1. Buckley inequality

The average of a summable positive function (a *weight*) w over an interval I will be denoted by the symbol $\langle w\rangle_{I}$:

$$\langle w\rangle_{I} \stackrel{\text{def}}{=} \frac{1}{|I|}\int_I w(t)\,dt\,,$$

where $|I|$ stands for the Lebesgue measure of I. For an interval J, the symbol $A_\infty(J,\delta)$ denotes the δ-ball in the Muckenhoupt class A_∞:

$$A_\infty(J,\delta) \stackrel{\text{def}}{=} \left\{w\colon w\in L^1(J),\ w\ge 0,\ \langle w\rangle_{I} \le \delta e^{\langle \log w\rangle_I}\ \forall I\subset J\right\}\,. \tag{1.1}$$

We denote by $\mathcal{D}_J$ the set of all dyadic subintervals of J and by $A^d_\infty(J,\delta)$ the dyadic analogue of (1.1), i.e. in the definition of $A^d_\infty(J,\delta)$ we consider only $I\in\mathcal{D}_J$.

Theorem (Buckley [1])**.** *There exists a constant $c=c(\delta)$ such that*

$$\sum_{I\in\mathcal{D}_J}|I|\Big(\frac{\langle w\rangle_{I_+}-\langle w\rangle_{I_-}}{\langle w\rangle_{I}}\Big)^2\le c(\delta)|J|$$

for any weight w from $A^d_\infty(J,\delta)$.

In the statement of the theorem we use notation $I_\pm$ to mean the right and left halves of I, respectively. By $\mathcal{D}^n_J$ we denote the n-th generation of the dyadic intervals nested in J, i.e. $\mathcal{D}^0_J=\{J\}$, $\mathcal{D}^1_J=\{J_\pm\}$, etc.

Now, we are ready to introduce the main object of our consideration, the so-called Bellman function of the problem.

$$\mathbf{B}(x)=\mathbf{B}(x_1,x_2;\delta)$$
$$\stackrel{\text{def}}{=}\sup_{w\in A^d_\infty(J,\delta)}\left\{\frac{1}{|J|}\sum_{I\in\mathcal{D}_J}|I|\Big(\frac{\langle w\rangle_{I_+}-\langle w\rangle_{I_-}}{\langle w\rangle_{I}}\Big)^2\colon \langle w\rangle_{J}=x_1,\ \langle \log w\rangle_{J}=x_2\right\}\,.$$

This function is defined on the domain

$$\Omega_\delta \stackrel{\text{def}}{=}\left\{x=(x_1,x_2)\colon\ \log\frac{x_1}{\delta}\le x_2\le \log x_1\right\}\,.$$

Indeed, the right bound is simply Jensen's inequality and the left one means that our weight w is from $A_\infty(J,\delta)$. The parameter δ is fixed throughout. Let us note that we did not assign the index J to $\mathbf{B}$, despite the fact that all test functions w in its definition are considered on J. This omission is not due to our desire to simplify notation, but rather an indication of the very important fact that the function $\mathbf{B}$ does not depend on J.

A bit more notation. For a given weight $w\in A_\infty(J,\delta)$ and any subinterval $I\subset J$, there corresponds the following point of Ω_δ: $x^I=(\langle w\rangle_{I},\langle \log w\rangle_{I})$.

(**Homework assignment:** *Check that the function* $\mathbf{B}$ *defined on the whole domain* Ω_δ*, i.e. for every point x, $x\in\Omega_\delta$, there exists a function $w\in A_\infty(J,\delta)$ such that $x=x^J$.*)

Let us now consider some properties of $\mathbf{B}$ that are clear from its definition; these properties will help us find $\mathbf{B}$ explicitly.

Lemma 1.1 (Main inequality)**.** *For every pair of points $x^\pm$ from Ω_δ such that their mean $x = (x^+ + x^-)/2$ is also in Ω_δ, the following inequality holds*

$$\mathbf{B}(x) \geq \frac{\mathbf{B}(x^+) + \mathbf{B}(x^-)}{2} + \Big(\frac{x_1^+ - x_1^-}{x_1}\Big)^2. \tag{1.2}$$

Proof. Let us split the sum in the definition of $\mathbf{B}$ into three parts: the sum over $\mathcal{D}_{J_+}$, the sum over $\mathcal{D}_{J_-}$, and an additional term, corresponding to J itself:

$$\begin{aligned}\frac{1}{|J|}\sum_{I\in\mathcal{D}_J}|I|\Big(&\frac{\langle w\rangle_{I_+}-\langle w\rangle_{I_-}}{\langle w\rangle_I}\Big)^2\\ &=\frac{1}{2|J_+|}\sum_{I\in\mathcal{D}_{J_+}}|I|\Big(\frac{\langle w\rangle_{I_+}-\langle w\rangle_{I_-}}{\langle w\rangle_I}\Big)^2\\ &+\frac{1}{2|J_-|}\sum_{I\in\mathcal{D}_{J_-}}|I|\Big(\frac{\langle w\rangle_{I_+}-\langle w\rangle_{I_-}}{\langle w\rangle_I}\Big)^2\\ &+\Big(\frac{\langle w\rangle_{J_+}-\langle w\rangle_{J_-}}{\langle w\rangle_J}\Big)^2.\end{aligned}$$

Now we choose the weights $w^\pm$ on the intervals $J_\pm$ that almost give us the supremum in the definition of $\mathbf{B}(x^\pm)$, i.e.

$$\frac{1}{|J_\pm|}\sum_{I\in\mathcal{D}_{J_\pm}}|I|\Big(\frac{\langle w^\pm\rangle_{I_+}-\langle w^\pm\rangle_{I_-}}{\langle w^\pm\rangle_I}\Big)^2 \geq \mathbf{B}(x^\pm)-\eta,$$

for an arbitrary fixed small $\eta > 0$. Then for the weight w on J, defined as w^+ on J_+ and w^- on J_-, we obtain the inequality

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}_J}|I|\Big(\frac{\langle w\rangle_{I_+}-\langle w\rangle_{I_-}}{\langle w\rangle_I}\Big)^2 \geq \frac{\mathbf{B}(x^+)+\mathbf{B}(x^-)}{2}-\eta+\Big(\frac{x_1^+-x_1^-}{x_1}\Big)^2. \tag{1.3}$$

Observe that the compound weight w is an admissible weight, corresponding to the point x. Indeed, $x^\pm = x^{J_\pm}$ and by construction $w^\pm \in A_\infty^d(J_\pm,\delta)$; therefore, the weight w satisfies the inequality $\langle w\rangle_I \leq \delta e^{\langle \log w\rangle_I}$ for all $I \in \mathcal{D}_{J_+}$, since w^+ does, and for all $I \in \mathcal{D}_{J_-}$, since w^- does. Lastly, $\langle w\rangle_J \leq \delta e^{\langle \log w\rangle_J}$, because, by assumption, $x \in \Omega_\delta$.

We can now take supremum in (1.3) over all admissible weights w, which yields

$$\mathbf{B}(x) \geq \frac{\mathbf{B}(x^+)+\mathbf{B}(x^-)}{2}-\varepsilon+\Big(\frac{x_1^+-x_1^-}{x_1}\Big)^2,$$

which proves the main inequality because η is arbitrarily small. □

Lemma 1.2 (Boundary condition)**.**

$$\mathbf{B}(x_1, \log x_1) = 0\,.$$

Proof. Let us take a boundary point x of our domain Ω_δ, that is a point with $x_2 = \log x_1$. Since the equality in Jensen's inequality $e^{\langle w\rangle} \leq \langle e^w\rangle$ occurs only for constant functions w, the only test function corresponding to x is the constant weight $w = x_1$. So, on this boundary we have $\mathbf{B}(x) = 0$. □

Lemma 1.3 (Homogeneity)**.** *There is a function g on $[1,\delta]$ satisfying $g(1)=0$ and such that*

$$\mathbf{B}(x)=\mathbf{B}(x_1e^{-x_2},0)=g(x_1e^{-x_2})\,.$$

Proof. For a weight w on an interval J and a positive number τ consider a new weight, $\tilde{w}=\tau w$. If x is a point from Ω_δ corresponding to w and J, i.e. $x_1=\langle w\rangle_{_J}$, $x_2=\langle \log w\rangle_{_J}$, then the point $\tilde{x}=(\tilde{x}_1,\tilde{x}_2)$, $\tilde{x}_1=\tau x_1$, $\tilde{x}_2=x_2+\log\tau$, corresponds to $\tilde{w}$. Note that the expression in the definition of $\mathbf{B}$ is homogeneous of order 0 with respect to w, i.e. it does not depend on τ. Since the weights w and $\tilde{w}$ run over the whole set $A_\infty^d(J,\delta)$ simultaneously, we get $\mathbf{B}(x)=\mathbf{B}(\tilde{x})$. Choosing $\tau=e^{-x_2}$, we obtain

$$\mathbf{B}(x)=\mathbf{B}(x_1e^{-x_2},0)\,.$$

To complete the proof, it suffices to take $g(s)=\mathbf{B}(s,0)$. The boundary condition $g(1)=0$ holds due to Lemma 1.2. □

We are now ready to demonstrate how the Bellman function method works.

Lemma 1.4 (Bellman induction)**.** *Let g be a nonnegative function on $[1,\delta]$ such that the function $B(x)\stackrel{\text{def}}{=}g(x_1e^{x_2})$ satisfies inequality* (1.2) *in Ω_δ. Then Buckley's inequality holds with the constant $c(\delta)=\|g\|_{L^\infty([1,\delta])}$.*

Proof. Fix an interval J and a weight $w\in A_\infty^d(J,\delta)$. Let us repeatedly use the main inequality in the form

$$|I|\,B(x^I)\ge|I_+|\,B(x^{I_+})+|I_-|\,B(x^{I_-})+|I|\Big(\frac{x_1^{I_+}-x_1^{I_-}}{x_1^I}\Big)^2,$$

applying it first to J, then to the intervals of the first generation (that is $J_\pm$), and so on until $\mathcal{D}_J^n$:

$$\begin{aligned}|J|\,B(x^J)&\ge|J_+|\,B(x^{J_+})+|J_-|\,B(x^{J_-})+|J|\Big(\frac{x_1^{J_+}-x_1^{J_-}}{x_1^J}\Big)^2\\&\ge\sum_{I\in\mathcal{D}_J^n}|I|\,B(x^I)+\sum_{k=0}^{n-1}\sum_{I\in\mathcal{D}_J^k}|I|\Big(\frac{x_1^{I_+}-x_1^{I_-}}{x_1^I}\Big)^2.\end{aligned}$$

Therefore,

$$\sum_{k=0}^{n-1}\sum_{I\in\mathcal{D}_J^k}|I|\Big(\frac{x_1^{I_+}-x_1^{I_-}}{x_1^I}\Big)^2\le|J|\,B(x^J)\,,$$

and, passing to the limit as $n\to\infty$, we get

$$\sum_{I\in\mathcal{D}_J}|I|\Big(\frac{x_1^{I_+}-x_1^{I_-}}{x_1^I}\Big)^2\le|J|\,B(x^J)=|J|\,g(x_1e^{-x_2})\le|J|\sup_{s\in[1,\delta]}g(s)\,.$$

□

A natural question arises: how to find such a function g? To answer it, we first replace our main inequality, which is an inequality in finite differences, by a differential inequality. Let us denote the difference between x^+ and x^- by 2Δ, then $x^\pm = x \pm \Delta$ and the Taylor expansion around the point x gives

$$B(x^\pm) = B(x) \pm \frac{\partial B}{\partial x_1}\Delta_1 \pm \frac{\partial B}{\partial x_2}\Delta_2 + \frac{1}{2}\frac{\partial^2 B}{\partial x_1^2}\Delta_1^2 + \frac{\partial^2 B}{\partial x_1 \partial x_2}\Delta_1\Delta_2 + \frac{1}{2}\frac{\partial^2 B}{\partial x_2^2}\Delta_2^2 + o(|\Delta|^2)\,,$$

and, therefore,

$$\begin{aligned}\frac{B(x^+) + B(x^-)}{2} &+ \Big(\frac{x_1^+ - x_1^-}{x_1}\Big)^2 - B(x)\\ &= \frac{1}{2}\frac{\partial^2 B}{\partial x_1^2}\Delta_1^2 + \frac{\partial^2 B}{\partial x_1 \partial x_2}\Delta_1\Delta_2 + \frac{1}{2}\frac{\partial^2 B}{\partial x_2^2}\Delta_2^2 + 4\Big(\frac{\Delta_1}{x_1}\Big)^2 + o(|\Delta|^2)\,.\end{aligned}$$

Thus, under the assumption that our candidate B is sufficiently smooth, the main inequality (1.2) implies the following matrix differential inequality

$$\begin{pmatrix} \dfrac{\partial^2 B}{\partial x_1^2} + \dfrac{8}{x_1^2} & \dfrac{\partial^2 B}{\partial x_1 \partial x_2} \\[2ex] \dfrac{\partial^2 B}{\partial x_1 \partial x_2} & \dfrac{\partial^2 B}{\partial x_2^2} \end{pmatrix} \le 0\,. \tag{1.4}$$

By the preceding two lemmata, we can restrict our search to functions B of the form $B(x_1, x_2) = g(x_1 e^{-x_2})$, where g is a function on the interval $[1, \delta]$. In terms of g, our condition (1.4) can be rewritten as follows:

$$\begin{pmatrix} e^{-2x_2}\Big(g'' + \dfrac{8}{s^2}\Big) & -e^{-x_2}(sg')' \\[2ex] -e^{-x_2}(sg')' & s(sg')' \end{pmatrix} \le 0\,,$$

where $g = g(s)$ and $s = x_1 e^{-x_2}$. This matrix inequality is equivalent to three scalar inequalities:

$$g'' + \frac{8}{s^2} \le 0, \tag{1.5}$$

$$(sg')' \le 0, \tag{1.6}$$

and the condition that the determinant of the matrix must be nonnegative. However, we replace the last requirement by a stronger one — we require the determinant to be identically zero. This requirement comes from our desire to find the best possible estimate: if we take an extremal weight w, i.e. a weight on which the supremum in the definition of the Bellman function is attained, then we must have equalities on each step of the Bellman induction; therefore, on each step the main inequality (1.2) becomes equality. Thus, for each dyadic subinterval I of J there exists a direction through the point x^I in Ω_δ along which the quadratic form given by (1.4) is identically zero. Hence, the matrix (1.4) has a non-trivial kernel and so must have a zero determinant.

