Stability and reconstruction for some inverse problems on the identification of boundary terms and inverse scattering

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Outline

The corrosion problem : physical models

The voltage potential u satisfies

$$\Delta u = 0 \text{ in } \Omega \ ,$$

where Ω represents the electrostatic conductor.

The boundary conditions

L) A *linear* boundary condition

$$\frac{\partial u}{\partial \nu} = -\varphi u \quad ,$$

where $\varphi \geqslant 0$ is the *Robin coefficient*.

NL) A *nonlinear* boundary condition due to *Butler* and *Volmer*

$$\frac{\partial u}{\partial \nu} = \lambda(\exp(\alpha u) - \exp(-(1-\alpha)u)) \ .$$

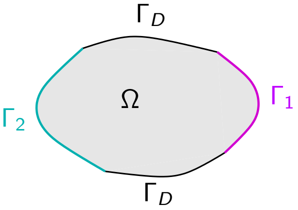
Some results on the *linear* model

- P.G. Kaup, F. Santosa, M. Vogelius (1995) - Reconstruction of the profile loss by thin plate approximation.
- D.Fasino, G.Inglese (1999, 2001) - Reconstruction of the Robin coefficient by thin plate approximation and Galerkin method.
- S.Chaabane, I.Fellah, M.Jaoua, J.Leblond (1999, 2003) - Logarithmic stability in 2D and directional Lipschitz stability for the Robin coefficient.
- G.Alessandrini, L.Del Piero, L.Rondi (2003) - Logarithmic stability result in 2D for the Robin coefficient.
- E.S.(2007) - Lipschitz stability for a piecewise constant Robin coefficient.

The Robin problem

The simplified mathematical model which describes the electrochemical phenomenon of surface *corrosion* in metals is the following

$$
\begin{cases}
\Delta u = 0 & \text{in } \Omega \, , \\
\dfrac{\partial u}{\partial \nu} = g & \text{on } \Gamma_2 \, , \\
\dfrac{\partial u}{\partial \nu} = -\gamma(x) u & \text{on } \Gamma_1 \, , \\
u = 0 & \text{on } \Gamma_D \, .
\end{cases}
\qquad \textbf{(R)}
$$



where Ω=electrostatic conductor, u=electrostatic potential, g=prescribed current density, γ=*Robin coefficient* due to the corrosion damage, Γ_2=*accessible* portion, Γ_1=*inaccessible* portion, Γ_D=grounded portion.

An inverse problem for a piece-wise constant Robin coefficient

To determine the **Robin coefficient** γ from the knowledge of the *electrostatic potential* $u|_{\Gamma_2}$ and from the *current density* $\frac{\partial u}{\partial \nu}|_{\Gamma_2}$ provided the following *a priori* hypothesis hold,

i) some bounds on the current density :
$\|g\|_{C^{0,\alpha}(\Gamma_2)} \leqslant G$, $\|g\|_{L^\infty(\Gamma_2)} \geqslant m$;

ii) a priori assumption on the *Robin* term :

$$\gamma(x) = \sum_{j=1}^{N} \gamma^j \chi_{\Gamma_1^j}(x) , \quad 0 \leqslant \gamma^j < J \quad , \qquad j = 1, \cdots , N$$

where, for any $j = 1, \ldots, N$, γ^j are real *unknown* numbers and Γ_1^j are *known* and disjoint portions of Γ_1 such that $\cup_{\mathbf{j=1}}^{\mathbf{N}} \overline{\mathbf{\Gamma_1^j}} = \overline{\mathbf{\Gamma_1}}$.

The stability result

Theorem (E.S. - Inverse problem, 2007)

Let $\gamma_i,\ i = 1, 2$ be two piecewise constant Robin coefficients of the form

$$\gamma_i(x) = \sum_{j=1}^{N} \gamma_i^j \chi_{\Gamma_1^j}(x)\ ,\quad x \in \Gamma_1,\quad i = 1, 2\ .$$

Let $u_i \in H^1(\Omega),\ i = 1, 2$ be the two weak solutions to the problem **(R)** *with $\gamma = \gamma_i$ respectively.*
Then there exists a constant $C > 0$ depending on the a-priori data only such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Gamma_1)} \leqslant C\|u_1 - u_2\|_{L^2(\Gamma_2)}\ .$$

The exponential behavior of the Lipschitz constant

Corollary (E.S. - Inverse problems, 2007)

There exists $N_0 \in \mathbb{N}$ such that for any $N \geqslant N_0$ we have that

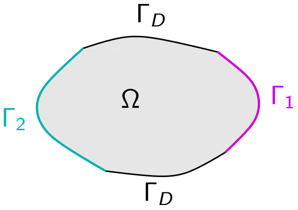
$$\mathbf{C_N} \geqslant \mathbf{k_1} \exp(\mathbf{k_2 N}^{\frac{1}{2n-1}})$$

where k_1, k_2 positive constants depending on the a priori data only and $C = C_N$ is the Lipschitz constant in the previous Lipschitz stability estimate.

