Efficient feedbacks for low thrust transfers

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Abstract

Motivated by the implementation of time optimal control law for low thrust orbital transfers of the controlled Keplerian system, we study the possibility to use stabilizing feedbacks to achieve time efficient transfer. We use a family of stabilizing feedback laws based on Lyapunov functions of first integrals. This idea is not new but it turns out that with a proper choice of these Lyapunov functions the feedback can be really simple and efficient. We proposed a method to select a good Lyapunov function and applied it to a wide range of time optimal transfer cases toward the geosynchronous orbit. The difference of transfer times between a good interpolating feedback and the optimal control is outstandingly small.

1 Introduction

Finding a control that minimizes some criterion (here, the transfer time), while performing an assigned task (here, reaching the target orbit) has always been a key question in control engineering. A lot of research has been done on the development of optimization algorithms to overcome the inherent difficulties of optimal control for this particular problem. The computations become very sensitive when the thrust tends to zero. But if control theory [1, 2] provides tools to select among all trajectories the candidates to be optimal, only few theoretical results are known about time optimal orbital transfers [3][4]. Many practitioners duly object that, apart from being difficult to compute, optimal control provides an open-loop control which in practice should be recomputed often because of its high sensitivity to the model, the initial condition and the possible perturbations. In particular the time optimal transfers depend both on the initial and final orbits and on the characteristics of the satellite. More over we may not apply approximate linearization techniques in order to track the optimal trajectory because time optimal control are saturated and the linearized system is uncontrollable along time optimal trajectories.

Instead of using optimal control algorithms, one may design control laws that perform the desired task (here, reach the target orbit), robustly, and *then*, if the criterion is vital, chose, among these particular controls, one that performs reasonably good with respect to the criterion. Lyapunov control (or "Jurdjevic-Quinn control", or "damping control", see [5, 6]) has already been used on this problem, see e.g. [7] (that references the private report [8]) or [9], leading to very simple, naturally robust asymtptotiques stabilizing feedbacks and to transfer strategies with reasonable performances. These authors however did not insist on achieving better performances. There is however a wide choice of Lyapunov functions provided by the first integrals of the two body problem that yield to different feedbacks with different performances.

The present paper is devoted to exploring the possible choices of Lyapunov functions and show, based mainly on numerical experiments, that the minimum time may be very closely approached by these feedbacks. In order to select a good feedback a possibility is to start with a referenced trajectory computed with an optimazation software such as *Mipelec* [10], and to sellecte the feedback which as clsoe as possible to the optimal control along the optimal trajectory. It is Lyapunov interpolation. Applying this method to some referenced trajectories [10, 3] computed with optimization programs, it appears that we obtain final times very close to the optimal ones.

The paper is organized as follows. In Section 2 we present the controlled 2-body system and the transfer problem. In Section 4 we introduce a simple function which measure the distance from an elliptic orbit to the equatorial geosynchronous orbit and use it to built a feedback control law. We apply it to two Ariane's GTO to GEO transfer cases referenced in literature for which an estimate of the optimal time has been computed. It turns out that this feedback law is really efficient to achieve these time optimal transfers. The important point is that this distance does not depend on the characteristics of the satellite but only of the initial and final orbital parameters. Finally in the Section 5 we present a general method to select a good distance knowing a time optimal transfer between two elliptic orbits. We illustrate this method on the planar problem and apply it to a large range of transfers. The results of these simulations are extremely conclusive.

