On the convergence of Padé approximants to rational perturbations of Markov functions

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Definition

The $[L/M]$ Pade approximant for $F(\lambda) = \sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}}$ is a ratio

$$F^{[L/M]}(\lambda) = \frac{A^{[L/M]}(1/\lambda)}{B^{[L/M]}(1/\lambda)}$$

of polynomials $A^{[L/M]}$, $B^{[L/M]}$ of formal degree L, M, respectively, such that $B^{[L/M]}(0) \neq 0$ and

$$F(\lambda) - F^{[L/M]}(\lambda) = O\left(\frac{1}{\lambda^{L+M+1}}\right).$$

In the case when $L = M = n$, the $[n/n]$ Padé approximant is called the nth diagonal Padé approximant.

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Example

Let us consider

$$F(\lambda) = \lambda \ln\left(1 - \frac{1}{\lambda}\right) = 1 - \frac{1}{2\lambda} + \frac{1}{3\lambda^2} - \dots, \quad \lambda \to \infty.$$

Then

$$F^{[1/0]}(\lambda) = 1 - \frac{1}{2\lambda} = F(\lambda) + O\left(\frac{1}{\lambda^2}\right),$$

$$F^{[0/1]}(\lambda) = \frac{1}{1 + \frac{1}{2\lambda}} = F(\lambda) + O\left(\frac{1}{\lambda^2}\right),$$

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Theorem (A.Markov, 1895)

Let σ be a nonnegative measure on $[-1,1]$,

$$F(\lambda)=\widehat{\sigma}:=\int_{-1}^{1}\frac{d\sigma(t)}{t-\lambda}.$$

Then the $[n/n]$ Pade approximants for F exist for every $n\in\mathbb{N}$ and converge to F locally uniformly in $\mathbb{C}\setminus[-1,1]$.

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In the case where $\operatorname{supp}\sigma=[-1,\alpha]\cup[\beta,1]$, there are examples which show that there is no uniform convergence in the gap (α,β) (A. Markov).

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$$F(\lambda) = -\frac{1}{\lambda - a_0} - \frac{b_0^2}{\lambda - a_1} - \frac{b_1^2}{\lambda - a_2} - \cdots .$$

$$J = \begin{pmatrix} a_0 & b_0 & \\ b_0 & a_1 & \ddots \\ & \ddots & \ddots \end{pmatrix}$$

$$J_{[0,n]} = \begin{pmatrix} a_0 & b_0 & \mathbf{0} \\ b_0 & \ddots & \\ \mathbf{0} & & a_n \end{pmatrix}$$

Proof.

$$F^{[n/n]}(\lambda) = \left((J_{[0,n-1]} - \lambda)^{-1} e, e\right)_{\ell^2} \to \left((J - \lambda)^{-1} e, e\right)_{\ell^2} = F(\lambda)$$

where $e := e_0 = (100\dots)^\top$.

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$$\varphi(\lambda) = r_1(\lambda) \int_a^b \frac{d\mu(t)}{t-\lambda} + r_2(\lambda)$$

where μ is a positive Borel measure, $r_1 = q_1/\omega_1$ is a rational function, nonnegative for $\lambda \in \mathbb{R}$ ($\deg q_1 \leq \deg \omega_1$), and r_2 is a proper rational function.

$$\varphi(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_n}{\lambda^{n+1}} - \ldots \quad |\lambda| > R.$$

Let $S_n := (s_{i+j})_{i,j=0}^n$. It was shown by M.G. Krein and H. Langer that the number of the negative eigenvalues

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Let φ_0 be a function in question and let n be such that $\det S_n \neq 0$ $(\varphi_0 \in \mathbf{N}_\kappa)$.

$$\varphi_0(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_{2n}}{\lambda^{2n+1}} + o\left(\frac{1}{\lambda^{2n+1}}\right) \quad (\lambda\to\infty).$$

- Let n_1 be the smallest natural number j such that $\det S_{j-1} \neq 0$.
- There exist a monic polynomial p_0 of degree $k_0 = n_1$ and a function $\varphi_1 \in \mathbf{N}_{\kappa-\nu(S_{n_1-1})}$ such that

$$-\frac{1}{\varphi_0(\lambda)} = \varepsilon_0 p_0(\lambda) + b_0^2\varphi_1(\lambda), \quad b_0 > 0,\, \varepsilon_0 = \pm 1.$$

Moreover, we have

$$\varphi_1(\lambda) = -\frac{s_0^{(1)}}{\lambda} - \frac{s_1^{(1)}}{\lambda^2} - \cdots - \frac{s_{2(n-n_1)}^{(1)}}{\lambda^{2(n-n_1)+1}} + o\left(\frac{1}{\lambda^{2(n-n_1)+1}}\right) \quad (\lambda\to\infty)$$

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$$C_p = \begin{pmatrix} 0 & \dots & 0 & -p_0 \\ 1 & & \mathbf{0} & -p_1 \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & -p_{k-1} \end{pmatrix}.$$

It is well known that $\det(\lambda - C_p) = p(\lambda)$ and the spectrum $\sigma(C_p)$ of C_p is simple.

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Let $p(\lambda) = p_k\lambda^k + \cdots + p_1\lambda + p_0$ be a monic polynomial of degree k (i.e. $p_k = 1$). The companion matrix C_p for p has the following form

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Let $H_{[0,\infty)}$ be a bounded self-adjoint GJM . Then the m-function $m_{[0,\infty)}(\cdot) \in \mathbf{N}_\kappa$, where $\kappa = \nu(G)$. Moreover, in this case we have that $s_j = \left[H^j_{[0,\infty)} e, e\right]$.

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where μ is a positive measure, $r_1 = q_1/\omega_1$ is a rational function, nonnegative for $\lambda \in \mathbb{R} \setminus \mathcal{P}(r_1)$ ($\deg q_1 \le \deg \omega_1$), and r_2 is a real rational proper function. Then the diagonal Padé approximants for φ converge to φ locally uniformly in $\mathbb{C} \setminus ([a,b] \bigcup \mathcal{P}(\varphi))$.

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where ν is a signed measure on $[-1, 1]$.

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The following relation holds true

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(ii) If the corresponding sequence $\left\{ \frac{d_{n+1}}{d_n} \right\}_{d_n \neq o}$ is bounded then the sequence of $[n/n]$ Padé approximants $\mathfrak{F}^{[n/n]}$ $(d_n \neq 0)$ converges to $\mathfrak{F}$ locally uniformly in $\mathbb{C} \setminus ([a-\varepsilon, b+\varepsilon] \cup \mathcal{P}(\varphi))$ for some $\varepsilon > 0$.

Consider the function $\mathfrak{F}$ of the form

$$\mathfrak{F}(\lambda) = r_1(\lambda)\int_a^b \frac{td\sigma(t)}{t-\lambda} + r_2(\lambda),$$

where σ is a finite nonnegative measure on $[a,b] \ni 0$, r_j are real rational functions, such that $r_j(\lambda) = O(1/\lambda)$ for $\lambda \to \infty$ $(j = 1, 2)$ and $r_1(\lambda)$ is nonnegative for real λ.

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Great thanks for your attention!