Calculating the determinant, we get the equation

$$\Big(g' - \frac{8}{s}\Big)(sg')' = 0\,.$$

The general solution of this equation is $g(s) = c\log s + c_1$. Due to the boundary condition $g(1) = 0$, we have to take $c_1 = 0$.

Now we need to chose another constant, c. To this end, we return to the necessary conditions (1.5)–(1.6). The second inequality is fulfilled for all c, because the expression is identically zero, while the first one gives $c \ge 8$. Since we would like to have g as small as possible (as it gives the upper bound in Buckley's inequality), it is natural to take $c = 8$. Finally, we get

$$g(s) = 8\log s \qquad \text{and} \qquad B(x_1, x_2) = 8(\log x_1 - x_2)\,.$$

Lemma 1.5. *The function*

$$B(x_1, x_2) = 8(\log x_1 - x_2)$$

satisfies the main inequality (1.2).

Proof. Put, as before, $\Delta = \frac{1}{2}(\,x^+ - x^-)$, so $x^\pm = x \pm \Delta$. Then

$$\begin{aligned}
B(x) - &\frac{B(\,x^+) + B(\,x^-)}{2} - \Big(\frac{x_1^+ - x_1^-}{x_1}\Big)^2 \\
&= 8\log x_1 - 8x_2 - 4\log(\,x_1^+x_1^-) + 4(\,x_2^+ + x_2^-) - \Big(\frac{x_1^+ - x_1^-}{x_1}\Big)^2 \\
&= 4\log\frac{x_1^2}{(x_1+\Delta_1)(x_1-\Delta_1)} - 4\Big(\frac{\Delta_1}{x_1}\Big)^2 \\
&= -4\left[\log\left(1 - \Big(\frac{\Delta_1}{x_1}\Big)^2\right) + \Big(\frac{\Delta_1}{x_1}\Big)^2\right] \ge 0\,.
\end{aligned}$$

□

Now we can apply Lemma 1.4 to $g(s) = 8\log s$, which yields the following

Theorem. *The estimate*

$$\sum_{I\in\mathcal{D}_J} |I|\Big(\frac{\langle w\rangle_{I_+} - \langle w\rangle_{I_-}}{\langle w\rangle_{I}}\Big)^2 \le 8\log\delta\,|J|$$

holds for any weight $w \in A_\infty^d(J,\delta)$.

Concluding this section, I would like to emphasize that we still have not found the Bellman function $\mathbf{B}$. The theorem just proved guarantees only the estimate

$$\mathbf{B}(x) \le 8(\log x_1 - x_2).$$

2. Homework assignment: A simple two-weight inequality

As an exercise, verify every step, outlined below, of the proof of this theorem:

Theorem. *If two weights* $u, v \in L^1(J)$ *satisfy the condition*

$$\sup_{I\in\mathcal{D}_J} \langle u\rangle_I \langle v\rangle_I \le M^2\,,$$

then

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}_J} |I|\,|\langle u\rangle_{I_+} - \langle u\rangle_{I_-}|\,|\langle v\rangle_{I_+} - \langle v\rangle_{I_-}| \le 16M\sqrt{\langle u\rangle_J\langle v\rangle_J}\,.$$

2.1. **Remark on the Haar functions.** *If we introduce the normalized Haar system*

$$h_{_I}(t)=\frac{1}{\sqrt{|I|}}\begin{cases}-1 & \text{if } t\in I_-,\\ \ \ 1 & \text{if } t\in I_+,\end{cases}$$

then $\sqrt{|I|}(\langle w\rangle_{_{I_+}}-\langle w\rangle_{_{I_-}})=2(w,h_{_I})$*. Thus the statement of the Theorem above can be rewritten in the form*

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}_J}|(u,h_{_I})|\,|(v,h_{_I})|\le 4M\sqrt{\langle u\rangle_{_J}\langle v\rangle_{_J}}$$

and that of Buckley's inequality, in the form

$$\sum_{I\in\mathcal{D}_J}\left(\frac{(w,h_{_I})}{\langle w\rangle_{_I}}\right)^2\le 2\log\delta\,|J|\,.$$

2.2. **The Bellman function of the problem.**

$$\mathbf{B}(x;m,M)\stackrel{\text{def}}{=}\sup_{u,v}\left\{\frac{1}{|J|}\sum_{I\in\mathcal{D}_J}|(u,h_{_I})|\,|(v,h_{_I})|\right\},$$

where the supremum is taken over the set of all admissible pairs of weights, i.e. such pairs u,v *that* $\langle u\rangle_{_J}=x_1$, $\langle v\rangle_{_J}=x_2$, *and* $m^2\le\langle u\rangle_{_I}\langle v\rangle_{_I}\le M^2$, $\forall I\in\mathcal{D}_J$*. To prove the theorem means to prove the inequality*

$$\mathbf{B}(x;0,M)\le 4M\sqrt{x_1x_2}\,.$$

The domain of $\mathbf{B}$ *is*

$$\Omega=\left\{x=(x_1,x_2)\colon m^2\le x_1x_2\le M^2\right\}.$$

2.3. **Properties.**

- *The function* $\mathbf{B}$ *does not depend on* J.
- *Homogeneity:* $\mathbf{B}(x_1,x_2)=\mathbf{B}(x_1x_2,1)\stackrel{\text{def}}{=}g(x_1x_2)$.
- *Boundary condition:* $\mathbf{B}|_{x_1x_2=m^2}=g(m^2)=0$.

2.4. **Main inequality.**
For every pair $x^\pm\in\Omega$ *such that* $x=\frac12x^++\frac12x^-\in\Omega$, *we have*

$$\mathbf{B}(x)\ge\frac{\mathbf{B}(x^+)+\mathbf{B}(x^-)}{2}+\frac{|x_1^+-x_1^-|\,|x_2^+-x_2^-|}{4}.$$

In the differential form,

$$\begin{pmatrix}\dfrac{\partial^2B}{\partial x_1^2} & \dfrac{\partial^2B}{\partial x_1\partial x_2}\pm1\\[2ex] \dfrac{\partial^2B}{\partial x_1\partial x_2}\pm1 & \dfrac{\partial^2B}{\partial x_2^2}\end{pmatrix}\le0\,,$$

or, in terms of g,

$$\begin{pmatrix}x_2^2g'' & g'+x_1x_2g''+\sigma\\[2ex] g'+x_1x_2g''+\sigma & x_1^2g''\end{pmatrix}\le0\,,$$

where $\sigma=\pm1$.

The condition that this matrix be degenerate gives us a differential equation, whose general solution is $g(s)=2c\sqrt{s}-s+c_1$ *(this is quite a bit of work). The constant* c_1 *can be* *found from the boundary condition:* $c_1 = m^2 - 2cm$*, and the constant* c *has to be chosen as small as possible to obtain the best estimate:* $c = 2M$*. Thus, the answer is*

$$\mathbf{B}(x; m, M) \le 4M\sqrt{x_1 x_2} - x_1 x_2 + m^2 - 4mM\,.$$

All details of the proof that, in fact, we have found the true Bellman function, i.e.

$$\mathbf{B}(x; m, M) = 4M\sqrt{x_1 x_2} - x_1 x_2 + m^2 - 4mM\,,$$

can be found in [11].

3. John–Nirenberg inequality, Part I

A function $\varphi \in L^1(J)$ is said to belong to the space BMO(J) if

$$\sup_I \langle |\varphi(s) - \langle\varphi\rangle_I|\rangle_I < \infty$$

for all subintervals $I \subset J$. If this condition holds only for the dyadic subintervals $I \in \mathcal{D}_J$, we will write $\varphi \in \mathrm{BMO}^d(J)$. In fact, the following is true

$$\varphi \in \mathrm{BMO}(J) \iff \Big(\int_I |\varphi(s) - \langle\varphi\rangle_I|^p\, ds\Big)^{\frac{1}{p}} < \infty, \quad \forall p \in (0, \infty),\ I \subset J\,.$$

If we factor over the constants, we get a normed space, where the expression on the right-hand side can be taken as one of the equivalent norms for any $p \in [1, \infty)$. In what follows, we will use the L^2-based norm:

$$\|\varphi\|^2_{\mathrm{BMO}(J)} = \sup_{I\subset J} \frac{1}{|I|}\int_I |\varphi(s) - \langle\varphi\rangle_I|^2\, ds = \sup_{I\subset J}\big(\langle\varphi^2\rangle_I - \langle\varphi\rangle_I^2\big)\,.$$

The BMO ball of radius ε centered at 0 will be denoted by BMO_ε. Using the Haar decomposition

$$\varphi(s) = \langle\varphi\rangle_J + \sum_{I\in\mathcal{D}_J} (\varphi, h_I) h_I(s)\,,$$

we can write down the expression for the norm in the following way

$$\|\varphi\|^2_{\mathrm{BMO}(J)} = \sup_{I\subset J}\frac{1}{|I|}\sum_{L\in\mathcal{D}_I} |(\varphi, h_L)|^2 = \frac{1}{4}\sup_{I\subset J}\frac{1}{|I|}\sum_{L\in\mathcal{D}_I} |L|\,\big(\langle\varphi\rangle_{L^+} - \langle\varphi\rangle_{L^-}\big)^2.$$

Theorem (John–Nirenberg [2])**.** *There exist absolute constants* c_1 *and* c_2 *such that*

$$|\{s \in J \colon |\varphi(s) - \langle\varphi\rangle_J| \ge \lambda\}| \le c_1 e^{-c_2\frac{\lambda}{\|\varphi\|}}|J|$$

for all $\varphi \in \mathrm{BMO}_\varepsilon(J)$.

An equivalent, integral form of the same assertion is the following

Theorem. *There exists an absolute constant* ε_0 *such that for any* $\varphi \in \mathrm{BMO}_\varepsilon(J)$ *with* $\varepsilon < \varepsilon_0$ *the inequality*

$$\langle e^\varphi\rangle_J \le c\, e^{\langle\varphi\rangle_J}$$

holds with a constant $c = c(\varepsilon)$ *not depending on* φ*.*

We shall prove the theorem in this integral form and find the sharp constant $c(\varepsilon)$. Our Bellman function,

$$\mathbf{B}(x;\varepsilon) \stackrel{\text{def}}{=} \sup_{\varphi\in\mathrm{BMO}_\varepsilon(J)} \left\{\langle e^{\varphi}\rangle_J \colon \langle\varphi\rangle_J = x_1,\ \langle\varphi^2\rangle_J = x_2\right\},$$

is well-defined on the domain

$$\Omega_\varepsilon \stackrel{\text{def}}{=} \left\{x=(x_1,x_2)\colon x_1^2\le x_2\le x_1^2+\varepsilon^2\right\}.$$

First, we will consider the dyadic problem and deduce the main inequality for the dyadic Bellman function.