The nonlinear profile problem

The more accurate mathematical model which describes the electrochemical phenomenon of surface *corrosion* in metals is the following

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \ , \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \Gamma_2 \ , \\ \dfrac{\partial u}{\partial \nu} = f(u) & \text{on } \Gamma_1 \ , \\ u = 0 & \text{on } \Gamma_D \ . \end{cases} \qquad \textbf{(C)}$$



where Ω=electrostatic conductor, u=electrostatic potential, g=prescribed current density, f=*nonlinear term* due to the corrosion damage, Γ_2=*accessible* portion, Γ_1=*inaccessible* portion, Γ_D=grounded portion.
The boundary value problem **(C)** *might not be well posed*, indeed this the case when $g = 0$ and $f(u) = pu$, where $p > 0$ is an eigenvalue of a Steklov type eigenvalue problem.

Some results on the *nonlinear* model

- K. Bryan, O. Kavian, M.Vogelius, J.M.Xu (1998-2002) - Existence and uniqueness of solutions to the *direct* problem.
- G. Alessandrini, E.S. (2005-2006) - Logarithmic stability and reconstruction for the nonlinear corrosion profile.
- D. Fasino, G. Inglese (2005) - Logarithmic stability in 2D and numerical approximation of the nonlinear corrosion profile.
- S. Chaabane, M. El Guénichi, J. Leblond, M. Zghal (2006) - Identification, stability and BEP algorithm for the nonlinear term in 2D.
- H. Cao, V. Pereverzev, E.S. (2007) - Regularized reconstruction algorithm for the identification of the nonlinear term.
- P. Kügler, E.S. (2008) - Tikhonov regularization and convergence rates for the determination of the nonlinear term.

The inverse problem

To determine the **nonlinear coefficient** f by the knowledge of the *voltage potential* $u|_{\Gamma_2}$ and the *current density* $\frac{\partial u}{\partial \nu}|_{\Gamma_2}$ provided the following *a priori* assumptions hold,

i) an energy bound :
$\int_\Omega |\nabla u|^2 \leqslant E^2$;

ii) some bounds on the current density :
$\|g\|_{C^{0,\alpha}(\Gamma_2)} \leqslant G$, $\qquad \|g\|_{L^\infty(\Gamma_2)} \geqslant m$;

iii) a priori assumptions on the *nonlinear* term :
$f : \mathbb{R} \to \mathbb{R}, \quad f(0) = 0$,
$|f(u) - f(v)| \leqslant L|u - v|$, for every $u, v \in \mathbb{R}$.

The stability result for f

Theorem (G. Alessandrini, E.S. - Applicable Analysis, 2006)

Let $u_i \in H^1(\Omega)$ *be weak solutions to the problem* **(C)**, *with* $f = f_i$ *and* $g = g_i$ *respectively and let* $\psi_i = u_i|_{\Gamma_2}$, $i = 1, 2$. *If*

$$\|\psi_1 - \psi_2\|_{L^2(\Gamma_2)} \leqslant \varepsilon \ ,$$

$$\|g_1 - g_2\|_{L^2(\Gamma_2)} \leqslant \varepsilon \ ,$$

then

$$\|f_1 - f_2\|_{L^\infty(V)} \leqslant C|\log(\varepsilon)|^{-\theta} \ ,$$

where

$$V = (\alpha, \beta) \subseteq [-CE, CE] \ ,$$

is such that

$$\beta - \alpha > \frac{\exp[-(m/c)^{-\gamma}]}{2}$$

with $0 < \theta < 1, C, c > 0, \gamma > 1$.