2 The controlled Keplerian system

A satellite of mass m in rotation around the Earth is classically modeled as a mass point subject to the Earth gravitation and the thrust. According to Newton dynamical laws, the position and velocity vectors $(r, v) \in T\mathbb{R}^3_*$ ($\mathbb{R}^3_* = \mathbb{R}^3 \setminus \{0\}$), are solution of the system (1).

$$\begin{cases} \dot{r} = v\,, \\ \dot{v} = -\mu \frac{r}{|r|^3} + \frac{u}{m}\,. \end{cases} \tag{1}$$

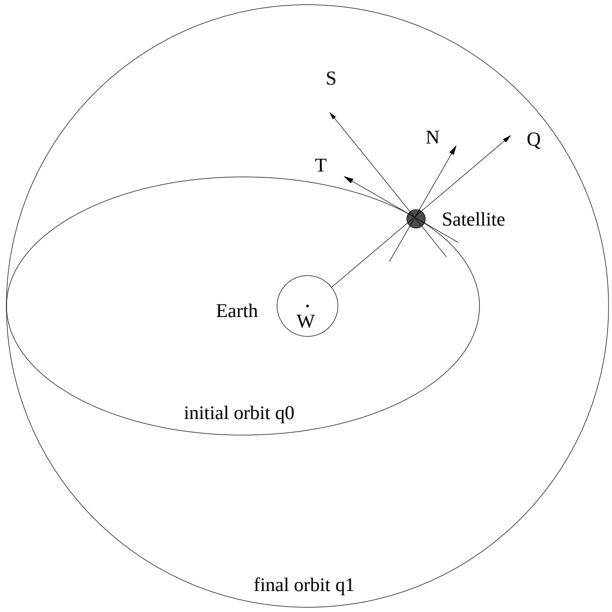

Fig. 1. The frame QSW. Q is the radial unit vector, S is orthogonal to Q in the plane of the osculating orbit, and W is orthogonal to this plane.

This control u is physically provided by some engine inboard the satellite, that produces some thrust of limited magnitude ϵ. The control must satisfy the constraint (2).

$$\|u\| \leq \epsilon. \tag{2}$$

The mass of the satellite decreases as fuel is consumed by the engine, according to the law

$$\dot{m} = -\frac{\|u\|}{g_0 I_{sp}}\,. \tag{3}$$

The specific impulse I_{sp} and the thrust modulus ϵ characterise the satellite engine, while g_0 and μ are the classical physical constants associated to the Earth's gravitation. Modern (electric) engines have a higher I_{sp}; this allows to produce the same thrust while consuming a notably smaller mass of propellant. The drawback is that the maximum thrust they can produce (ϵ) is much smaller than for classical chemical engines, "low thrust" means that ϵ is a small paramteter in the equations. The impulsive approximations that enable Hohmann's transfer is not valid for "low thrust", the control laws to achieved orbital transfers have to be changed.

The uncontrolled Hamiltonian system, with $H = \frac{v^2}{2} - \frac{1}{\|r\|}$, is integrable and even presents five first integrals in the 3 dimensional case [11]. For example consider the kinetic momentum

$$h(r, v) = r \times v\,, \tag{4}$$

and the eccentricity vector

$$A(r,v) = \frac{v \times h}{\mu} - \frac{r}{|r|}. \tag{5}$$

The 2 vectors h and A are not independent since $A \times h = 0$, but, it can be proven that any five components are independent functions of (r, v).

Let $\{q_i\}_{i\in[1..5]}$ be a choice of independent first integrals that characterized the Keplerian orbit and ϕ an angle, that fixes the position on the orbit. Let (ξ, η, ζ) be an orthonormal frame and (u_ξ, u_η, u_ζ) be the components of the control vector u in this frame. The controlled Keplerian system (1) may be express this action-angle coordinates by the equation (6).

$$\frac{d}{dt}(q,\phi)^T = f_0(q,\phi) + u_\xi f_\xi(q,\phi) + u_\eta f_\eta(q,\phi) + u_\zeta f_\zeta(q,\phi), \tag{6}$$

where f_i are vector fields on $I\!R^6$. The drift f_0 describes the Keplerian motion, almost all its components are null, $f_0 = (0, \omega(q,\phi))^T$. If you choose to work with the classical five independent first integrals $q = \{c,\, e_x,\, e_y,\, h_x,\, h_y\}$, with L and to decompose the control u in the frame QSW associated to the satellite (see figure 1), then $\omega(q,L) = \frac{\mu^2 Z^2}{c^3}$ and