Lemma 3.1 (Main inequality)**.** *For every pair of points $x^\pm$ from Ω_ε such that their mean $x=(x^++x^-)/2$ is also in Ω_ε, the following inequality holds*

$$\mathbf{B}(x)\ge\frac{\mathbf{B}(x^+)+\mathbf{B}(x^-)}{2}. \tag{3.1}$$

Proof. The proof repeats almost verbatim the proof of the main inequality for the Buckley Bellman function. We split the integral in the definition of $\mathbf{B}$ into two parts, the integral over J_+ and the one over J_- :

$$\int_J e^{\varphi(s)}\,ds=\int_{J_+}e^{\varphi(s)}\,ds+\int_{J_-}e^{\varphi(s)}\,ds.$$

Now we choose such functions $\varphi^\pm$ on the intervals $J_\pm$ that they almost give us the supremum in the definition of $\mathbf{B}(x^\pm)$, i.e.

$$\frac{1}{|J_\pm|}\int_{J_\pm}e^{\varphi(s)}\,ds\ge\mathbf{B}(x^\pm)-\eta,$$

for a fixed small $\eta>0$. Then for the function φ on J, defined as φ^+ on J_+ and φ^- on J_-, we obtain the inequality

$$\frac{1}{|J|}\int_J e^{\varphi(s)}\,ds\ge\frac{\mathbf{B}(x^+)+\mathbf{B}(x^-)}{2}-\eta. \tag{3.2}$$

Observe that the compound function φ is an admissible test function corresponding to the point x. Indeed, $x^\pm=x^{J_\pm}$ and by construction $\varphi^\pm\in\mathrm{BMO}_\varepsilon^d(J_\pm)$; therefore, the function φ satisfies the inequality $\langle\varphi^2\rangle_I-\langle\varphi\rangle_I^2\le\varepsilon^2$ for all $I\in\mathcal{D}_{J_+}$, since φ^+ does, and for all $I\in\mathcal{D}_{J_-}$, since φ^- does. Lastly, $\langle\varphi^2\rangle_J-\langle\varphi\rangle_J^2\le\varepsilon^2$, because, by assumption, $x\in\Omega_\varepsilon$.

We can now take supremum in (3.2) over all admissible functions φ which yields

$$\mathbf{B}(x)\ge\frac{\mathbf{B}(x^+)+\mathbf{B}(x^-)}{2}-\eta,$$

which proves the main inequality because η is arbitrarily small. □

As in the case of the Buckley inequality, the next our step is to derive a boundary condition for $\mathbf{B}$.

Lemma 3.2 (Boundary condition)**.**

$$\mathbf{B}(x_1,x_1^2)=e^{x_1}. \tag{3.3}$$

Proof. The function $\varphi(s) = x_1$ is the only test function corresponding to the point $x = (x_1, x_1^2)$, because the equality in the Hölder inequality $x_2 \ge x_1^2$ occurs only for constant functions. Hence, $e^{\varphi} = e^{x_1}$. $\square$

Now we are ready to describe super-solutions as functions verifying the main inequality and the boundary conditions.

Lemma 3.3 (Bellman induction)**.** *If B is a continuous function on the domain Ω_ε, satisfying the main inequality* (3.1) *for any pair $x^\pm$ of points from Ω_ε such that $x \stackrel{\text{def}}{=} \frac{x^+ + x^-}{2} \in \Omega_\varepsilon$, as well as the boundary condition* (3.3)*, then $\mathbf{B}(x) \le B(x)$.*

Proof. Fix a bounded function $\varphi \in \mathrm{BMO}_\varepsilon(J)$. By the main inequality we have

$$|J|B(x^J)| \ge |J_+|B(x^{J_+})| + |J_-|B(x^{J_-})| \ge \sum_{I \in \mathcal{D}_J^n} |I|B(x^I) = \int_J B(x^{(n)}(s))\, ds\,,$$

where $x^{(n)}(s) = x^I$, when $s \in I$, $I \in \mathcal{D}_J^n$. (Recall that $\mathcal{D}_J^n$ stands for the set of subintervals of n-th generation.) By the Lebesgue differentiation theorem we have $x^{(n)}(s) \to (\varphi(s), \varphi^2(s))$ almost everywhere. Now, we can pass to the limit in this inequality as $n \to \infty$. Since φ is assumed to be bounded, $x^{(n)}(s)$ runs in a bounded — and, therefore, compact — subdomain of Ω_ε. Since B is continuous, it is bounded on any compact set and so, by the Lebesgue dominated convergence theorem, we can pass to the limit in the integral using the boundary condition (3.3):

$$|J|B(x^J) \ge \int_J B(\varphi(s), \varphi^2(s))\, ds = \int_J e^{\varphi(s)} ds = |J| \langle e^{\varphi} \rangle_{J}\,. \tag{3.4}$$

To complete the proof of the lemma, we need to pass from bounded to arbitrary BMO test functions. To this end, we will use the following result:

Lemma 3.4 (Cut-off Lemma)**.** *Fix $\varphi \in \mathrm{BMO}(J)$ and two real numbers c, d such that $c < d$. Let $\varphi_{c,d}$ be the cut-off of φ at heights c and d :*

$$\varphi_{c,d}(s) = \begin{cases} c, & if\ \varphi(s) \le c; \\ \varphi(s), & if\ c < \varphi(s) < d; \\ d, & if\ \varphi(s) \ge d. \end{cases} \tag{3.5}$$

Then

$$\langle \varphi_{c,d}^2 \rangle_{I} - \langle \varphi_{c,d} \rangle_{I}^2 \le \langle \varphi^2 \rangle_{I} - \langle \varphi \rangle_{I}^2, \quad \forall I,\ I \subset J,$$

and, consequently,

$$\|\varphi_{c,d}\|_{\mathrm{BMO}} \le \|\varphi\|_{\mathrm{BMO}}.$$

Proof. First, let us note that it is sufficient to prove this lemma for a one-sided cut, for example, for $c = -\infty$. We then get the full statement by applying this argument twice. Indeed, if we denote by $C_d\varphi$ the cut-off of φ from above at height d, i.e. $C_d\varphi = \varphi_{-\infty,d}$, then $\varphi_{c,d} = -C_{-c}(-C_d\varphi)$.

Take a measurable subset $I \subset J$ and let $I_1 = \{s \in I \colon \varphi(s) < d\}$ and $I_2 = \{s \in I \colon \varphi(s) \ge d\}$. Let $\beta_k = |I_k|/|I|, k = 1, 2$. We have the following identity:

$$\begin{aligned} &\big[\langle \varphi^2 \rangle_{I} - \langle \varphi \rangle_{I}^2\big] - \big[\langle (C_d\varphi)^2 \rangle_{I} - \langle C_d\varphi \rangle_{I}^2\big] \\ &= \beta_2 \big[\langle \varphi^2 \rangle_{I_2} - \langle \varphi \rangle_{I_2}^2\big] + \beta_1\beta_2 \big[\langle \varphi \rangle_{I_2} - d\big]\big[\langle \varphi \rangle_{I_2} + d - 2\langle \varphi \rangle_{I_1}\big], \end{aligned}$$

which proves the lemma, because $\langle\varphi\rangle_{I_1} \le d \le \langle\varphi\rangle_{I_2}$. □

Now, let $\varphi \in \mathrm{BMO}_\varepsilon(J)$ be a function bounded from above. Then, by the above lemma, $\varphi_n \stackrel{\text{def}}{=} \varphi_{-n,\infty} \in \mathrm{BMO}_\varepsilon(J)$, and, according to (3.4), we have

$$B(\langle\varphi_n\rangle_J, \langle\varphi_n^2\rangle_J) \ge \langle e^{\varphi_n}\rangle_J .$$

Since e^φ is a summable majorant for e^{φ_n} and B is continuous, we can pass to the limit and obtain the estimate (3.4) for any function φ bounded from above. Finally, we repeat this approximation procedure for an arbitrary φ. Now, we take $\varphi_n = \varphi_{-\infty,n}$ and use the monotone convergence theorem to pass to the limit in the right-hand side of the inequality.

So, we have proved the inequality

$$B(x^J) \ge \langle e^\varphi\rangle_J$$

for arbitrary $\varphi \in \mathrm{BMO}_\varepsilon(J)$. Taking supremum over all admissible test functions corresponding to the point x, we get $B(x) \ge \mathbf{B}(x)$. □

As before, we pass from the finite-difference inequality (3.1) to the infinitesimal one:

$$\frac{d^2B}{dx^2} \stackrel{\text{def}}{=} \begin{pmatrix} \dfrac{\partial^2 B}{\partial x_1^2} & \dfrac{\partial^2 B}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 B}{\partial x_1 \partial x_2} & \dfrac{\partial^2 B}{\partial x_2^2} \end{pmatrix} \le 0 , \tag{3.6}$$

and we will require this Hessian matrix to be degenerate, i.e. $\det(\frac{d^2B}{dx^2}) = 0$. Again, to solve this PDE, we use a homogeneity property to reduce the problem to an ODE.

Lemma 3.5 (Homogeneity)**.** *There exists a function G on the interval $[0, \varepsilon^2]$ such that*

$$\mathbf{B}(x;\varepsilon) = e^{x_1} G(x_2 - x_1^2) , \qquad G(0) = 1 .$$

Proof. Let φ be an arbitrary test function and $x = (\langle\varphi\rangle_J, \langle\varphi^2\rangle_J)$ its Bellman point on J. Then the function $\tilde\varphi \stackrel{\text{def}}{=} \varphi + \tau$ is also a test function with the same norm, and its Bellman point is $\tilde x = (x_1 + \tau, x_2 + 2\tau x_1 + \tau^2)$. Therefore,

$$\mathbf{B}(\tilde x) = \sup_{\tilde\varphi} \langle e^{\tilde\varphi}\rangle_J = e^\tau \sup_{\varphi} \langle e^\varphi\rangle_J = e^\tau \mathbf{B}(x) .$$

Choosing $\tau = -x_1$ we get

$$\mathbf{B}(x) = e^{-\tau}\mathbf{B}(x_1 + \tau, x_2 + 2\tau x_1 + \tau^2) = e^{x_1}\mathbf{B}(0, x_2 - x_1^2) .$$

Setting $G(t) = \mathbf{B}(0,t)$ completes the proof. □

Since $G > 0$, we can introduce $g(t) = \log G(t)$ and look for a function B of the form

$$B(x_1, x_2) = e^{x_1 + g(x_2 - x_1^2)} .$$

By direct calculation, we get

$$\frac{\partial^2 B}{\partial x_1^2} = \left(1 - 4x_1 g' + 4x_1^2 (g')^2 - 2g' + 4x_1^2 g''\right) B\,,$$
$$\frac{\partial^2 B}{\partial x_1 \partial x_2} = \left(g' - 2x_1 (g')^2 - 2x_1 g''\right) B\,,$$
$$\frac{\partial^2 B}{\partial x_2^2} = \left((g')^2 + g''\right) B.$$

The partial differential equation $\det(\frac{d^2B}{dx^2}) = 0$ then turns into the following ordinary differential equation:

$$\left(1 - 4x_1 g' + 4x_1^2 (g')^2 - 2g' + 4x_1^2 g''\right)\left((g')^2 + g''\right) = \left(g' - 2x_1 (g')^2 - 2x_1 g''\right)^2,$$

which reduces to

$$g'' - 2g'g'' - 2(g')^3 = 0\,.$$

Dividing by $2(g')^3$ (since we are not interested in constant solutions), we get

$$\left(\frac{1}{g'} - \frac{1}{4(g')^2}\right)' = 1\,,$$

which yields

$$\frac{1}{g'} - \frac{1}{4(g')^2} = t + \text{const}$$

or, equivalently,

$$-\left(1 - \frac{1}{2g'}\right)^2 = t + \text{const}, \qquad \forall s \in [0, \varepsilon^2]\,.$$

Since the left-hand side is non-positive, the constant cannot be greater than $-\varepsilon^2$. Let us denote it by $-\delta^2$, where $\delta \geq \varepsilon$.

Thus, we have two possible solutions:

$$1 - \frac{1}{2g'_\pm} = \pm\sqrt{\delta^2 - t}\,.$$

Using the boundary condition $g(0) = 0$, we obtain

$$g_\pm(t) = \frac{1}{2}\int_0^t \frac{ds}{1 \mp \sqrt{\delta^2 - s}} = \log\frac{1 \mp \sqrt{\delta^2 - t}}{1 \mp \delta} \pm \sqrt{\delta^2 - t} \mp \delta\,.$$

This yields two solutions for B :

$$B_\pm(x) = \frac{1 \mp \sqrt{\delta^2 - x_2 + x_1^2}}{1 \mp \delta} \exp\left\{x_1 \pm \sqrt{\delta^2 - x_2 + x_1^2} \mp \delta\right\}\,.$$

Homework assignment.