The reconstruction result for f - 1st approach

By the approximate electrostatic measurements $\{\psi_\varepsilon, g_\varepsilon\}$ of $\left\{u|_{\Gamma_2}, \frac{\partial u}{\partial \nu}|_{\Gamma_2}\right\}$ we want to recover an approximate profile f_ε. We reformulate the Cauchy problem associated to the **(C)** in terms of the regularized inversion of the following compact operator

$$T : \frac{\partial u}{\partial \nu}\Big|_{\Gamma_1} \rightarrow \frac{\partial u}{\partial \nu}\Big|_{\Gamma_2} .$$

Theorem (G. Alessandrini, E.S. - J. Comput. Appl. Math., 2007)

If

$$\|\psi - \psi_\varepsilon\|_{H^{\frac{1}{2}}_{00}(\Gamma_2)} \leqslant \varepsilon \quad and \quad \|g - g_\varepsilon\|_{H^{\frac{1}{2}}_{00}(\tilde{\Gamma_2})^*} \leqslant \varepsilon ,$$

then

$$\|u_\varepsilon - u\|_{C^1(\Gamma_1)} \rightarrow 0 \quad as\ \varepsilon \rightarrow 0$$

where $u_\varepsilon \in H^1(\Omega)$ is a weak solution to a mixed boundary value problem defined by means of a regularization strategy R^ε for the compact operator T.

The reconstruction result for f - 1st approach

Theorem (G. Alessandrini, E.S. - J. Comput. Appl. Math., 2007)

Let $u \in H^1(\Omega)$ be a weak solution to the problem **(C)**, *with $\psi = u|_{\Gamma_2}$. If, given $\varepsilon > 0$, we have that $\psi_\varepsilon \in H^{\frac{1}{2}}_{00}(\Gamma_2)$ and $g_\varepsilon \in H^{\frac{1}{2}}_{00}(\Gamma_2^\rho)^*$*

$$\|\psi - \psi_\varepsilon\|_{H^{\frac{1}{2}}_{00}(\Gamma_2)} \leqslant \varepsilon \ ,$$

$$\|g - g_\varepsilon\|_{H^{\frac{1}{2}}_{00}(\Gamma_2^\rho)^*} \leqslant \varepsilon \ ,$$

then there exist an interval V such for a.e. $t \in V$

$$f_\varepsilon(t) \to f(t) \quad as \quad \varepsilon \to \ 0 \ ,$$

where

$$f_\varepsilon(t) = \frac{1}{\int_{\{x\in\Gamma_1 : u_{\varepsilon=t}\}} |\nabla_{x'} u_\varepsilon|^{-1} d\sigma_{n-2}} \int_{\{x\in\Gamma_1 : u_{\varepsilon=t}\}} \frac{\partial u_\varepsilon}{\partial \nu} |\nabla_{x'} u_\varepsilon|^{-1} d\sigma_{n-2} \ .$$

The reconstruction result for f - 2nd approach

(H. Cao, S. Pereverzev, E.S. - Ricam Report, submitted)

We split the original *nonlinear* problem in two *linear* problems.

a) We reduce the resolution of the Cauchy problem into the resolution of a linear operator equation which is regularized by discretization

b) Once we know by the step a) the Dirichlet and the Neumann traces of u on Γ_1 we can define the linear operator

$$B : f \to f(u(x))$$

and we solve the equation $Bf = \dfrac{\partial u(x)}{\partial \nu}$.

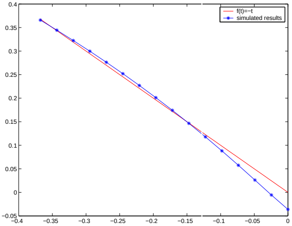
The reconstruction result for f - 2nd approach

(H. Cao, S. Pereverzev, E.S. - Ricam Report, submitted)

i) The linear case

$$\Omega = [0,\pi] \times [0,1]$$
$$\Gamma_2 = [0,\pi] \times [0] \ ,$$
$$\Gamma_1 = [0,\pi] \times [1] \ ,$$
$$\Gamma_D = \{0\} \times [0,1], \{\pi\} \times [0,1] \ .$$

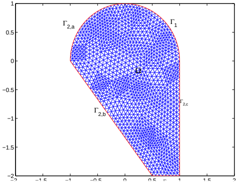
$$g = -\sin(x) \ ,$$
$$f(t) = -t \ ,$$
$$u(x,y) = [\sinh(y) - \cosh(y)] \sin(x) \ .$$

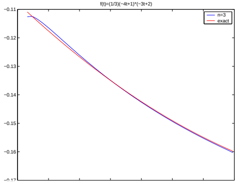


The reconstruction result for f - 2nd approach

(H. Cao, S. Pereverzev, E.S. - Ricam Report, submitted)

ii) The nonlinear case





$$g = -\frac{5y+4}{(4y+5)^2} \;,$$

$$f(t) = \begin{cases} \frac{1}{3}(-4t+1)(-3t+2) \text{ if } t \le \frac{5}{12} \\ -\frac{1}{3}t - \frac{1}{36} \text{ if } t \ge \frac{5}{12} \end{cases} \;,$$

$$u(x,y) = \frac{y+2}{(y+2)^2 + x^2} \;.$$

The reconstruction result for f - 3rd approach

(P.Kügler, E.S. - in preparation)

