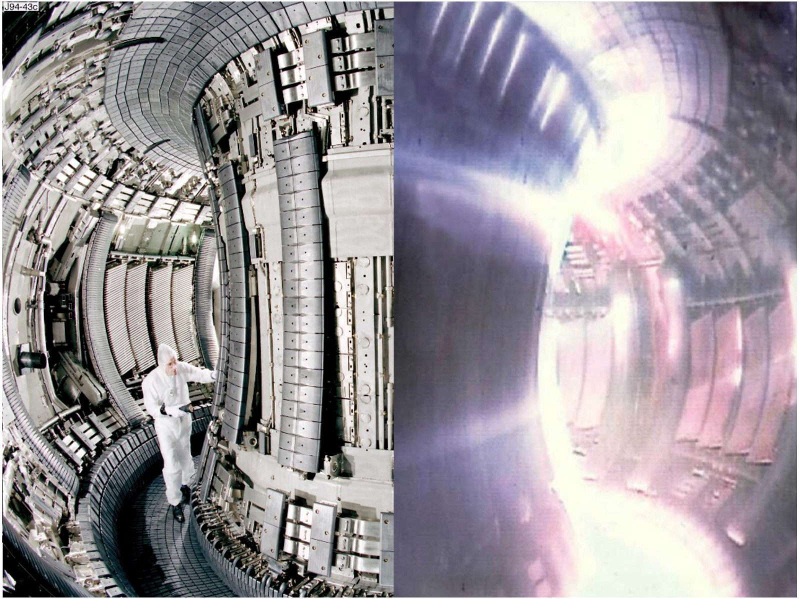
Outline

JET Tokamak



JET : vacuum vessel and plasma



Introduction

- Equilibrium of a plasma : a free boundary problem
- Equilibrium equation inside the plasma, in an axisymmetric configuration : Grad-Shafranov equation
- Right-hand side of this equation is a non-linear source : the toroidal component of the plasma current density j_ϕ

Aim of this work

Perform the real-time identification of this non-linearity from experimental measurements

Mathematical modelling of the equilibrium

- Momentum equation :

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}) + \nabla p = \mathbf{j} \times \mathbf{B}$$

- At the slow resistive diffusion time scale

$$\rho(\frac{\partial u}{\partial t} + \mathbf{u}.\nabla\mathbf{u}) \ll \nabla p$$

Equilibrium equations

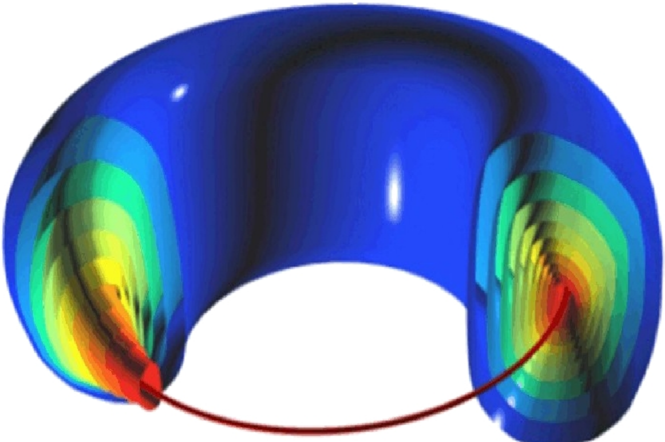
$$\begin{cases} \nabla p = \mathbf{j} \times \mathbf{B} & \text{(Conservation of momentum)} \\ \nabla.\mathbf{B} = 0 & \text{(Conservation of } \mathbf{B}\text{)} \\ \nabla \times \mathbf{B} = \mu\mathbf{j} & \text{(Ampere's law)} \end{cases}$$

$+$ axisymmetric assumption $=>$ Grad-Shafranov equation

Magnetic surfaces

$$\nabla p \cdot \mathbf{B} = 0 \text{ and } \nabla p \cdot \mathbf{j} = 0$$

=> Field lines and current lines are on isobaric surfaces ($p = cst$) = magnetic surfaces