(1) *Check that the quadratic form of the Hessian is:*

$$\sum_{i,j=1}^{2} \frac{\partial^2 B_\pm}{\partial x_i \partial x_j} \Delta_i \Delta_j = \mp \frac{\left(\left(x_1 \pm \sqrt{\delta^2 - x_2 + x_1^2}\right)\Delta_1 - \frac{1}{2}\Delta_2\right)^2}{\sqrt{\delta^2 - x_2 + x_1^2}\,(1 \mp \delta)} \exp\left\{x_1 \pm \sqrt{\delta^2 - x_2 + x_1^2} \mp \delta\right\}.$$

(2) *Find the extremal trajectories along which the Hessian degenerates.*

4. Homogeneous Monge–Ampère equation

Now, we change the subject of our consideration for a while and look for the solutions of the equation

$$B_{x_1x_1}B_{x_2x_2}=(B_{x_1x_2})^2 \tag{4.1}$$

in a general setting.

Linear functions always satisfy (4.1). Since we are looking for the smallest possible concave function B, it always will be linear, if a linear function satisfies the required boundary conditions. It is a simple case, and in what follows we assume that B is not linear. This means that in each point x of the domain there exists a unique (up to a scalar coefficient) vector, say $\Theta(x)$, from the kernel of the matrix $\frac{d^2B}{dx^2}$.

Let us check that functions B_{x_i} are constant along the vector field Θ. The tangent vector field to the level set $f(x_1,x_2)=\text{const}$ has the form $\begin{pmatrix}-f_{x_2}\\ f_{x_1}\end{pmatrix}$ (it is orthogonal to $\operatorname{grad} f=\begin{pmatrix}f_{x_1}\\ f_{x_2}\end{pmatrix}$). Thus, we need to check that the both vectors $\begin{pmatrix}-(B_{x_i})_{x_2}\\ (B_{x_i})_{x_1}\end{pmatrix}$ are in the kernel of the Hessian (i.e. proportional the kernel vector Θ). This is a direct consequence of (4.1). For example, for $i=1$ we have:

$$\begin{pmatrix}B_{x_1,x_1} & B_{x_1x_2}\\ B_{x_2x_1} & B_{x_2x_2}\end{pmatrix}\begin{pmatrix}-(B_{x_1})_{x_2}\\ (B_{x_1})_{x_1}\end{pmatrix}=\begin{pmatrix}-B_{x_1x_1}B_{x_1x_2}+B_{x_1x_2}B_{x_1x_1}\\ -B_{x_2x_1}B_{x_1x_2}+B_{x_2x_2}B_{x_1x_1}\end{pmatrix}=0.$$

If we parameterize the integral curves of the field Θ by some parameter s we can write $B_{x_i}\stackrel{\text{def}}{=}t_i(s)$, $s=s(x_1,x_2)$. Any B_{x_i} that is not identically constant can itself be taken as s. However, usually it is more convenient to parameterize the integral curves by some other parameter with a clear geometrical meaning.

Now, we check that the function $t_0\stackrel{\text{def}}{=}B-x_1t_1-x_2t_2$ is also constant along the integral curves. Since

$$-\frac{\partial t_0}{\partial x_2}=-B_{x_2}+x_1\frac{\partial t_1}{\partial x_2}+x_2\frac{\partial t_2}{\partial x_2}+t_2=x_1B_{x_1x_2}+x_2B_{x_2x_2}$$

and

$$\frac{\partial t_0}{\partial x_1}=B_{x_1}-t_1-x_1\frac{\partial t_1}{\partial x_1}-x_2\frac{\partial t_2}{\partial x_1}=-x_1B_{x_1x_1}-x_2B_{x_1x_2},$$

we have

$$\begin{pmatrix}-(t_0)_{x_2}\\ (t_0)_{x_1}\end{pmatrix}=-x_1\begin{pmatrix}-(t_1)_{x_2}\\ (t_1)_{x_1}\end{pmatrix}-x_2\begin{pmatrix}-(t_2)_{x_2}\\ (t_2)_{x_1}\end{pmatrix}\in\operatorname{Ker}\frac{d^2B}{dx^2}.$$

So, we have proved that in the representation

$$B=t_0+x_1t_1+x_2t_2 \tag{4.2}$$

of a solution of the homogeneous Monge–Ampère equation, the coefficients t_i are constant along the vector field generated by the kernel of the Hessian. Now we prove that the integral curves of this vector field are in fact straight lines given by the equation

$$dt_0+x_1dt_1+x_2dt_2=0\,. \tag{4.3}$$

This is, indeed, the equation of a straight line, because all the differentials are constant along the trajectory. In a parametrization of the trajectories is chosen, this equation can be rewritten as a usual linear equation with constant coefficients. For example, let us take $s = t_0$; then (4.3) turns into

$$1 + x_1\frac{dt_1}{dt_0} + x_2\frac{dt_2}{dt_0} = 0\,,$$

where the coefficients $\frac{dt_i}{dt_0}$, being functions of t_0, are constant on each trajectory.

Now, let us deduce equation (4.3). On one hand,

$$dB = B_{x_1}dx_1 + B_{x_2}dx_2 = t_1dx_1 + t_2dx_2\,. \tag{4.4}$$

On the other hand, from representation (4.2) we have

$$dB = dt_0 + t_1dx_1 + x_1dt_1 + t_2dx_2 + x_2dt_2\,. \tag{4.5}$$

A comparison of (4.4) and (4.5) yields (4.3).

More details about solutions of the homogeneous Monge–Ampère equation, together with an example of its application to the John–Nirenberg inequality, can be found in [12]. In the following section, we just consider this alternative method of finding a candidate for the role of the Bellman function for the integral John–Nirenberg inequality.

5. John–Nirenberg inequality, Part II

Let us now re-solve the Monge–Ampère boundary value problem for the John–Nirenberg inequality using the method described in the previous section. We are looking for a solution of the form $B(x) = t_0 + x_1t_1 + x_2t_2$ satisfying the boundary condition

$$B(x_1, x_1^2) = e^{x_1} \tag{5.1}$$

and the homogeneity condition

$$B(x_1 + \tau, x_2 + 2\tau x_1 + \tau^2) = e^\tau B(x)\,.$$

This time, instead of using this identity to reduce the number of variables, we differentiate it with respect to τ,

$$\frac{\partial B}{\partial x_1} + (2x_1 + 2\tau)\frac{\partial B}{\partial x_2} = e^\tau B(x)\,,$$

and set $\tau = 0$:

$$t_1 + 2x_1t_2 = t_0 + x_1t_1 + x_2t_2\,.$$

Thus, we obtain an equation of a straight line:

$$(t_0 - t_1) + x_1(t_1 - 2t_2) + x_2t_2 = 0\,. \tag{5.2}$$

Since our B cannot be a linear function (a linear function cannot satisfy the boundary condition), we have only one extremal line passing through a given point. Therefore, this line must coincide with (4.3), which yields proportionality of the coefficients:

$$\frac{dt_0}{t_0 - t_1} = \frac{dt_1}{t_1 - 2t_2} = \frac{dt_2}{t_2}\,. \tag{5.3}$$

Using the second equality, we express t_1 in terms of t_2 :

$$\begin{aligned} t_2 dt_1 &= (t_1 - 2t_2)dt_2 \\ t_2 dt_1 - t_1 dt_2 &= -2t_2 dt_2 \\ d(\frac{t_1}{t_2}) &= -2\frac{dt_2}{t_2} \\ \frac{t_1}{t_2} &= -2\log|t_2| + 2c_1 \\ t_1 &= -2t_2\log|t_2| + 2c_1 t_2\,. \end{aligned}$$

Now, we use the equality between the first and third terms in (5.3):

$$\begin{aligned} t_2 dt_0 &= (t_0 - t_1)dt_2 \\ t_2 dt_0 - t_0 dt_2 &= -t_1 dt_2 \\ d(\frac{t_0}{t_2}) &= -\frac{t_1}{t_2}\,\frac{dt_2}{t_2} \\ &= 2(\log|t_2| - c_1)d\log|t_2| \\ &= d(\log^2|t_2| - 2c_1\log|t_2|) \\ \frac{t_0}{t_2} &= \log^2|t_2| - 2c_1\log|t_2| + c_2 \\ t_0 &= t_2\log^2|t_2| - 2c_1 t_2\log|t_2| + c_2 t_2\,. \end{aligned}$$

Dividing (5.2) by t_2 gives

$$\Big(\frac{t_0}{t_2} - \frac{t_1}{t_2}\Big) + x_1\Big(\frac{t_1}{t_2} - 2\Big) + x_2 = 0.$$

Plugging into this equality the earlier expressions for t_1 and t_0, we get

$$(5.4)\quad \big(\log^2|t_2| - 2c_1\log|t_2| + c_2 + 2\log|t_2| - 2c_1\big) + x_1\big(-2\log|t_2| + 2c_1 - 2\big) + x_2 = 0\,.$$

From this expression, it is clear that it is convenient to introduce a new parametrization of our extremal trajectories:

$$a = \log|t_2| - c_1 + 1\,.$$

The equation of the extremal trajectory (5.4) then takes the form

$$a^2 - 2ax_1 + x_2 - 1 + c_2 - c_1^2 = 0\,.$$

Since

$$c_1^2 + 1 - c_2 = a^2 - 2ax_1 + x_2 = (a - x_1)^2 + (x_2 - x_1^2) \geq 0\,,$$

we can introduce a new positive constant $\delta \overset{\text{def}}{=} (c_1^2 + 1 - c_2)^{1/2}$. In this notation (5.4) becomes

$$(5.5)\qquad x_2 = 2ax_1 - a^2 + \delta^2\,.$$

Note that this is an equation of the line tangent to the parabola $x_2 = x_1^2 + \delta^2$ at the point $(a, a^2 + \delta^2)$.

Now, let us collect everything and write down a formula for B :

$$\begin{aligned} t_0 &= (a^2-2a+2-\delta^2)t_2 \\ t_1 &= -2(a-1)t_2 \\ t_2 &= \pm e^{a+c_1-1} = ce^a \\ B &= t_0 + x_1t_1 + x_2t_2 \\ &= (a^2-2a+2-\delta^2-2(a-1)x_1+x_2)t_2 \\ &= 2c(1-a-x_1)e^a\,. \end{aligned}$$

From the equation of the extremal line (5.5), we can express a as a function of x :

$$a = a(x) = x_1 \pm \sqrt{\delta^2 - x_2 + x_1^2}.$$

Therefore,

$$B(x) = 2c\Big(1 \mp \sqrt{\delta^2 - x_2 + x_1^2}\Big)\exp\Big\{x_1 \pm \sqrt{\delta^2 - x_2 + x_1^2}\Big\}\,.$$

We find the constant c from the boundary condition (5.1), and we choose the sign by checking the sign of the Hessian. Finally, we obtain

$$B(x) = \frac{1-\sqrt{\delta^2 - x_2 + x_1^2}}{1-\delta}\exp\Big\{x_1 + \sqrt{\delta^2 - x_2 + x_1^2} - \delta\Big\}\,. \tag{5.6}$$

Were we a bit more clever, we could realize from the beginning that any extremal trajectory must touch the upper boundary tangentially, because, when splitting the interval, a boundary point x can be split into $x^{\pm}$ only along the tangential direction. With that realization all calculations become much simpler.

Indeed, take an extremal line given by (5.5). It intersects the lower boundary $x_2 = x_1^2$ at two points (u, u^2) with $u = a \mp \varepsilon$. Since B has to be linear on the extremal line, we have

$$B(x) = k(u)(u - x_1) + f(u),$$

where $f(u)$ is the boundary value of B. We will not specify this value until the very end of calculation. This will serve to demonstrate that knowing extremal trajectories in advance allows one to solve some rather general problems, and not just this specific one.

Let us calculate the partial derivative of B with respect to either coordinate, say x_2 :

$$t_2 = B_{x_2} = \big(k'(u)(x_1-u) - k(u) + f'(u)\big)\frac{\partial u}{\partial x_2}\,.$$

Using (5.5), we get

$$\frac{\partial u}{\partial x_2} = \frac{\partial a}{\partial x_2} = \frac{1}{2(x_1-a)}$$

and

$$t_2 = \frac{1}{2}k'(u) + \frac{\pm\varepsilon k'(u) - k(u) + f'(u)}{2(x_1-a)}\,.$$

Since t_2 has to be constant on the extremal line, we conclude that

$$t_2 = \frac{1}{2}k'(u)$$

and the coefficient k satisfies the equation

$$\mp\varepsilon k'(u) - k(u) + f'(u) = 0,$$

whose general solution is

$$k(u) = \pm\frac{1}{\varepsilon}\int_u^{\pm\infty} e^{-|t-u|/\varepsilon} f'(t)\,dt + C_\pm e^{\pm u/\varepsilon}\,.$$

(The same conclusion could be reached by considering B_{x_1}.) Let us not discuss at this point why the constant $C_\pm$ should be chosen equal to zero, other than say that it is a consequence of our trying to find the best possible estimate. Rewriting the last formula for our case, $f(u) = e^u$, we get

$$k(u) = \frac{1}{1\mp\varepsilon}e^u\,,$$

and, therefore,

$$\begin{aligned} B(x) &= e^u\Big(\frac{x_1 - u}{1\mp\varepsilon} + 1\Big) \\ &= \frac{1\mp\sqrt{\varepsilon^2 - x_2 + x_1^2}}{1\mp\varepsilon}\exp\Big\{x_1 \pm \sqrt{\varepsilon^2 - x_2 + x_1^2} \mp \varepsilon\Big\}\,. \end{aligned}$$

Taking the upper sign throughout, we get (5.6) with $\delta = \varepsilon$.