We consider the set of admissible profiles f

$$K = \{ f \in H^1(I) \ : \ f(0) = 0 \ , -L < f' < 0 \}$$

where $L > 0$ is a constant and the interval $I = [u_{min}, u_{max}]$ is such that $u_{min} < u_f < u_{max}$ holds for u_f solution to the direct problem **(C)** for any $f \in K$.
We denote with z_δ the noisy data

$$\|z - z_\delta\|_{L^2(\Gamma_2)} \leqslant \delta.$$

We assume that the exact data z is attainable from $f^\dagger \in K$.
Problem: For $\beta > 0$, find a parameter $f_\beta^\delta \in K$ that minimizes

$$J_\beta(f) = \int_{\Gamma_2} |u_f - z^\delta|^2 + \beta \|f - f^*\|_I^2$$

over K for a suitable choice of β and f^*.

The reconstruction result for f - 3rd approach

(P.Kügler, E.S. - in preparation)

We proved the

- Existence: a minimizer f_β^δ exists for any $z^\delta \in L^2(\Gamma_2)$;
- Stability: for a fixed regularization parameter β, the minimizers of J_β depend continuously on the data z^δ;
- Convergence: the regularized solutions f_β^δ converge toward the true parameter $f^\dagger$ as both the noise level δ and the regularization parameter β (chosen by an a priori rule) tend to zero;

  We found the following convergence rates when $\beta \sim \delta$

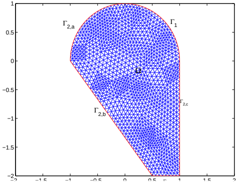
  - $\|u_{f_\beta^\delta} - z^\delta\|^2_{L^2(\Gamma_2)} = O(\delta)$;
  - $\|f_\beta^\delta - f^\dagger\|^2_{H^1(I)} = O(|\log(\delta)|^{-\theta})$;

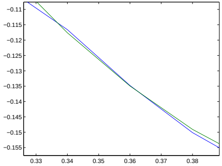
  where $0 < \theta < 1$.

The reconstruction result for f - 3rd approach

(P.Kügler, E.S. - in preparation)

ii) **The numerical test**





$$g = -\frac{5y+4}{(4y+5)^2} \ ,$$

$$f(t) = \begin{cases} \frac{1}{3}(-4t+1)(-3t+2) \text{ if } t \leq \frac{5}{12} \\ -\frac{1}{3}t - \frac{1}{36} \text{ if } t \geq \frac{5}{12} \end{cases} \ ,$$

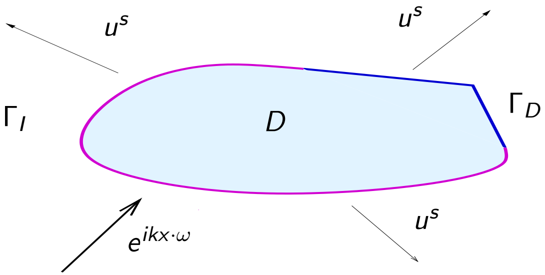
$$u(x,y) = \frac{y+2}{(y+2)^2 + x^2} \ .$$

Impedance scattering problem - Formulation

The scattering of an acoustic time-harmonic plane wave by an obstacle partially coated by a material with surface *impedance* λ is modeled by the following mixed boundary value problem for the Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ u = 0 & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} + i\lambda(x) u = 0 & \text{on } \Gamma_I, \\ \lim_{r\to\infty} r \left(\dfrac{\partial u^s}{\partial r} - iku^s \right) = 0, & r = \|x\|. \end{cases} \qquad \textbf{(Sc)}$$

Impedance scattering problem - Formulation



where D=obstacle, $u = e^{ikx\cdot\omega} + u^s$=acoustic field, $e^{ikx\cdot\omega}$=incident wave, u^s=scattered wave, k=wave number, ω=incident direction, λ=surface *impedance*, Γ_I=*coated* portion, Γ_D=remaining portion. Moreover Γ_I is $C^{1,1}$ smooth.

Remark:The *direct* problem is well-posed.

F. Cakoni. - D. Colton. - P. Monk, Inverse Problems 2001, $N = 2$, $\lambda = const. > 0$

Impedance scattering problem - The inverse problem

The scattered field u^s has the following asymptotic behavior

$$u^s(x) = \frac{\exp(ikr)}{r} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{r}\right) \right\} \quad ,$$

as r tends to ∞, uniformly with respect to $\hat{x} = \frac{x}{\|x\|}$.