Axisymmetric configuration

$\nabla \mathbf{B} = 0$, conservation of $\mathbf{B}$

- Cylindrical coordinates $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)$
- $\mathbf{B}$ independent of ϕ
- $\mathbf{B} = (B_r \mathbf{e}_r + B_z \mathbf{e}_z) + B_\phi \mathbf{e}_\phi = \mathbf{B}_p + \mathbf{B}_\phi$
- Poloidal flux : $\psi(r, z) = \frac{1}{2\pi} \int_D \mathbf{B} d s = \int_0^r B_z r dr$

- $\mathbf{B}_p = \frac{1}{r} \nabla \psi^\perp = \frac{1}{r} (\nabla \psi \times \mathbf{e}_\phi)$
- $\mathbf{B}.\nabla \psi = 0$. $\psi = cst$ on each magnetic surface. $p = p(\psi)$

$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, Ampere's law

$\nabla \mathbf{j} = 0$

- $\mathbf{j} = \mathbf{j}_p + \mathbf{j}_\phi$.
- $\mathbf{j}_p = \frac{1}{r}(\nabla(\frac{f}{\mu_0}) \times \mathbf{e}_\phi)$
- $\mathbf{j}.\nabla p = 0$. $\nabla f \times \nabla p = 0$. $f = f(\psi)$ diamagnetic function

- $\mathbf{B}_\phi = \frac{f}{r}\mathbf{e}_\phi$
- $\mathbf{j}_\phi = (-\Delta^* \psi)\mathbf{e}_\phi$

with

$$\Delta^*. = \frac{\partial}{\partial r}(\frac{1}{\mu_0 r}\frac{\partial .}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{\mu_0 r}\frac{\partial .}{\partial z})$$

Grad-Shafranov equation

$\mathbf{j} \times \mathbf{B} = \nabla p$, equilibrium equation

- $(\mathbf{j}_p + j_\phi \mathbf{e}_\phi) \times (\mathbf{B}_p + B_\phi \mathbf{e}_\phi) = -\frac{1}{\mu_0 r} B_\phi \nabla f + j_\phi \frac{1}{r} \nabla \psi = \nabla p.$
- $\nabla p = p'(\psi) \nabla \psi$ and $\nabla f = f'(\psi) \nabla \psi$