Homework assignment. *To emphasize the dependence on parameter, let us refer to the function* (5.6) *as $B(x;\delta)$. Check that $B(x;\varepsilon)$ does not satisfy the main inequality in the domain Ω_ε, but $B(x;\delta)$ does, provided $\delta \geq \frac{3}{2\sqrt{2}}\varepsilon$.*

6. John–Nirenberg inequality, Part III

We now prove a geometric result that is crucial to applying the Bellman function method to the usual, non-dyadic BMO (recall that up to this point all discussions were about the dyadic space). Let $[x,y]$ denote the straight-line segment connecting two points x and y in the plane. Then we have the following lemma.

Lemma 6.1 (Splitting lemma)**.** *Fix two positive numbers ε,δ, with $\varepsilon < \delta$. For an arbitrary interval I and any function $\varphi \in \mathrm{BMO}_\varepsilon(I)$, there exists a splitting $I = I_+ \cup I_-$ such that the whole straight-line segment $[x^{I_-}, x^{I_+}]$ is inside Ω_δ. Moreover, the parameters of splitting $\alpha_\pm \stackrel{\mathrm{def}}{=} |I_\pm|/|I|$ are separated form 0 and 1 by constants depending on ε and δ only, i.e. uniformly with respect to the choice of I and φ.*

Proof. Fix an interval I and a function $\varphi \in \mathrm{BMO}_\varepsilon(I)$. We now demonstrate an algorithm to find a splitting $I = I_- \cup I_+$ (i.e. choose the splitting parameters $\alpha_\pm = |I_\pm|/|I|$) so that the statement of the lemma holds. For simplicity, put $x^0 = x^I$ and $x^\pm = x^{I_\pm}$.

First, we take $\alpha_- = \alpha_+ = \frac{1}{2}$ (see Fig. 1). If the whole segment $[x^-, x^+]$ is in Ω_δ, we fix this splitting. Assuming it is not the case, there exists a point x on this segment with $x_2 - x_1^2 > \delta^2$. Observe that only one of the segments, either $[x^-, x^0]$ or $[x^+, x^0]$, contains such points. Denote the corresponding endpoint (x^- or x^+) by ξ and define a function ρ by

$$\rho(\alpha_+) = \max_{x\in[x^-,x^+]}\{x_2 - x_1^2\} = \max_{x\in[\xi,x^0]}\{x_2 - x_1^2\}.$$

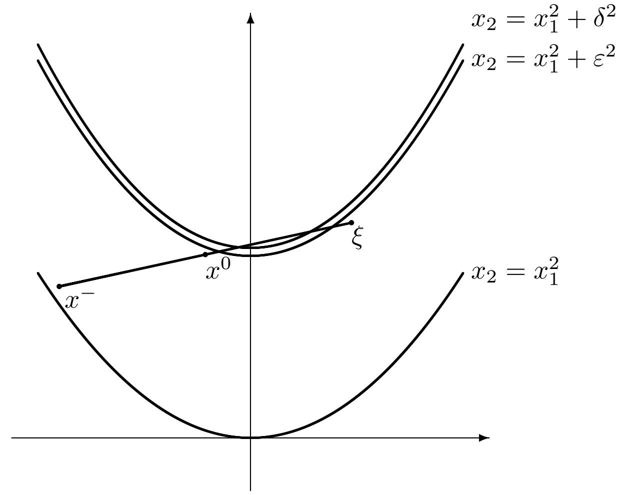


FIGURE 1. The initial splitting: $\alpha_- = \alpha_+ = \frac{1}{2}$, $\xi = x^+$.

By assumption, $\rho\left(\frac{1}{2}\right) > \delta^2$. We will now change α_+ so that ξ approaches x^0, i.e. we will increase α_+ if $\xi = x^+$ and decrease it if $\xi = x^-$. We stop when $\rho(\alpha_+) = \delta^2$ and fix that splitting. It remains to check that such a moment occurs and that the corresponding α_+ is separated from 0 and 1.

Without loss of generality, assume that $\xi = x^+$. Since the function $x^+(\alpha_+)$ is continuous on the interval $(0, 1]$ and $x^+(1) = x^0$, ρ is continuous on $[\frac{1}{2}, 1]$. We have $\rho\left(\frac{1}{2}\right) > \delta^2$ and we also know that $\rho(1) \le \varepsilon^2 < \delta^2$ (because $x^0 \in \Omega_\varepsilon$). Therefore, there is a point $\alpha_+ \in \left[\frac{1}{2}, 1\right]$ with $\rho(\alpha_+) = \delta^2$ (Fig. 2).

Having just proved that the desired point exists, we need to check that the corresponding α_+ is not too close to 0 or 1. If $\xi = x^+$, we have $\alpha_+ > \frac{1}{2}$ and $\xi_1 - x_1^0 = x_1^+ - x_1^0 = \alpha_-(x_1^+ - x_1^-)$. Similarly, if $\xi = x^-$, we have $\alpha_- > \frac{1}{2}$ and $\xi_1 - x_1^0 = x_1^- - x_1^0 = \alpha_+(x_1^- - x_1^+)$. Thus, $|\xi_1 - x_1^0| = \min\{\alpha_\pm\}|x_1^- - x_1^+|$. For the stopping value of α_+, the

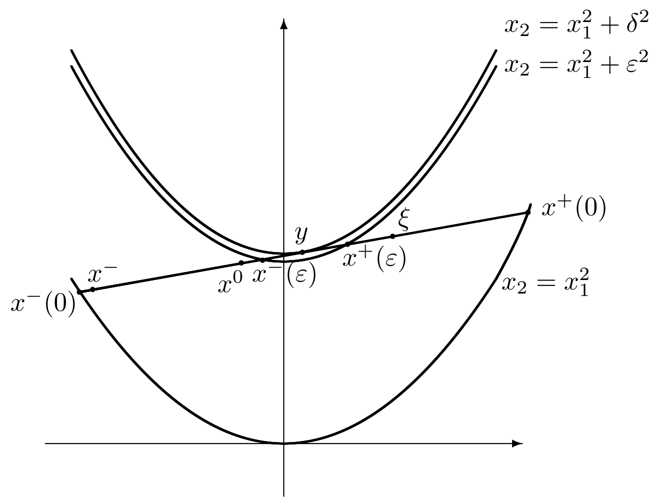


FIGURE 2. The stopping time: $[x^-, \xi]$ is tangent to the parabola $x_2 = x_1^2 + \varepsilon^2$.

straight line through the points x^-, x^+, and x^0 is tangent to the parabola $x_2 = x_1^2 + \delta^2$ at some point y. The equation of this line is, therefore, $x_2 = 2x_1y_1 - y_1^2 + \delta^2$. The line intersects the graph of $x_2 = x_1^2 + s^2$ at the points

$$x^\pm(s) = \left(y_1 \pm \sqrt{\delta^2 - s^2},\, y_2 \pm 2y_1\sqrt{\delta^2 - s^2}\right).$$

Let us focus on the points $x^\pm(0)$ and $x^\pm(\varepsilon)$. We have

$$[x^-(\varepsilon),\, x^+(\varepsilon)] \subset [x^0,\, \xi] \subset [x^-,\, x^+] \subset [x^-(0),\, x^+(0)]$$

and, therefore,

$$\begin{aligned} 2\sqrt{\delta^2 - \varepsilon^2} &= |x_1^+(\varepsilon) - x_1^-(\varepsilon)| \le |x_1^0 - \xi_1| = \min\{\alpha_\pm\}|x_1^+ - x_1^-| \\ &\le \min\{\alpha_\pm\}|x_1^+(0) - x_1^-(0)| = \min\{\alpha_\pm\}2\delta, \end{aligned}$$

which implies

$$\sqrt{1 - \left(\frac{\varepsilon}{\delta}\right)^2} \le \alpha_+ \le 1 - \sqrt{1 - \left(\frac{\varepsilon}{\delta}\right)^2}.$$

As promised, this estimate does not depend on φ or I. □

From now on, we shall consider not the dyadic Bellman function $\mathbf{B}$, but the "true" one:

$$\mathbf{B}(x;\varepsilon) \stackrel{\text{def}}{=} \sup_{\varphi\in \mathrm{BMO}_\varepsilon(J)} \left\{\langle e^\varphi\rangle_J \colon \langle\varphi\rangle_J = x_1,\ \langle\varphi^2\rangle_J = x_2\right\}.$$

The test functions now run over the ε-ball of the non-dyadic BMO.

Using the splitting lemma, we are able to make the Bellman induction work in the non-dyadic case.

Lemma 6.2 (Bellman induction). *If B is a continuous, locally concave function on the domain Ω_δ, satisfying the boundary condition* (3.3), *then $\mathbf{B}(x;\varepsilon) \le B(x)$ for all $\varepsilon < \delta$.*

Proof. Fix a function $\varphi \in \mathrm{BMO}_\varepsilon(J)$. By the splitting lemma we can split every subinterval $I \subset J$, in such a way that the segment $[x^{I_-}, x^{I_+}]$ is inside Ω_δ. Since B is locally concave, we have

$$|I|B(x^I) \ge |I_+|B(x^{I_+}) + |I_-|B(x^{I_-})$$

for any such splitting. Now we can repeat, word for word, the arguments used in the dyadic case. If $\mathcal{D}_n$ is the set of intervals of n-th generation, then

$$|J|B(x^J) \ge |J_+|B(x^{J_+}) + |J_-|B(x^{J_-}) \ge \sum_{I\in\mathcal{D}_n} |I|B(x^I) = \int_J B(x^{(n)}(s))\,ds\,,$$

where $x^{(n)}(s) = x^I$, when $s \in I$, $I \in \mathcal{D}_n$. By the Lebesgue differentiation theorem we have $x^{(n)}(s) \to (\varphi(s), \varphi^2(s))$ almost everywhere. (We have used here the fact that we split the intervals so that all coefficients $\alpha_\pm$ are uniformly separated from 0 and 1, and, therefore, $\max\{|I| \colon I \in \mathcal{D}_n\} \to 0$ as $n \to \infty$). Now, we can pass to the limit in this inequality as $n \to \infty$. Again, first we assume φ to be bounded and, by the

Lebesgue dominated convergence theorem, pass to the limit in the integral using the boundary condition (3.3):

$$|J|B(x^J) \geq \int_J B(\varphi(s), \varphi^2(s))\,ds = \int_J e^{\varphi(s)} ds = |J|\langle e^{\varphi}\rangle_J\,.$$

Then using the cut-off approximation, we get the same inequality for an arbitrary $\varphi \in \mathrm{BMO}_\varepsilon(J)$. □

Corollary 6.3.

$$\mathbf{B}(x;\varepsilon) \leq B(x;\delta) \qquad \varepsilon < \delta < 1\,.$$

Proof. The function $B(x;\delta)$ was constructed as a locally concave function satisfying boundary condition (3.3). □

Corollary 6.4.

$$\mathbf{B}(x;\varepsilon) \leq B(x;\varepsilon)\,. \tag{6.1}$$

Proof. Since the function $B(x;\delta)$ is continuous with respect to the parameter $\delta \in (0,1)$, we can pass to the limit $\delta \to \varepsilon$ in the preceding corollary. □

Now, we would like to prove the inequality converse to (6.1). To this end, for every point x of Ω_ε we construct a test function φ on any interval with BMO norm ε, satisfying $\langle e^{\varphi}\rangle = B(x;\varepsilon)$, and such that its Bellman point is x (let us call such a function an *optimizer* for the point x). This would imply the inequality $\mathbf{B}(x;\varepsilon) \geq B(x;\varepsilon)$.