$$\begin{aligned}
f_S &= \left(\tfrac{c^2}{\mu Z}, \tfrac{cA}{\mu Z}, \tfrac{cB}{\mu Z}, 0, 0, 0\right)^T, \\
f_Q &= \left(0, \tfrac{c}{\mu}\sin L, -\tfrac{c}{\mu}\cos L, 0, 0, 0\right)^T, \\
f_W &= \left(0, -\tfrac{cY}{\mu Z}e_y, \tfrac{cY}{\mu Z}e_x, \tfrac{cX}{2\mu Z}\cos L, \tfrac{cX}{2\mu Z}\sin L, \tfrac{cY}{\mu Z}\right)^T,
\end{aligned}$$

$$\text{with} \begin{cases}
Z &= 1 + e_x \cos L + e_y \sin L\,, \\
A &= e_x + (1+Z)\cos L\,, \\
B &= e_y + (1+Z)\sin L\,, \\
X &= 1 + h_x^2 + h_y^2\,, \\
Y &= h_x \sin L - h_y \cos L\,.
\end{cases}$$

Note that the mapping $(r,v) \mapsto (q,\phi)$, from $TI\!R^3_*$ to a subset Ω of $I\!R^5 \times \mathbb{S}$ may present some singularities. But on the elliptic domain,

$$\mathcal{E} = \{(r,v) \in TI\!R^3_*, H < 0 \,\text{and}\, \|h\| \neq 0\}, \tag{7}$$

and

$$\mathbb{E}_q = \{(c,(e_x,e_y),(h_x,h_y)) \in I\!R_* \times \mathbb{D}^2\}, \tag{8}$$

with $\mathbb{D} = \{x \in I\!R^2, \|x\| < 1\}$, it is a classical result that the mapping $(r,v) \mapsto (c, e_x, e_y, h_x, h_y, L)$ is a diffeomorphism from $\mathcal{E}$ to $\mathbb{E}_q \times \mathbb{S}$.

As most of dynamical systems, the satellite admits various coordinates representation for example the phase space (r, v) or the action-angle (q, ϕ). Does there exist a good coordinate chart to represent the dynamic of the controlled 2-body system? The question is open. We will see in Section 3 that the first integrals can be used to build stabilizing feedback laws.

In general, the osculating orbit can be an ellipse, a parabola or an hyperbola. The elliptic region is the set of state where the Hamiltonian H is negative, equivalent to $e = \sqrt{e_x^2 + e_y^2} < 1$. We will not investigate here what happens outside this region. A transfer is the operation of reaching a target orbit q_1 from an initial orbit q_0, with no requirement on the final longitude. The transfer problem is the one of designing a control law, either feedback or open-loop, that fulfills this operation. Under the assumption that the mass is constant, it has been shown that the controlled 2-body problem is controllable, there exists an admissible path that joins two orbits [12]. Given two elliptic orbits the existence of a transfer constrained to stay in the elliptic domain is given in [3] as an application of a theorem on affine system with recurrent drift [2].

The two main issues relevant for space industry in terms of a criterion to optimize, are to characterize the time minimal transfer and the final mass maximal transfer (maximizing the final mass means minimizing the fuel consumption). The optimal control theory provides tools to solve some path-planing problem subject to constrains such as the transfer problem in optimal time. But very few is known about time optimal transfer [13]. Programs such as Mipelec [10] or Tfmin [3] have been developed to find numerical solutions.

3 Jurdjevic-Quinn method

For a control system a feedback, a control law function of the state, is classically used to stabilize a desired state. The Jurdjevic-Quinn method gives sufficient conditions to strengthen the stability property of an affine system which admits a first integral, hence stable in the sense of Lyapunov. In [7] J.-M. Coron noted that the controlled 2-body system, described by the equations (1) or (6), satisfies to the hypotheses of the Jurdjevic-Quinn method. In this section we present this method taking in account the constraint (2). It should be notice that in [14] L. Praly and C. M. Kellett built a feedback, subject to such a constraint, that stabilizes a solution of the Keplerian motion.

