BELLMAN FUNCTIONS IN HARMONIC ANALYSIS

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In essence, the Bellman function method in Harmonic Analysis was introduced by Donald Burkholder for finding the norm in L^p of the martingale transform. Apparently Burkholder either did not notice or did not want to stress the link between his technique and Optimal Control, but it becomes quite transparent if one carefully analyzes his proofs.

This unusual (for analysts) approach/insight of Burkholder can be made into a method. This method lays down a bridge between (stochastic) Optimal Control and classical Harmonic Analysis, it allows to set up a clear analogy between certain questions in Harmonic Analysis and well studied methods in Optimal Control.

Soon it became clear that the scope of the method is quite wide, in particular Nazarov–Treil–Volberg obtained by this method a necessary and sufficient condition for the two weight martingale transform to be bounded. This was used later in Bellman estimates of Ahlfors-Beurling transform by Petermichl and Volberg, which in turn did answer an old problem in regularity theory of elliptic PDE.

Over last years, the Bellman function technique can be credited for helping to solve several old Harmonic Analysis problems while proposing a unified approach to many others. In the first category one could name the sharp weighted estimates of such classical operators as the Hilbert and Riesz transforms (Petermichl) and the Ahlfors-Beurling transform (Petermichl-Volberg). It was used also by Nazarov–Volberg to solve a problem of Cohn about embedding of the model space K_θ into $L^2(\mu)$. Recently it was shown that the Bellman function method can be used to solve another famous Harmonic Analysis problem: the so-called A_2 conjecture on sharp weighted estimates of *arbitrary* Calderón–Zygmund operators. In the second category one can name all kind of dimension free estimates of weighted and unweighted Riesz transforms. Roughly speaking, the Bellman function method makes apparent the hidden multiscale properties of Harmonic Analysis problems. It often replaces sophisticated stopping time arguments of combinatorial nature by relatively simple cookbook recipes. Conversely, given a Harmonic Analysis problem with certain scaling properties, one can associate with it a non-linear PDE, the so-called stochastic Bellman equation of the problem.

Untill recently, in each case it was a matter of luck, experience and tenacity to find a supersolution , or even better a solution of this PDE (*i.e.* the Bellman function of the problem). Recently, a method to find such a solution for a whole class of problems has emerged, largely due to the efforts of Vasyunin. So, the Bellman function technique is now turning into a theory which can be learned and applied to a relatively wide range of issues.

The number of people interested in this method grows. Terry Tao has a couple of notes, where he uses the method of Bellman function. In October 2010 there was a Fall School devoted to applications of Bellman functions in Harmonic Analysis held at Lake Arrowhead, CA. This school was organized by Christoph Thiele, Ignacio Uriarte–Tuero and Alexander Volberg. It gathered approximately 18 young researchers. It was be followed by a special session at an AMS meeting at UCLA. This area is very suitable for the young researchers, because

a) it shows how methods from one seemingly distant field (Optimal Control) can be used in Analysis,

b) it generate progress in both fields,

c) it allows to quickly develop a working knowledge of some of the most powerful tools in Analysis.

We plan to have approximately 8 lectures (by Alexander Volberg) mixed with recitations (by Vasily Vasyunin), where the participants will be taught "hands-on" to build Bellman functions.