First, we construct an optimizer φ_0 for the point $(0,\varepsilon^2)$. Without loss of generality, we can work on $I \stackrel{\mathrm{def}}{=} [0,1]$. Note that the function $\varphi_a \stackrel{\mathrm{def}}{=} \varphi_0 + a$ will then be an optimizer for the point $(a, a^2+\varepsilon^2)$. Indeed, φ_a has the same norm as φ_0, and if

$$\langle e^{\varphi_0}\rangle = B(0,\varepsilon^2;\varepsilon) = \frac{e^{-\varepsilon}}{1-\varepsilon}\,,$$

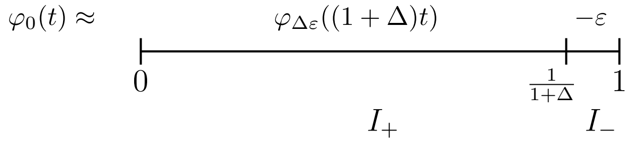
then

$$\langle e^{\varphi_a}\rangle = \frac{e^{a-\varepsilon}}{1-\varepsilon} = B(a, a^2+\varepsilon^2;\varepsilon)\,.$$

The point $(0,\varepsilon^2)$ is on the extremal line starting at $(-\varepsilon,\varepsilon^2)$. To keep equality on each step of the Bellman induction, when we split I into two subintervals I_- and I_+, the segment $[x^-, x^+]$ has to be contained in the extremal line along which our function B is linear. Since x is a convex combination of x^- and x^+, one of these points, say x^+, has to be to the right of x. However, the extremal line ends at $x = (0,\varepsilon^2)$, and so there seems to be nowhere to place that point. We circumvent this difficulty by placing x^+ infinitesimally close to x and using an approximation procedure. Where should x^- be placed? We already know optimizers for points on the lower boundary $x_2 = x_1^2$, since the only test function there are constants. Thus, it is convenient to put x^- there. Therefore, we set

$$x^- = (-\varepsilon,\varepsilon^2) \qquad \text{and} \qquad x^+ = (\Delta\varepsilon,\varepsilon^2)\,,$$

for small Δ. To get these two points, we have to split I in proportion $1:\Delta$, that is we take $I_+ = [0,\frac{1}{1+\Delta}]$ and $I_- = [\frac{1}{1+\Delta},1]$. To get the point x^-, we have to put $\varphi_0(t) = -\varepsilon$ on I_-. On I_+, we put a function corresponding not to the point x^+, but to the point $(\Delta\varepsilon,(1+\Delta^2)\varepsilon)$ on the upper boundary, which is close to x^+ (the distance



between these two points is of order Δ^2). For such a point the extremal function is $\varphi_{\Delta\varepsilon}(t) = \varphi_0(t) + \Delta\varepsilon$. Therefore, this function, when properly rescaled, can be placed on I_+. As a result, we obtain

$$\varphi_0(t) \approx \varphi_0\big((1+\Delta)t\big) + \Delta\varepsilon \approx \varphi_0(t) + \varphi_0'(t)\Delta t + \Delta\varepsilon\,,$$

which yields

$$\varphi_0'(t) = -\frac{\varepsilon}{t}\,.$$

Taking into account the boundary condition $\varphi_0(1) = -\varepsilon$, we get

$$\varphi_0(t) = \varepsilon \log\frac{1}{t} - \varepsilon\,.$$

Let us check that we have found what we need:

$$\langle e^{\varphi_0}\rangle_{[0,1]} = \int_0^1 e^{-\varepsilon}\frac{dt}{t^\varepsilon} = \frac{e^{-\varepsilon}}{1-\varepsilon} = B(0,\varepsilon^2;\varepsilon)\,.$$

It easy now to get an extremal function for an arbitrary point x in Ω_ε. First of all, we draw the extremal line through x. It touches the upper boundary at the point $(a, a^2+\varepsilon^2)$ with $a = x_1 + \sqrt{\varepsilon^2 - x_2 + x_1^2}$ and intersects the lower boundary at the point (u, u^2) with $u = a - \varepsilon$. Now, we split the interval $[0,1]$ in proportion $(x_1-u):(a-x_1)$ and concatenate the two known optimizers, $\varphi = u$ for the $x^- = (u,u^2)$ and $\varphi = \varphi_a$ for $x^+ = (a, a^2+\varepsilon^2)$. This gives the following function:

$$\varphi(t) = \begin{cases} \varepsilon\log\frac{x_1-u}{t} + u & 0 \le t \le x_1 - u \\ u & x_1 - u \le t \le 1 \end{cases}, \quad \text{where} \quad u = x_1 + \sqrt{\varepsilon^2 - x_2 + x_1^2} - \varepsilon\,.$$

This is a function from BMO_ε satisfying the required property $\langle e^{\varphi}\rangle_{[0,1]} = B(x;\varepsilon)$ (see the homework assignment below).

This completes the proof of the following theorem

Theorem. *If* $\varepsilon < 1$, *then*

$$\mathbf{B}(x;\varepsilon) = \frac{1 - \sqrt{\varepsilon^2 - x_2 + x_1^2}}{1-\varepsilon}\exp\Big\{x_1 + \sqrt{\varepsilon^2 - x_2 + x_1^2} - \varepsilon\Big\}\,;$$

if $\varepsilon \ge 1$, *then* $\mathbf{B}(x;\varepsilon) = \infty$.

Indeed, the second statement can be verified by the same extremal function φ, because e^φ is not summable on $[0,1]$ for $\varepsilon \ge 1$.

The first proof of the theorem above appeared in [5] and [10]; a complete proof of this result together with the estimate from below (i.e. the lower Bellman function) and consideration of the dyadic version of the problem can be found in [8] (the online version of this paper is [9]).

Homework assignment 1. *Verify the following properties of the extremal function* φ *:*

- $\langle\varphi\rangle_{[0,1]} = x_1$;
- $\langle\varphi^2\rangle_{[0,1]} = x_2$;
- $\langle e^\varphi\rangle_{[0,1]} = B(x_1, x_2; \varepsilon)$;
- $\varphi \in \mathrm{BMO}_\varepsilon$. [1]

Homework assignment 2. *Recall that we also obtained a second solution,*

$$b(x;\varepsilon) = \frac{1+\sqrt{\varepsilon^2 - x_2 + x_1^2}}{1+\varepsilon}\, \exp\Big\{x_1 - \sqrt{\varepsilon^2 - x_2 + x_1^2} + \varepsilon\Big\}\,.$$

Check that this is the solution of the following extremal problem:

$$\mathbf{b}(x;\varepsilon) \overset{\text{def}}{=} \inf_{\varphi\in\mathrm{BMO}_\varepsilon(J)} \left\{\langle e^\varphi\rangle_J \colon \langle\varphi\rangle_J = x_1,\ \langle\varphi^2\rangle_J = x_2\right\},$$

that is check that the Bellman induction works and construct an extremal function for every $x \in \Omega_\varepsilon$.

7. Dyadic maximal operator

Let us define the dyadic maximal operator on the set of positive locally summable functions w, as follows:

$$(Mw)(t) \overset{\text{def}}{=} \sup_{I\in\mathcal{D}_\mathbb{R},\, t\in I} \langle w\rangle_I\,.$$

We would like to estimate the norm of M as an operator acting from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Even though the operator is defined on the whole line, we first localize its action to a fixed dyadic interval J; we will pass to all of $\mathbb{R}$ at the end. Thus, we are looking for the function

$$\mathbf{B}(x_1,x_2;L) \overset{\text{def}}{=} \sup_{w\ge0}\left\{\langle (Mw)^2\rangle_J \colon\ \langle w\rangle_J = x_1,\ \langle w^2\rangle_J = x_2,\ \sup_{I\supset J,\ I\in\mathcal{D}_\mathbb{R}} \langle w\rangle_I = L\right\}.$$

We need the "external" parameter L because M is not truly local: the value of Mw on an interval J depends not only on the behavior of w on J, but also on that on the whole line $\mathbb{R}$. The function $\mathbf{B}$ depends on three variables, and each of them can change when we split the interval of definition. Nevertheless, we will consider L as a parameter. The reason will become clear a bit later.

As before, $\mathbf{B}$ does not depend on J. Its domain is

$$\Omega \overset{\text{def}}{=} \left\{(x_1,x_2;L)\colon 0 < x_1 \le L,\ x_1^2 \le x_2\right\},$$

or, if we consider L as a fixed parameter,

$$\Omega_L \overset{\text{def}}{=} \left\{(x_1,x_2)\colon 0 < x_1 \le L,\ x_1^2 \le x_2\right\}.$$

Lemma 7.1 (Main inequality)**.** *Take $(x;L) \in \Omega$ and let the points $(x^\pm;L^\pm) \in \Omega$ be such that $x = (x^+ + x^-)/2$ and $L^\pm = \max\{x_1^\pm, L\}$. Then the following inequality holds*:

$$\mathbf{B}(x;L) \ge \frac{\mathbf{B}(x^+;L^+) + \mathbf{B}(x^-;L^-)}{2}\,. \tag{7.1}$$

[1] *Hint*: Due to the cut-off lemma (Lemma 3.4), it is sufficient to check that $\log t \in \mathrm{BMO}_1$, which follows from $\langle \log^2 t\rangle_{[c,d]} - \langle\log t\rangle_{[c,d]}^2 = 1 - \frac{cd}{(d-c)^2}\Big(\log\frac{d}{c}\Big)^2$.

Proof. The proof is now standard. Fixing an interval J and a small number $\eta > 0$, we take a pair of functions $w^\pm$ such that

$$\mathbf{B}(x^\pm; L^\pm) \ge \langle (Mw^\pm)^2 \rangle_{J_\pm} - \eta$$

and set

$$w(t) = \begin{cases} w^\pm(t), & \text{if } t \in J_\pm, \\ L, & \text{if } t \notin J\,. \end{cases}$$

Then w is a test function corresponding to the Bellman point $(x; L)$ with $(Mw)(t) = (Mw^\pm)(t)$ for $t \in J_\pm$. Therefore,

$$\begin{aligned} \mathbf{B}(x; L) \ge \langle (Mw)^2 \rangle_J &= \frac{1}{2}\big(\langle (Mw^+)^2 \rangle_{J_+} + \langle (Mw^-)^2 \rangle_{J_-}\big) \\ &\ge \frac{1}{2}\big(\mathbf{B}(x^+; L^+) + \mathbf{B}(x^-; L^-)\big) - \eta\,, \end{aligned}$$

which proves the lemma. □

Corollary 7.2 (Concavity)**.** *For a fixed L, the function $\mathbf{B}$ is concave on Ω_L.*

Proof. For any pair $x^\pm \in \Omega_L$, we have $L^\pm = L$ and (7.1) becomes the usual concavity condition. □

Corollary 7.3 (Boundary condition)**.** *If the function $\mathbf{B}$ is sufficiently smooth, then*

$$\frac{\partial \mathbf{B}}{\partial L}(x; x_1) = 0\,. \tag{7.2}$$

Proof. First of all, we note that the definition of $\mathbf{B}$ immediately yields the inequality $\frac{\partial \mathbf{B}}{\partial L} \ge 0$. Now, take an arbitrary point x on the boundary $x_1 = L$ and a pair $x^\pm$ such that $x = (x^+ + x^-)/2$. Let $\Delta_k = (x_k^+ - x_k^-)/2, k = 1, 2$, and assume, without loss of generality, that $\Delta_1 > 0$. Then $x_k^\pm = x_k \pm \Delta_k$ and $x_1^- < x_1 = L < x_1^+$; therefore, $L^+ = x_1^+$ and $L^- = L$. Writing the main inequality up to the terms of first order in Δ, we get

$$\begin{aligned} 0 &\le \mathbf{B}(x; L) - \frac{1}{2}\big(\mathbf{B}(x^+; L^+) + \mathbf{B}(x^-; L^-)\big) \\ &= \mathbf{B}(x_1, x_2; x_1) - \frac{1}{2}\big(\mathbf{B}(x_1 + \Delta_1, x_2 + \Delta_2; x_1 + \Delta_1) + \mathbf{B}(x_1 - \Delta_1, x_2 - \Delta_2; x_1)\big) \\ &\approx \mathbf{B}(x; x_1) - \frac{1}{2}\big(\mathbf{B}(x; x_1) + \mathbf{B}_{x_1}\Delta_1 + \mathbf{B}_{x_2}\Delta_2 + \mathbf{B}_L\Delta_1 + \mathbf{B}(x; x_1) - \mathbf{B}_{x_1}\Delta_1 - \mathbf{B}_{x_2}\Delta_2\big) \\ &= -\frac{1}{2}\mathbf{B}_L(x; x_1)\Delta_1\,. \end{aligned}$$

Since $\mathbf{B}_L(x; x_1) \ge 0$, the last inequality is possible only if $\mathbf{B}_L(x; x_1) = 0$. □

Lemma 7.4 (Homogeneity)**.** *If the function $\mathbf{B}$ is sufficiently smooth, then*

$$\mathbf{B}(x; L) = \frac{1}{2}x_1\mathbf{B}_{x_1} + x_2\mathbf{B}_{x_2} + \frac{1}{2}L\,\mathbf{B}_L\,. \tag{7.3}$$

Proof. As before, together with a test function w we consider the function $\tilde{w} = \tau w$ for $\tau > 0$. Comparing the Bellman functions at the corresponding Bellman points gives us the equality

$$\mathbf{B}(\tau x_1, \tau^2 x_2; \tau L) = \tau^2 \mathbf{B}(x_1, x_2; L).$$

Differentiating this identity with respect to τ at the point $\tau = 1$ proves the lemma. □

Before we start looking for a Bellman candidate, let us state one more boundary condition — in fact, the principal one.