Let q^1 be a target ellipse and q the current orbit. Fix $k_i > 0$ and define

$$V_k(q) = \sum_{i=1}^{5} k_i (q_i - q_i^1)^2. \tag{9}$$

The function V_k defines a distance from the current orbit q to q^1, but depends of the choice of coordinates. Note that V_k is a first integral of the Keplerian motion, the Lie derivative of V_k along the drift f_0 is null

$$L_{f_0} V_k(q, \phi) = 0. \tag{10}$$

The function V_k is proper, i.e.

$$\forall \beta \in I\!R, V_k^\beta = \{(q, \phi), V_k(q, \phi) \leq \beta\} \text{ is compact,} \tag{11}$$

and has no critical points outside the ellipse q^1:

$$dV_k(q, \phi) = 0 \Rightarrow q = q^1. \tag{12}$$

These two conditions imply that V reaches its minimum on q^1.

We define the distribution $\mathcal{F}^2$: the Lie brackets of length 2 from the drift f_0 against the controlled vector fields f_i by

$$\mathcal{F}^2(q, \phi) = \mathrm{Span}_{I\!R} \left\{ f_0(q, \phi), (\mathrm{ad}\ f_0)^k f_j(q, \phi), j \in \{\xi, \eta, \zeta\}, 0 \leq k \leq 2. \right\}.$$

This distribution is independent of the choice of coordinates, more over

$$\dim \mathcal{F}^2(q, \phi) = 6, \text{ for all } (q, \phi). \tag{13}$$

The following theorem is classical. Let $|L_f V_k| = \sqrt{\sum_{i \in \{\xi,\eta,\zeta\}} L_{f_i} V_k^2}$.

Theorem 1 *The conditions (10), (11), (12) and (13) imply that for all $\delta > 0$ the smooth feedback u^δ defined by*

$$\begin{cases} |L_f V_k| > \delta, \ u_i = -\epsilon \frac{L_{f_i} V_k}{|L_f V_k|}, \\ |L_f V_k| \leq \delta, \ u_i = -\epsilon \frac{L_{f_i} V_k}{\delta}, \end{cases} \tag{14}$$

asymptotically stabilizes the elliptic orbit q^1.

A sketch of proof goes as follows. The map V_k is a Lyapunov function for the vector field

$$h = f_0 + u_\xi^\delta f_\xi + u_\eta^\delta f_\eta + u_\zeta^\delta f_\zeta.$$

indeed

$$\begin{aligned} \dot{V}_k &= L_h V_k, \\ &= -\epsilon/\delta \min(\delta, 1/|L_f V_k|) \textstyle\sum_{i \in \{\xi,\eta,\zeta\}}^m L_{f_i} V_k^2, \\ &\leq 0. \end{aligned}$$

Applying the LaSalle invariance principle [15]: the trajectories starting in V_α are converging to the biggest positive h-invariant set I included in

$$W = \{(q, \phi) \in \mathbb{E}_q,\ L_h V(q, \phi) = 0\}$$

It turns out that conditions (13), (10), (11) and (12) implies that I is equal to q^1. $\square$

We do not investigate here the whole domain of stability. Nevertheless note that for β small enough V_k^β is a subset of the elliptic domain $\mathbb{E}_q$. Then, for all initial orbits in V_k^β the feedback u^δ asymptotically stabilizes q_1.

To be more precise this feedback converges exponentially fast toward the target. Then, even if this continuous feedback converges in infinite time, the mass converges to a finite limit (see for example the Figure 7). This limit can be used to define the transfer duration T_f (15) of a feedback u^{delta}.

$$T_f = \frac{g_0 I_{sp}}{u_{max}}(m_0 - m_f). \tag{15}$$

More over since the linearized system along the target orbit is controllable, according to fundamental properties of the reachable set [15], any time optimal ball, the set of point that can join the target orbit in a time less or equal to a positive constant time, cover a ball centered on the target with radius proportional to ϵ.