To determine the **surface impedance** λ by the knowledge of the *far field pattern* u_∞ provided the following *a priori* assumptions hold,

i) bound on the Lipschitz continuity of the *impedance* :
$\|\lambda\|_{C^{0,1}(\Gamma_I)} \leqslant \Lambda$;

ii) uniform lower bound :
$\lambda(x) \geqslant \lambda_0 > 0$.

Impedance scattering problem - The stability theorem

By *stability* we mean the *quantitative* evaluation of the *continuous dependence* of the unknown impedance λ upon the far field measurement u_∞.

Theorem (E.S. - SIAM J. Math. Anal., 2006)

Let $u_i \in H^1_{loc}(\mathbb{R}^3 \setminus \bar{D})$ be the weak solutions to the problem **(Sc)** *with $\lambda = \lambda_i$ and $u_\infty = u_{i,\infty}$, $i = 1, 2$ respectively. If, for ε sufficiently small we have*

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \leqslant \varepsilon,$$

then

$$\|\lambda_1 - \lambda_2\|_{L^\infty(\tilde{\Gamma}_l)} \leqslant const.|\log(\varepsilon)|^{-\theta} .$$

Sound soft problem - Formulation

The scattering of an acoustic time-harmonic plane wave, at a given number $k > 0$ and at a given direction $\omega \in \mathbb{S}^2$ by a sound soft obstacle $D \in \mathbb{R}^3$ is modeled by the following Dirichlet problem for the Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathbf{D}}, \\ u = 0 & \text{on } \partial \mathbf{D}, \\ \lim\limits_{r \to \infty} r \left(\dfrac{\partial u^s}{\partial r} - iku^s \right) = 0, & r = \|x\|. \end{cases} \qquad \textbf{(Ss)}$$

where $u = e^{ikx\cdot\omega} + u^s$.

Sound soft problem - The inverse problem : $u_\infty \longrightarrow D$

About the literature ...

Uniqueness

A *classical* result due to Schiffer (1966): the knowledge of $u_\infty(\omega, \hat{x})$ *for all* $\omega, \hat{x} \in \mathbb{S}^2$ and at a *fixed* $k > 0$ uniquely determines the scattering obstacle.

Conjecture: *Formally* the obstacle D should be determined from its scattering amplitude at a fixed energy $k > 0$ and at a *fixed* incident direction $\omega \in \mathbb{S}^2$.

- The conjecture is still unproven for general domains D.
- On the contrary there are several uniqueness results when D has a geometrical constraint imposed, for example ...

Sound soft problem - Uniqueness under geometrical constraints

Smallness condition : If D is constrained to lie in a disk with a sufficiently small radius which depends on the wave number k.

- D. Colton, B.D. Sleeman, IMA J. Appl. Math. 1983.

Closeness condition : If D is sufficiently close to an obstacle of a known shape.

- R. Kress, W. Rundell, Inverse Problems 1994.
- P. Stefanov, G. Uhlmann, Proc. Amer. Math. Soc. 2003.

Polyhedral condition : If D is a polyhedral scatterer.

- C. Liu, A. Nachman, 1994.
- J. Cheng, M. Yamamoto, Inverse Problems 2003.
- G. Alessandrini, L. Rondi, Proc. Amer. Math. Soc. 2005.

SSP - The inverse problem with the closeness condition

To determine *locally* the sound soft obstacle **D** by the knowledge of the *far field pattern* u_∞ at a fixed incident direction ω and at a fixed energy $k > 0$ provided

i) there exist two obstacles D_+ and D_- such that

$$\text{Vol}(D_+ \setminus D_-) < \frac{4\pi^4}{3} k^{-3} ,$$

ii) $D_- \subset D \subset D_+$.

Remark: Let k_0^2 be a Dirichlet eigenvalue of $-\Delta$ in a bounded domain G then by the Faber-Krahn inequality

$$k_0^3 \geqslant \frac{4\pi^4}{3\text{Vol}(G)} .$$

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Sound soft problem - Uniqueness and stability

The uniqueness holds under the *closeness* condition.

Theorem (Stefanov-Uhlmann, Proc.Amer.Math.Soc. 2004)

If D_1 and D_2 are two obstacles satisfying the above assumptions and such that $u_{1,\infty} = u_{2,\infty}$ then $D_1 = D_2$.

The stability holds under the *closeness* condition.

Theorem (E.S., M. Sini - Ricam Report, submitted)

If D_1 and D_2 are two obstacles satisfying the above assumptions and such that $u_{1,\infty} = u_{2,\infty}$

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \leqslant \varepsilon$$

then

$$d_H(D_1, D_2) \leqslant const. |\log(\varepsilon)|^{-\theta} \ .$$

Merci!