Lemma 7.5 (Boundary condition)**.**

$$\mathbf{B}(u,u^2;L)=L^2\,. \tag{7.4}$$

Proof. The only test function corresponding to the point $x=(u,u^2)$ is the function identically equal to u on the interval J. Hence, Mw is identically L on this interval. □

Remark 7.6. *Note that the boundary* $x_1=0$ *is not accessible, that is it does not belong to the domain: if* $x_1=0$, w *must be identically zero on* J, *which means that* $x_2=0$. *Therefore, no boundary condition can be stated on that boundary.*

We are now ready to search for a Bellman candidate. To this end, we will, as before, solve a Monge–Ampère boundary value problem. The arguments why we are looking for a solution of the Monge–Ampère equation are the same as before: the concavity condition forces us to look for a function whose Hessian is negative and the optimality condition requires the Hessian to be degenerate.

Again, we are looking for a solution in the form

$$B(x)=t_0+x_1t_1+x_2t_2$$

that is linear along extremal trajectories given by

$$dt_0+x_1dt_1+x_2dt_2=0.$$

Let us parameterize the extremal lines by the first coordinate of their points of intersection with the boundary $x_2=x_1^2$. Since the boundary $x_1=0$ is not accessible, such an extremal line can either be vertical (i.e. parallel to the x_2-axis) or slant to the right, in which case it intersects the boundary $x_1=L$ at a point, say, (L,v). (**A homework question**: *why can an extremal line never connect two points of the boundary* $x_2=x_1^2$*?*)

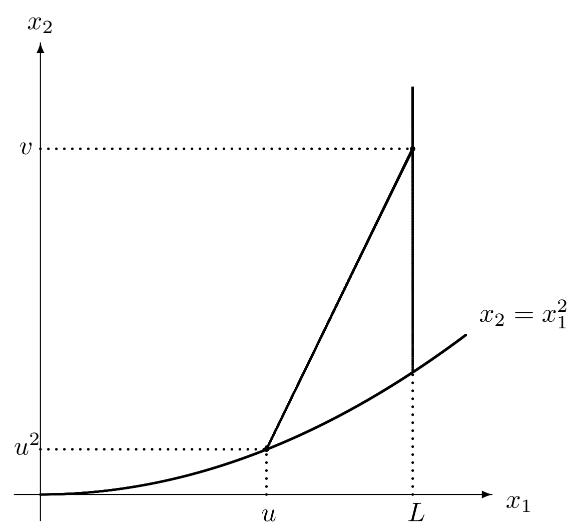


Figure 3. The extremal trajectory passing through (u,u^2) and (L,v)

The former case is very simple. Since B is linear on each vertical line and satisfies the boundary condition (7.4), it has the form

$$B(x; L) = k(x_1, L)(x_2 - x_1^2) + L^2.$$

Since $B_{x_2 x_2} = 0$ and the matrix $B_{x_i x_j}$ must be non-positive, we have $B_{x_1 x_2} = k_{x_1} = 0$, i.e. k does not depend on x_1, $k = k(L)$. Now we use the second boundary condition $B_L(x_1, x_2; x_1) = 0$, which turns into $k'(x_1)(x_2 - x_1^2) + 2x_1 = 0$. The last equation has no solution, therefore this case is impossible, at least in the whole domain Ω_L.

Consider the latter case, when the extremal line goes from the bottom boundary to the right boundary, as shown in the picture. The boundary condition (7.4) on the bottom boundary gives us

$$t_0 + ut_1 + u^2 t_2 = L^2, \tag{7.5}$$

and the condition (7.2) on the right boundary, together with (7.3), yields

$$t_0 + Lt_1 + vt_2 = \frac{1}{2} Lt_1 + vt_2. \tag{7.6}$$

From the last equation, we get

$$t_0 + \frac{1}{2} Lt_1 = 0. \tag{7.7}$$

Now we differentiate (7.5),

$$(dt_0 + udt_1 + u^2 dt_2) + (t_1 + 2ut_2)du = 0,$$

and use the fact that the point (u, u^2) is on the trajectory, i.e. $dt_0 + udt_1 + u^2 dt_2 = 0$. Thus,

$$(t_1 + 2ut_2)du = 0.$$

This equations gives us two possibilities: either $u = \text{const}$, producing a family of trajectories all passing through the point (u, u^2), or $t_1 + 2ut_2 = 0$. The first possibility cannot give a foliation of the whole Ω_L, since it would result in trajectories connecting two points of the bottom boundary (an impossibility, by the earlier homework question). Therefore, let us consider the second possibility, i.e.

$$t_1 + 2ut_2 = 0. \tag{7.8}$$

Solving the system of three linear equations, (7.5), (7.7), and (7.8), of three variables t_0, t_1, and t_2, we obtain:

$$t_0 = \frac{L^3}{L-u}, \qquad t_0' = \frac{L^3}{(L-u)^2},$$

$$t_1 = -\frac{2L^2}{L-u}, \qquad t_1' = -\frac{2L^2}{(L-u)^2},$$

$$t_2 = \frac{L^2}{u(L-u)}, \qquad t_2' = -\frac{L^2(L-2u)}{u^2(L-u)^2}.$$

Now we can plug the derivatives of t_i into the equation of extremal trajectories $dt_0 + x_1 dt_1 + x_2 dt_2 = 0:$

$$\frac{L^3}{(L-u)^2} - x_1 \frac{2L^2}{(L-u)^2} - x_2 \frac{L^2(L-2u)}{u^2(L-u)^2} = 0, \tag{7.9}$$

or

$$x_2 = \frac{2u^2}{2u - L}\Big(x_1 - \frac{L}{2}\Big).$$

We see that this is a "fan" of lines passing through the point $(L/2, 0)$. However, those elements of this fan that intersect the "forbidden" boundary $x_1 = 0$ cannot be extremal trajectories. Therefore, the acceptable lines foliate not the whole domain Ω_L, but only the sub-domain $x_1 \geq L/2$. To foliate the rest, we return to considering vertical lines. Earlier, we have refused this type of trajectories for the whole domain Ω_L, since the foliation so produced would not give a function satisfying the boundary condition on the line $x_1 = L$. However, such trajectories are perfectly suited for foliating the sub-domain $x_1 \leq L/2$, especially because the boundary of the two sub-domains, the vertical line $x_1 = L/2$, fits as an element of both foliations. On this line, we have

$$t_0 = 2L^2, \qquad t_1 = -4L, \qquad t_2 = 4,$$

and so

$$B(L/2, x_2; L) = 2L^2 - 4L(L/2) + 4x_2 = 4x_2.$$

As we have seen, the Bellman candidate on the vertical trajectories must be of the form

$$B(x; L) = k(L)(x_2 - x_1^2) + L^2.$$

To get $B = 4x_2$ on the line $x_1 = L/2$, we have to take $k(L) = 4$, which gives the following Bellman candidate in the left half of Ω_L :

$$B(x; L) = 4(x_2 - x_1^2) + L^2.$$

To have an explicit formula for the Bellman candidate in the right half of Ω_L, we need an expression for u, which we find solving equation (7.9):

$$u = \frac{\sqrt{x_2}\,L}{\sqrt{x_2} + \sqrt{x_2 - L(2x_1 - L)}}.$$

This yields

$$B(x; L) = \big(\sqrt{x_2} + \sqrt{x_2 - L(2x_1 - L)}\big)^2.$$

Finally, our Bellman candidate in Ω is given by

$$B(x; L) = \begin{cases} 4(x_2 - x_1^2) + L^2, & 0 < x_1 \leq \frac{L}{2},\ x_2 \geq x_1^2, \\ \big(\sqrt{x_2} + \sqrt{x_2 - L(2x_1 - L)}\big)^2, & \frac{L}{2} \leq x_1 \leq L,\ x_2 \geq x_1^2. \end{cases} \tag{7.10}$$

Now, we start proving that the Bellman candidate just found is indeed the Bellman function of our problem.

Lemma 7.7. *The function defined by* (7.10) *satisfies the main inequality* (7.1).

Proof. Let us define a new function $\tilde{B}$ in the domain $\tilde{\Omega} \overset{\text{def}}{=} \{x = (x_1, x_2) \colon\ x_1 > 0,\ x_2 \geq x_1^2\}$:

$$\tilde{B}(x; L) \overset{\text{def}}{=} \begin{cases} B(x; L), & 0 < x_1 \leq L,\ x_2 \geq x_1^2, \\ B(x; x_1), & x_1 \geq L,\ x_2 \geq x_1^2, \end{cases}$$

or

$$\tilde{B}(x;L)=\begin{cases}4(x_2-x_1^2)+L^2, & 0<x_1\le\frac{L}{2},\ x_2\ge x_1^2,\\ \big(\sqrt{x_2}+\sqrt{x_2-L(2x_1-L)}\,\big)^2, & \frac{L}{2}\le x_1\le L,\ x_2\ge x_1^2.\\ \big(\sqrt{x_2}+\sqrt{x_2-x_1^2)}\,\big)^2, & x_1\ge L,\ x_2\ge x_1^2.\end{cases}$$

Let us calculate the first partial derivatives:

$$\tilde{B}_{x_1}(x;L)=\begin{cases}-8Lx_1, & 0<x_1\le\frac{L}{2},\\ -2L\Big(1+\dfrac{\sqrt{x_2}}{\sqrt{x_2-L(2x_1-L)}}\Big), & \frac{L}{2}\le x_1\le L,\\ -2x_1\Big(1+\dfrac{\sqrt{x_2}}{\sqrt{x_2-x_1^2}}\Big), & x_1\ge L,\ x_2\ge x_1^2.\end{cases}$$

$$\tilde{B}_{x_2}(x;L)=\begin{cases}4, & 0<x_1\le\frac{L}{2},\\ \dfrac{\big(\sqrt{x_2}+\sqrt{x_2-L(2x_1-L)}\big)^2}{\sqrt{x_2}\sqrt{x_2-L(2x_1-L)}}, & \frac{L}{2}\le x_1\le L,\\ \dfrac{\big(\sqrt{x_2}+\sqrt{x_2-x_1^2}\big)^2}{\sqrt{x_2}\sqrt{x_2-x_1^2}}, & \frac{L}{2}\le x_1\le L.\end{cases}$$

From these expressions we see that our function $\tilde{B}$ is C^1-smooth. Since the second derivative

$$\tilde{B}_{x_1x_1}(x;L)=\begin{cases}-8L, & 0<x_1\le\frac{L}{2},\\ -\dfrac{2L^2\sqrt{x_2}}{\big(x_2-L(2x_1-L)\big)^{3/2}}, & \frac{L}{2}\le x_1\le L,\\ -2\Big(\dfrac{\sqrt{x_2}}{\sqrt{x_2-x_1^2}}\Big)^3-2, & x_1\ge L,\ x_2\ge x_1^2,\end{cases}$$

is negative, one can check the concavity of $\tilde{B}$ in the domain $\Omega_+\stackrel{\text{def}}{=}\{x:x_1>0,x_2\ge x_1^2\}$ by verifying that the determinant of the Hessian matrix is non-negative. We know that this determinant is zero in Ω_L and, therefore, need to calculate the second derivatives of $\tilde{B}$ only in the domain $x_1>L$, where $\tilde{B}(x_1,x_2)=B(x_1,x_2;x_1)$. In this domain, we have

$$\tilde{B}_{x_1x_2}=\frac{x_1^3}{\sqrt{x_2}}\big(x_2-x_1^2\big)^{-3/2},$$

$$\tilde{B}_{x_2x_2}=-\frac{x_1^4}{2x_2^{3/2}}\big(x_2-x_1^2\big)^{-3/2},$$

which yields

$$\tilde{B}_{x_1x_1}\tilde{B}_{x_2x_2}-\tilde{B}_{x_1x_2}^2=\frac{x_1^4}{x_2(x_2-x_1^2)^2}+\frac{x_1^4}{x_2^{3/2}(x_2-x_1^2)^{3/2}}>0\,.$$

The concavity just proved immediately implies (7.1). Indeed, we have proved that the function $\tilde{B}$ is locally concave in each sub-domain of Ω_+, as well as C^1-smooth in the whole domain; therefore, it is concave everywhere in Ω_+. Furthermore, relation (7.1) is a special case of the concavity condition on the function $\tilde{B}$: x^- and x are in the sub-domain Ω_L, while x^+ may be either in Ω_L or the sub-domain $x_1 > L$. □

Lemma 7.8 (Bellman induction)**.** *For any continuous function B satisfying the main inequality* (7.1) *and the boundary condition* (7.4), *we have*

$$\mathbf{B}(x;L) \le B(x;L).$$

Proof. The proof is standard. First, we fix a test function w on $\mathbb{R}$ and a dyadic interval J. This gives us a Bellman point $(x;L)$. Then we start splitting the interval J, while repeatedly applying the main inequality:

$$\begin{aligned} |J|B(x;L) &\ge |J_+|B(x^{J_+};L^{J_+}) + |J_-|B(x^{J_-};L^{J_-}) \\ &\ge \sum_{I\in\mathcal{D}_n} |I|B(x^I,L^I) = \int_J B\big(x^{(n)}(s);L^{(n)}(s)\big)\,ds\,, \end{aligned}$$

where $(x^{(n)}(s);L^{(n)}(s)) = (x^I;L^I)$, when $s\in I$, $I\in\mathcal{D}_n$. By the Lebesgue differentiation theorem, we have $x^{(n)}(s)\to\big(w(s),w^2(s)\big)$ and by the definition of the maximal function $L^{(n)}(s)\to(Mw)(s)$ almost everywhere. For bounded w we can pass to the limit and obtain

$$B(x;L) \ge \langle (Mw)^2(s)\rangle_J.$$

Then, approximating, as before, an arbitrary test function w by its bounded cut-offs, we get the same inequality for all w, which immediately gives the required property: $B(x;L)\ge\mathbf{B}(x;L)$. □

Corollary 7.9. *For the function B given by* (7.10), *the inequality*

$$B(x;L) \ge \mathbf{B}(x;L)$$

holds.