More over since this feedback is continuous and admits a Lyapunov function, it present robustness properties to perturbations of the vector field h such as eclipses, bias on the engine... If you want to achieve a Rendez-vous, i.e. you want to arrive at a precise longitude on the GEO orbit, you can easily estimate the final longitude, integrating the feedback law, and then perform a 1 dimensional shooting method on the final longitude. Remarkably the final longitude is much more sensitive to small variation of V_k than the "transfer" time.

4 GTO to GEO time optimal transfers

In this section we consider two referenced transfers from the geosynchronous transfer orbits (GTO) of the Ariane rocket to the equatorial geosynchronous orbit (GEO). It is a classical orbital transfer for the space industry. In the Table 1 we give the characteristic of the two transfers. The initial orbits a_0, e_0, i_0 are close to each other they correspond to the range of the Ariane's GTO orbits. At the contrary, the characteristics of the two satellites (initial mass m_0, Isp and ϵ) differ sensitively, in particular the ratios ϵ/m_0, which is the

Case	1 [10]	2 [3]
a_0 (m)	24505900	2.458e7
e_0	0.725	0.75
i_0 (deg)	5.2	7
m_0 (kg)	2000	1500
Isp (s)	2000	1994.75
ϵ (N)	0.35	0.2

Table 1
Two GTO-GEO transfers

maximal acceleration allowed by the engine, $1.75e^{-4}$ for the case 1 and $1.33e^{-4}$ for the case 2. In few words, the satellite 2 is less powerful than the satellite 1, hence the optimal transfer time of 2 is longer than the optimal transfer time of 1, respectively 177 days versus 137 days.

We consider the function

$$V(q) = 4(\frac{a}{a_{geo}} - 1)^2 + 3e^2 + i^2. \tag{16}$$

This function defines a distance from a current orbit of semi-major axis a, eccentricity e and inclination$i$ to the geosynchronous orbit. We can used only 3 first integrals because the target orbit, the geosynchronous orbit, is circular and is in the equatorial plane. Remind the following relations between the first integrals, $a = \frac{c^2}{\mu\sqrt{1-e^2}}$, $e = \sqrt{e_x^2 + e_y^2}$ and $(\tan\frac{i}{2})^2 = h_x^2 + h_y^2$.

Let $\delta = 10^{-5}$, according to the Theorem 1 the control u^δ defined by (14) is a continuous stabilizing feedback of the equatorial geosynchronous orbit ($a = a_{geo}, e = 0, i = 0$). In order to get a transfer in finite time, we apply the feedback as long as possible and when the satellite is close to the target we switch to the time optimal trajectory. This time optimal problem can easily be solved by numerical computation because "close" to the target, the control is not "small".

The final results are exposed in the Table 2. We compare the feedback transfer time with the optimal time, computed with the optimization programs Mipelec and Tfmin. The conclusion is that the feedback laws based on V (16) is really efficient to achieve the time optimal transfer from the Ariane's GTO orbit to the GEO orbit. Of course if you consider an other function instead of V, you will generally not get such an efficient transfer. In the next section we will present a method, called Lyapunov interpolation, to select a good function V.

Case	1	2
Reference transfer time (day)	137.38 [10]	177.36 [3]
Lyapunov functions	V	V
Total final transfer time	136.45	177.41
Time spent with feedback	134	175