Grad-Shafranov equation

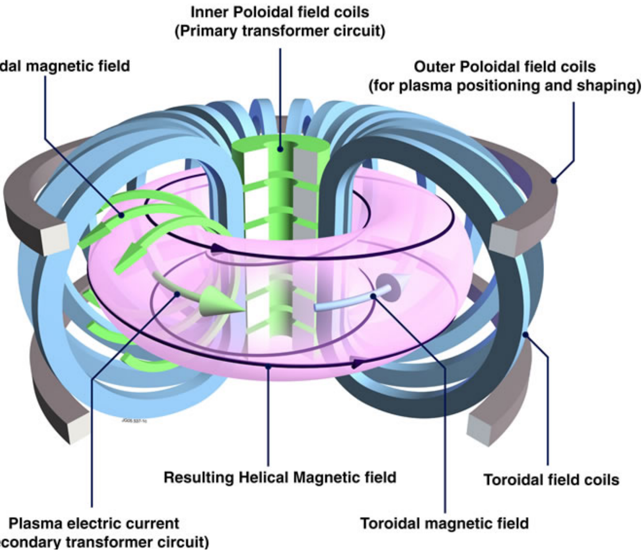
In the plasma

$$-\Delta^* \psi = r p'(\psi) + \frac{1}{\mu_0 r} (f f')(\psi)$$

In the vacuum

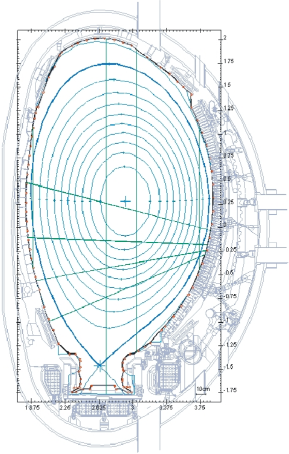
$$-\Delta^* \psi = 0$$

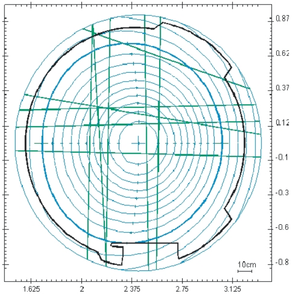
Tokamak



Inner Poloidal field coils
(Primary transformer circuit)
Poloidal magnetic field
Outer Poloidal field coils
(for plasma positioning and shaping)
Resulting Helical Magnetic field
Toroidal field coils
Plasma electric current
(secondary transformer circuit)
Toroidal magnetic field

Definition of the free plasma boundary





Two cases :

- magnetic separatrix : hyperbolic line with an X-point (left)
- outermost flux line inside the limiter (right)

Equation for $\psi(r, z)$ inside the vacuum vessel

$$\begin{cases} -\Delta^* \psi &= \lambda[\frac{r}{R_0} A(\bar{\psi}) + \frac{R_0}{r} B(\bar{\psi})]\chi_{\Omega_p} \quad \text{in } \Omega \\ \psi &= h \quad \text{on} \quad \partial\Omega \end{cases}$$

where

- R_0 major radius of the torus
- $p'(\bar{\psi}) = \lambda \frac{1}{R_0} A(\bar{\psi})$
- $\frac{1}{\mu_0}(ff')(\bar{\psi}) = \lambda R_0 B(\bar{\psi})$,
- $\bar{\psi} = \frac{\psi - \max_\Omega \psi}{\psi_b - \max_\Omega \psi} \in [0, 1]$ in Ω_p
- χ_{Ω_p} is the characteristic function of Ω_p

The inverse equilibrium problem : experimental measurements

- magnetic measurements on the vacuum vessel

$$\psi(M_i) = g_i \text{ and } \frac{1}{r}\frac{\partial \psi}{\partial n}(N_j) = h_j \text{ on } \partial\Omega,\ I_p = \int_{\Omega_p} j_\phi dx$$

- interferometry and polarimetry on several chords

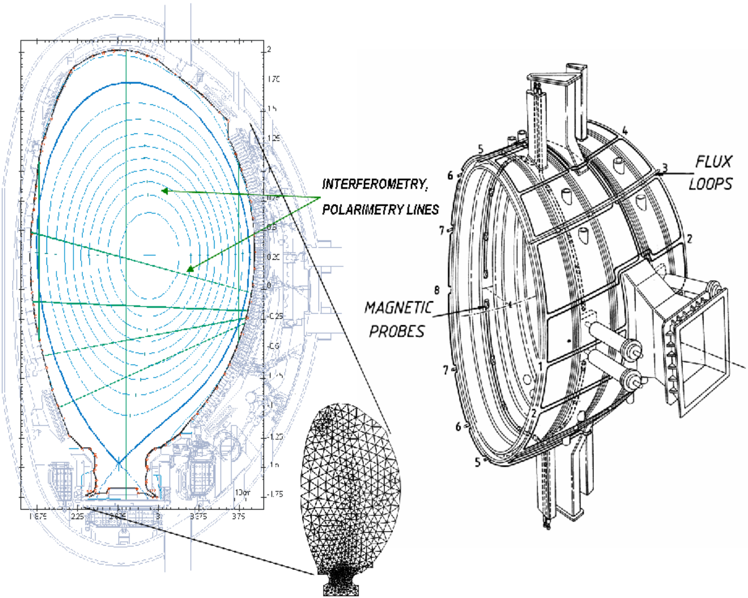
$$\int_{C_m} n_e(\psi) dl = \alpha_m,\ \int_{C_m} \frac{n_e(\psi)}{r}\frac{\partial \psi}{\partial n} dl = \beta_m$$