To prove the converse inequality, we need to construct an optimizer. However, in the present setting we have no test function realizing the supremum in the definition of the Bellman function. Thus, an optimizer will be given by a sequence of test functions.

First, we construct an optimizer on $(0,1)$ for the point (L,v). The extremal line passing through this point is $x_2=\frac{v}{L}(2x_1-L)$. It intersects the parabolic boundary at the point (u,u^2) with

$$u=\frac{v-\sqrt{v^2-L^2v}}{L}=\frac{L\sqrt{v}}{\sqrt{v}+\sqrt{v-L^2}}\,.$$

We need to split the interval $(0,1)$ in half, which splits the Bellman point $x=(L,v)$, into a pair of points $x^\pm$, $x=(x^-+x^+)/2$. We use the homogeneity of the problem in our construction. We know that the set of test functions for the point $\tilde{x}=(\tau x_1,\tau^2x_2)$ is the same as the set of test functions for the point x, each multiplied by τ. Therefore, if w is an optimizer for the point x, then τw is an optimizer for $\tilde{x}$. Hence, for the first splitting of $x=(L,v)$, we take the right point x^+ not on the continuation of the extremal line, but on the parabola $x_2=vL^{-2}x_1^2$, which is tangent to our extremal line at the point x. Then on the right half-interval $(\frac12,1)$ we can set the optimizer to be proportional to the appropriately scaled copy of itself: $w(t)=\beta w(2t-1)$ for $t\in(\frac12,1)$. What function do we need to take on the left half-interval? We can split the

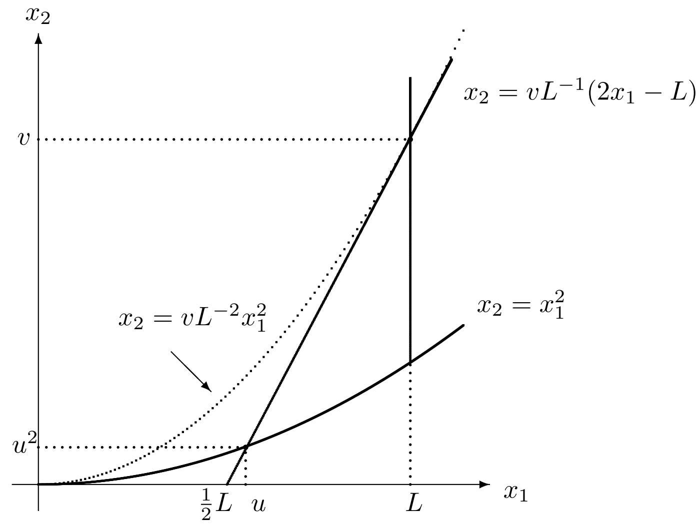


FIGURE 4. The extremal trajectory passing through (u, u^2) and (L, v) is tangent to the parabola $x_2 = vL^{-2}x_1^2$

corresponding Bellman point along the extremal line in such a manner that the right point $x^{(\frac{1}{4},\frac{1}{2})}$ returns to the initial point $x = (L, v)$ and, therefore, $w(t) = w(4t-1)$ for $t \in (\frac{1}{4}, \frac{1}{2})$. Continuing in this fashion, we put $w(t) = w(8t-1)$ for $t \in (\frac{1}{8}, \frac{1}{4})$ and so on. We can assume that the first splitting was chosen in such a way that after n steps the left point x^- lands precisely on the boundary $x_2 = x_1^2$, and, therefore, on the last interval $(0, 2^{-n})$ we have to set the optimizer w to be constant. Finally, our optimizing sequence will be given by

$$w_n(t) = \begin{cases} \alpha_n L & 0 < t < 2^{-n}, \\ w_n(2^k t - 1) & 2^{-k} < t < 2^{-k+1}, \ 1 < k < n, \\ \beta_n w_n(2t-1) & \frac{1}{2} < t < 1. \end{cases} \tag{7.11}$$

Let us verify that this recurrent relation defines the sequence $\{w_n\}$ correctly. To this end, let us introduce a sequence $\{w_{n,m}\}$ by induction:

$$w_{n,0}(t) = \begin{cases} \alpha_n L & 0 < t < 2^{-n}, \\ 0 & 2^{-n} < t < 1; \end{cases}$$

$$w_{n,m}(t) = \begin{cases} \alpha_n L & 0 < t < 2^{-n}, \\ w_{n,m-1}(2^k t - 1) & 2^{-k} < t < 2^{-k+1}, \ 1 < k < n, \\ \beta_n w_{n,m-1}(2t-1) & \frac{1}{2} < t < 1. \end{cases}$$

We see that $w_{n,m}(t) = w_{n,m-1}(t)$ for all t such that $w_{n,m-1}(t) \neq 0$, and the measure of the set where $w_{n,m-1}(t) = 0$ is $(1 - 2^{-n})^m$, i.e. it tends to zero as $m \to \infty$. Therefore, $w_{n,m}$ stabilizes almost everywhere as a sequence in m, and its limit w_n satisfies the recurrent relation (7.11).

Now, let us calculate the values of the parameters α_n and β_n. We choose them to get (L, v) as a Bellman point of w_n :

$$L = \langle w_n \rangle_{(0,1)} = 2^{-n}\alpha_n L + \big(\frac{1}{2} - 2^{-n}\big)L + \frac{1}{2}\beta_n L,$$
$$v = \langle w_n^2 \rangle_{(0,1)} = 2^{-n}\alpha_n^2 L^2 + \big(\frac{1}{2} - 2^{-n}\big)v + \frac{1}{2}\beta_n^2 v.$$

Solving this system yields

$$\alpha_n = \sqrt{1+2^{-n+1}}\,\frac{\sqrt{1+2^{-n+1}} - \sqrt{1-\frac{L^2}{v}}}{\frac{L^2}{v} + 2^{-n+1}} \xrightarrow[n\to\infty]{} \frac{v}{L^2}\Big(1 - \sqrt{1-\frac{L^2}{v}}\Big) = \frac{u}{L}\,.$$

When solving the quadratic equation for α_n, we chose the minus sign specifically to get this limit. Choosing the plus sign would produce, instead of u, the first coordinate of the second intersection point of the extremal line with the boundary $x_2 = x_1^2$.

Now, we need to calculate the maximal function for w_n, which is a simple matter:

$$Mw_n = \frac{w_n}{\alpha_n}\,.$$

It is easy to check by induction in m that $(Mw_{n,m})(t) = \frac{w_{n,m}(t)}{\alpha_n}$ for all t for which $w_{n,m}(t) \neq 0$. In the limit we obtain the required relation for Mw_n.

Finally, we have

$$\langle (Mw_n)^2 \rangle = \frac{\langle w_n^2 \rangle}{\alpha_n^2} = \frac{v}{\alpha_n^2} \longrightarrow \frac{vL^2}{u^2} = \big(\sqrt{v} + \sqrt{v-L^2}\big)^2 = B(L, v; L)\,.$$

Thus, we have proved the inequality $\mathbf{B}(x; L) \geq B(x; L)$ for x on the line $x_1 = L$. Now, take an arbitrary $x \in \Omega_L$ with $x_1 > L/2$. Let the extremal line passing through this point intersect the two boundaries of Ω_L at the points (u, u^2) and (L, v) and assume that the point x splits the segment between these two points in proportion $\alpha : (1-\alpha)$. Using the main inequality for $\mathbf{B}$, linearity of B on the extremal line, and the just-proved inequality $\mathbf{B}(L, v; L) \geq B(L, v; L)$, we can write down the following chain of estimates:

$$\begin{aligned}
\mathbf{B}(x; L) &\geq \alpha\mathbf{B}(L, v; L) + (1-\alpha)\mathbf{B}(u, u^2; L) \\
&= \alpha\mathbf{B}(L, v; L) + (1-\alpha)L^2 \\
&\geq \alpha B(L, v; L) + (1-\alpha)L^2 \\
&= \alpha B(L, v; L) + (1-\alpha)B(u, u^2; L) = B(x; L)\,.
\end{aligned}$$

We use the same trick to prove inequality $\mathbf{B}(x; L) \geq B(x; L)$ for $x_1 \leq L/2$, except now, instead of the vertical extremal line, we use a nearby line with a large slope. Take a number ξ close to x_1, $\xi < x_1$, and take the line passing through x and $(\xi, 0)$. Let (u, u^2) and (L, v) be the points where this line intersects the two boundaries of Ω_L. Then

$$v = \frac{L-\xi}{x_1 - \xi} \xrightarrow[\xi\to x_1]{} \infty\,, \qquad u = \frac{2\xi\sqrt{x_2}}{\sqrt{x_2} + \sqrt{x_2 - 4\xi(x_1-\xi)}} \xrightarrow[\xi\to x_1]{} x_1\,.$$

The concavity of $\mathbf{B}$ implies that

$$\mathbf{B}(x) \geq \frac{x_1-u}{L-u}\mathbf{B}(L,v) + \frac{L-x_1}{L-u}\mathbf{B}(u,u^2) \geq \frac{x_1-u}{L-u}\big(\sqrt{v}+\sqrt{v-L^2}\big)^2 + \frac{L-x_1}{L-u}L^2.$$

The limit of the second term in the last expression is L^2. To calculate the limit of the first term is a bit of work. First of all, note that

$$\big(\sqrt{v}+\sqrt{v-L^2}\,\big)^2 = v\Big(1+\sqrt{1-\frac{L^2}{v}}\,\Big)^2,$$

i.e. that term can be rewritten in the form

$$\frac{x_1-u}{L-u}\,v\left(1+\sqrt{1-\frac{L^2}{v}}\,\right)^2 = \frac{x_1-u}{x_1-\xi}\cdot\frac{L-\xi}{L-u}\,x_2\left(1+\sqrt{1-\frac{L^2}{v}}\,\right)^2.$$

The limit of the expression in parentheses is 2, the second fraction tends to 1, and for the first fraction we have

$$\frac{x_1-u}{x_1-\xi} = 1-\frac{u-\xi}{x_1-\xi} = 1-\frac{u^2}{x_2} \xrightarrow[\xi\to x_1]{} 1-\frac{x_1^2}{x_2}\,.$$

In the end, we have

$$\mathbf{B}(x;L) \geq \big(1-\frac{x_1^2}{x_2}\big)\cdot x_2\cdot 4 + L^2 = 4x_2-4x_1^2+L^2 = B(x;L)\,.$$

Thus, we have proved the following lemma.

Lemma 7.10.

$$\mathbf{B}(x;L) \geq B(x;L).$$

Taken together, this lemma and Corollary 7.9 prove the following theorem:

Theorem 7.11.

$$\mathbf{B}(x;L) = \begin{cases} 4(x_2-x_1^2)+L^2, & 0 < x_1 \leq \frac{L}{2},\ x_2 \geq x_1^2, \\ \big(\sqrt{x_2}+\sqrt{x_2-L(2x_1-L)}\,\big)^2, & \frac{L}{2} \leq x_1 \leq L,\ x_2 \geq x_1^2. \end{cases}$$

The Bellman setup of the problem discussed in this section was first stated in [4], the Bellman function above was found in [3] without solving the Monge–Ampère equation and without using the Bellman function method at all. The consideration presented here appears in [7] (an initial version in [6]).

Homework assignments:

(1) *Show that this theorem implies that*

$$\|Mw\|_{L^2(\mathbb{R})} \leq 2\|w\|_{L^2(\mathbb{R})}.$$

(2) *Follow the same steps to find the Bellman function for the dyadic maximal operator on L^p, for $p > 1$.*

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