Table 2
Time Efficient Feedbacks

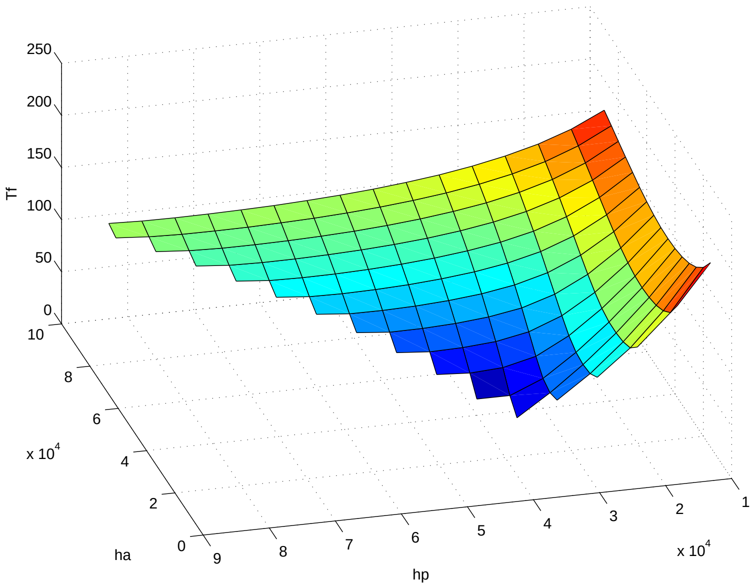

Fig. 2. Optimal transfer time computed with Mipelec, T_f^{mip}

5 Lyapunov interpolations of time optimal transfers

In this section we consider the planar system when the initial orbit is in the same plan than the target orbit . The initial orbits belong to the set of elliptic orbits with perigee's altitude h_p included in the interval $[10,000\,km\ 90,000\,km]$ and with apogee's altitude h_a less than $90,000\,km$. We studied planar transfers to the equatorial geosynchronous orbit $ha = hp = 36,000 km$. The satellite is characterized by an initial mass m_0 of 1000 kg, a specific impulse I_{sp} of 1500 s and a maximal thrust, ϵ of 0.1 N. We used the software Mipelec [10] to get referenced trajectories and referenced times T_f^{mip} (see the figure 2).

We consider the small class of distance (17), indexed with an angle θ in the interval $[0, \frac{\pi}{2}]$.

$$V_\theta = \cos\theta\,(\frac{a}{a_{geo}} - 1)^2 + \sin\theta\, e^2. \tag{17}$$

Let $\delta = 10^{-6}$, given a function V_θ, we defined the continuous feedback u_θ by (14). According to the Theorem 1, the control u_θ is a stabilizing feedback