- kinetic pressure

$$p(r,0) = p_d(r)$$

- motional Stark effect

$$f_j(B_r(M_j), B_z(M_j), B_\phi(M_j)) = \gamma_j$$


INTERFEROMETRY,
POLARIMETRY LINES
FLUX
LOOPS
MAGNETIC
PROBES

Statement of the inverse problem

State equation

$$\begin{cases} -\Delta^*\psi = \lambda[\frac{r}{R_0}A(\bar{\psi}) + \frac{R_0}{r}B(\bar{\psi})]1_{\Omega_p} & \text{in } \Omega \\ \psi = g & \text{on } \partial\Omega \end{cases}$$

Least square minimization

$$J(A, B, n_e) = J_0 + J_1 + J_2 + J_\epsilon$$

with

$$J_0 = \sum_k w_k^2 (\frac{1}{r}\frac{\partial\psi}{\partial n}(N_k) - h_k)^2$$

$$J_1 = \sum_k w_k^2 (\int_{C_k} \frac{n_e(\bar{\psi})}{r}\frac{\partial\psi}{\partial n} dl - \alpha_k)^2$$

$$J_2 = \sum_k w_k^2 (\int_{C_k} n_e(\bar{\psi}) dl - \beta_k)^2$$

Statement of the inverse problem (2)

Tikhonov regularization

$$J_\epsilon = \epsilon_A \int_0^1 (A''(x))^2 dx$$
$$+\epsilon_B \int_0^1 (B''(x))^2 dx$$
$$+\epsilon_{n_e} \int_0^1 (n_e''(x))^2 dx$$

Find A_0, B_0, n_{e0} such that :

$$J(A_0, B_0, n_{e0}) = inf \; J(A, B, n_e)$$

Numerical identification

Finite element resolution

$$\begin{cases} \text{Find } \psi \in H^1 \text{ with } \psi = g \text{ on } \partial\Omega \text{ such that} \\ \forall v \in H_0^1, \int_\Omega \frac{1}{\mu_0 r} \nabla\psi\nabla v dx = \int_{\Omega_p} \lambda[\frac{r}{R_0} A(\bar{\psi}) + \frac{R_0}{r} B(\bar{\psi})] v dx \end{cases}$$

with

$$A(x) = \sum_i a_i f_i(x), \quad B(\psi) = \sum_i b_i f_i(x), \; u = (a_i, b_i)$$

Fixed point

$$K\psi = Y(\psi)u + g$$

K modified stiffness matrix, u coefficients of A and B, g Dirichlet BC

Direct solver : $(\psi^n, u) \to \psi^{n+1}$

$$\psi^{n+1} = K^{-1}[Y(\psi^n)u + g]$$

Numerical identification (2)

Least-square minimization

$$J(u) = \|C(\psi)\psi - d\|^2 + u^T A u$$

- d : experimental measurements
- A : regularization terms

Approximation

$J(u) = \|C(\psi^n)\psi - d\|^2 + u^T Au$, with $\psi = K^{-1}[Y(\psi^n)u + g]$

$$\begin{aligned} J(u) &= \|C(\psi^n)K^{-1}Y(\psi^n)u + C(\psi^n)K^{-1}g - d\|^2 + u^T Au \\ &= \|E^n u - F^n\|^2 + u^T Au \end{aligned}$$

Normal equation. Inverse solver : $\psi^n \to u$

$$(E^{nT}E^n + A)u = E^{nT}F^n$$

EQUINOX

One equilibrium reconstruction

- First guess $(\psi^0, u^0, \lambda^0, \Omega_p^0)$
- Direct and Inverse problem fixed-point iterations :
  - n_e identification (normal equation, ψ^n)
  - Update $\lambda^{n+1} = I_p / \int_{\Omega_p^n} [\frac{r}{R_0} A^n(\bar{\psi}^n) + \frac{R_0}{r} B^n(\bar{\psi}^n)] dx$
    and scale DOFs $(m = \max |a_i|,\ u \leftarrow \frac{1}{m} u)$
  - Inverse solver : $\psi^n \rightarrow u^{n+1}$
  - Direct solver : $(\psi^n, u^{n+1}) \rightarrow \psi^{n+1},\ \Omega_p^{n+1}$
- Stopping condition $\frac{||\psi^{n+1} - \psi^n||}{||\psi^n||} < \epsilon$