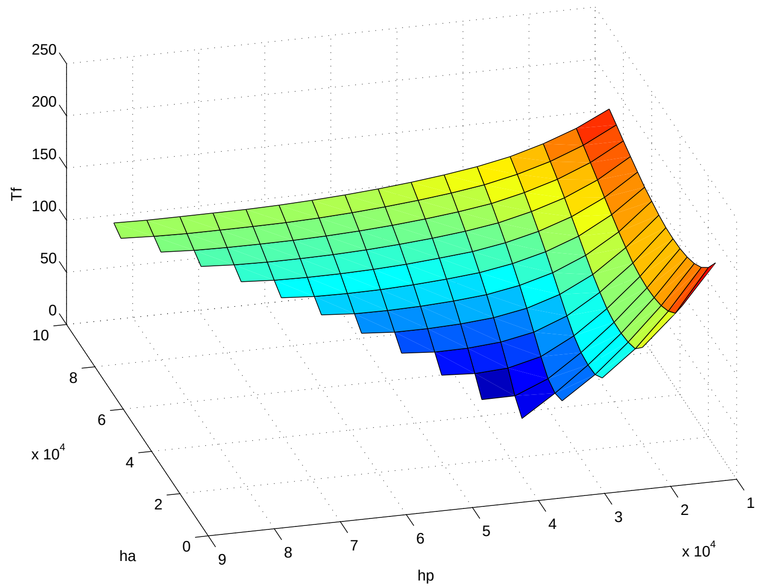

Fig. 3. Transfer time with Lyapunov's interpolation, T_f

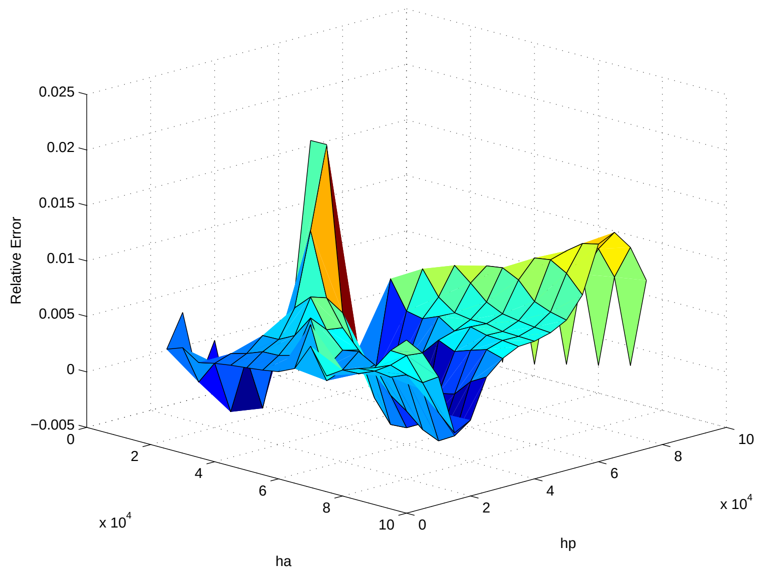

Fig. 4. Relative error, $|T_f^{mip} - T_f|/T_f^{mip}$

control law for all $\theta \in]0, \pi/2[$.

Given an initial orbit (h_a, h_p), in order to select a good distance V_θ, we look for the best interpolating feedback of the transfer trajectory $\gamma(t)$ computed with Mipelec. Let $\Gamma(t)$ be the referenced control define on $[0, T_f^{mip}]$ associated to $\gamma(t)$. We look for θ which minimizes the cost J (18).

$$J = \min_\theta \frac{1}{T_f^{mip}} \int_0^{T_f^{mip}} \|u_\theta(\gamma(t)) - \Gamma(t)\|^2 dt. \tag{18}$$

For a study of the notion of Lyapunov's interpolation the reader is referred to

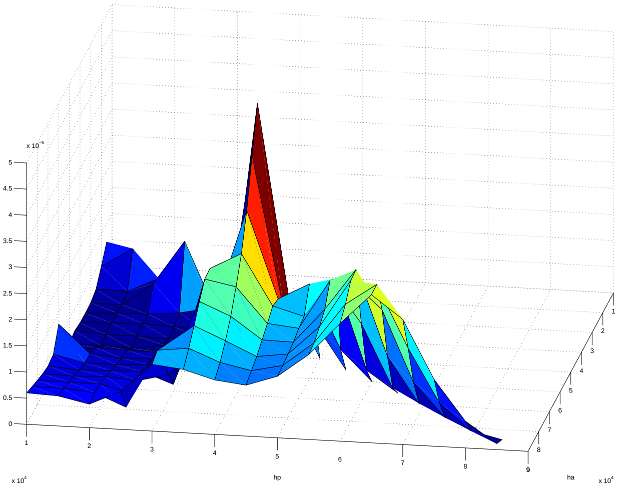

Fig. 5. The best cost J

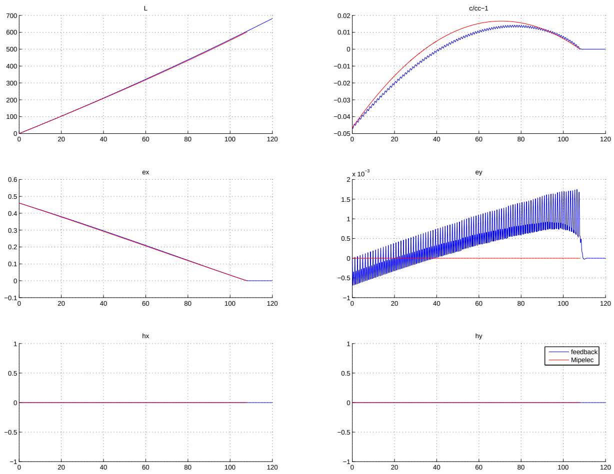

Fig. 6. Orbital parameter evolution, transfer ha=60000 hp=20000

the paper [16].

The main result of the present simulations can be set out as follow: the relative error between the optimal time T_f^{mip} and the feedback time T_f is less than 1% far from the target orbit (see figure 4). If we look at the cost of the best Lyapunov interpolation (see Figure 5) we observed that it is correlated with the relative error.