EQUINOX

Real-time

- Quasi-static approach. Maximum number of fixed-point iterations
- Cheap operations :
  - $K = LU$ and K^{-1} computed and stored once for all
  - Normal equation : ≈ 10 basis functions $\rightarrow$ small $\approx 20 \times 20$ linear system
  - Tikhonov regularization parameters unchanged (Lcurve)
- Expensive operations :
  - Update of matrices $C = \begin{pmatrix} C_{mag} \\ C_{polar}(\psi) \end{pmatrix}$ and $Y(\psi)$
  - Computation of products $C(\psi)K^{-1}$ and $C(\psi)K^{-1}Y(\psi)$

EQUINOX

- Available basis for A, B and n_e
  - Piecewise linear
  - Cubic B-splines
  - Scaling functions (Average Interpolating wavelets)

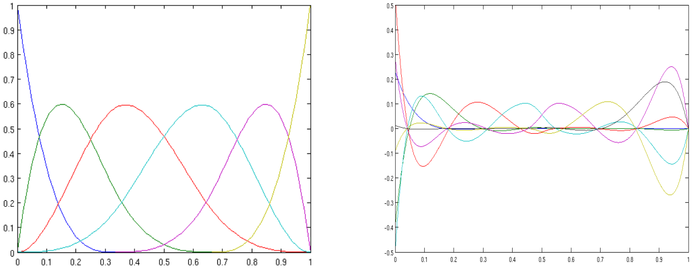


FIG.: Bsplines (left), Scaling functions (right)

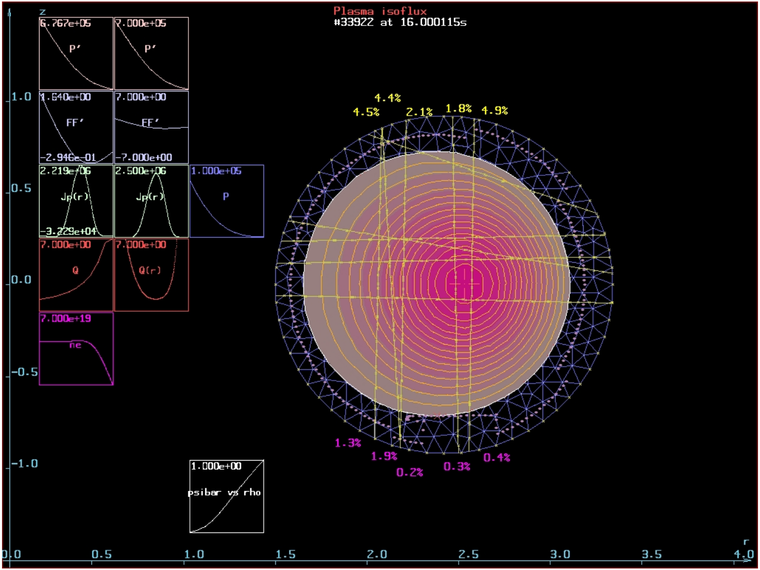
- C++, fully object-oriented

Numerical Results : Tore Supra and JET characteristics

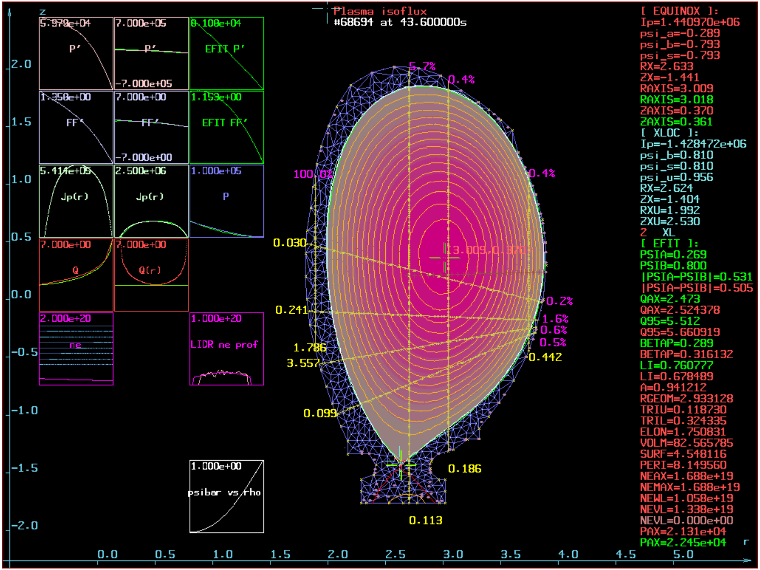
	ToreSupra	JET
Finite element mesh		
Number of triangles	1382	2871
Number of nodes	722	1470
functions A and B		
Basis type	Bspline	Bspline
Number of basis func.	8	8
Computation time (1.80GHz)		
One equilibrium	20 ms	60 ms

Real-time requirement : 100 ms

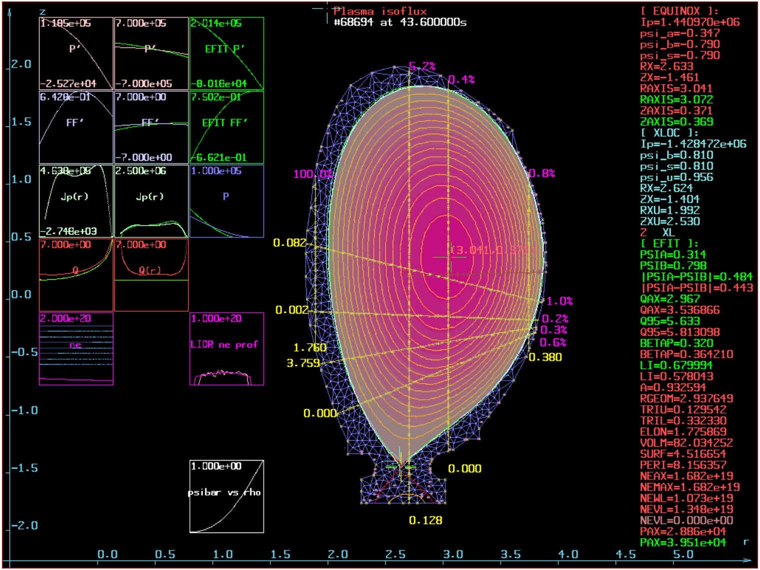
Tore Supra. Magnetics and polarimetry.



Jet 68694. Magnetics only.



Jet 68694. Magnetics and polarimetry.



Conclusion

- EQUINOX : Real-time identification of the current density and equilibrium reconstruction
- Makes possible future real-time control of the current profile

Some definitions

- Poloidal beta

  The poloidal beta is defined as the ratio of the mean kinetic pressure of the plasma to its magnetic pressure (Wesson p 116) :

$$\beta_p = \frac{\bar{p}}{B_{pa}^2/2\mu_0} \tag{5.1}$$

where

$$\bar{p} = \frac{\int_D p dv}{\int_D dv} = \frac{\int_{\Omega_p} prds}{\int_{\Omega_p} rds}$$

and

$$B_{pa} = \frac{\int_{\Gamma_p} B_p dl}{\int_{\Gamma_p} dl} = \frac{\mu_0 I_p}{L}$$

- Internal inductance

$$l_i = \frac{\bar{B}_p^2}{B_{pa}^2}$$

where

$$\bar{B}_p = \frac{\int_D B_p^2 dv}{\int_D dv}$$

- Safety factor
So called because of the role it plays in determining stability. In general terms higher values of q lead to greater stability.

$$q = \frac{\Delta\phi}{2\pi}$$

It represents the ratio of the variation of the toroidal angle needed for a magnetic field line to perform one poloidal turn.

$$q = \frac{1}{2\pi} \int_C \frac{B_\phi}{rB_p} dl$$

where C is a contour $\psi = cst$.