A deficiency in our algorithm to find the best Lyapunov interpolation may have explained why the error is not so smooth. But if we look at the application which associate to the initial orbit the best Lyapunov interpolation index by θ (see Figure 8) we observed two lines of discontinuities. One is the circular

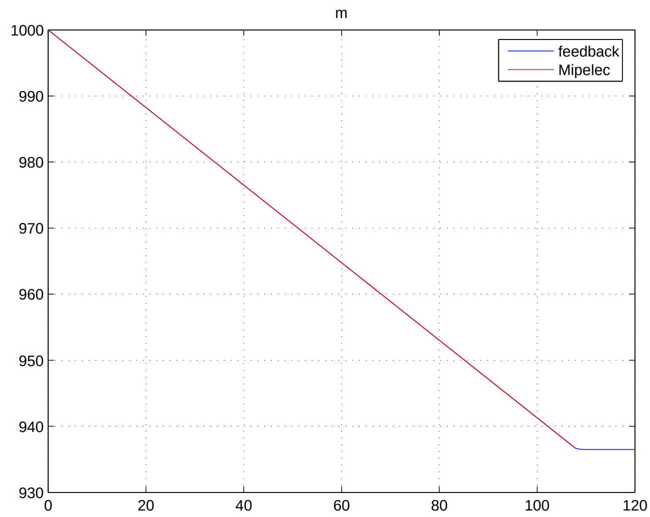

Fig. 7. Mass evolution, ha=60000 hp=20000

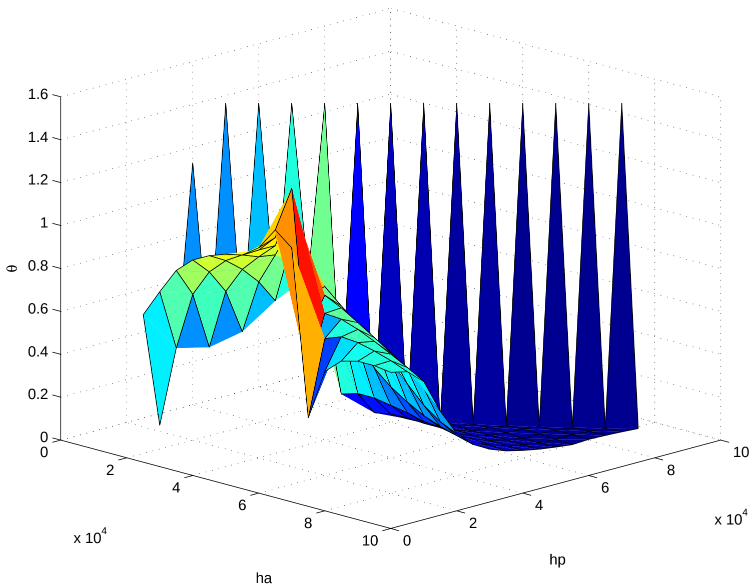

Fig. 8. The best Lyapunov interpolation, θ

orbit, $ha = hp$. The other one seems to follow the geosynchronous orbits, $\frac{ha+hp}{2} = 36,000km$. Hence the explanation of the variation are structural and not an artifact of the algorithm used to search for the best interpolation. In a forthcoming paper we will present a characterization of the asymptotic behavior of the time optimal transfer between two close coplanar circular orbits.

6 CONCLUSION

We show that controlled Lyapunov functions can be relevant to perform time efficient transfers. We think that the distances between orbits, the Lyapunov functions, are a powerful tool to handle low thrust satellite path-planing. Contrary to the optimization programs they give a control at any point having by nature some robustness properties to perturbations, such as eclipses or engine bias.

Following the results of the numerical simulation we may ask if the following proposition holds.

Conjecture 1 *For a given initial orbit q_0 and a target orbit q_1 it exists a stabilizing feedback generated by a function V_k such that the time T_k spend by this feedback to enter in a ball around the target orbit converge to the optimal time as the bound on the control ϵ tends to zero.*

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