Inria International programme
Associate Team
Final Report

Title: Inverse Magnetization Problems IN Geosciences

Associate Team acronym: IMPINGE

Principal investigator (Inria): L. Baratchart, Factas team, Inria Sophia Antipolis

Principal investigator (Main team): B. P. Weiss, Department of Earth, Atmospheric and Planetary Sciences (EAPS), Massachusetts Institute of Technology (MIT), Cambridge MA, USA.

Other participants: Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, Nashville TN, USA.

Key Words:

- **A- Research themes on digital science:** A6.1.1 Continuous modeling (PDE, ODE); A6.2.6 Optimization; A6.3.1 Inverse problems; A6.3.3 Data processing.
- **B- Other research themes and application areas:** B3.3.1 Geosciences, Earth and subsoils.

Website of Impinge: http://www-sop.inria.fr/apics/IMPINGE

1 Overview of the Activities

The associate team was composed of:

Inria: L. Baratchart (confirmed researcher), S. Chevillard (young researcher), J. Leblond (confirmed researcher), K. Mavreas (PhD student), D. Ponomarev (PhD student).

MIT: C. Borlina (PhD student), E. A. Lima (young researcher), B. P. Weiss (confirmed researcher).

Vanderbilt University (VU): D. P. Hardin (confirmed researcher), M. Northington (PhD student), E. B. Saff (confirmed researcher), C. Villalobos (PhD student).

1.1 Short Visits (duration $<$ 1 month)

2016, February 22 to March 6: L. Baratchart, S. Chevillard, J. Leblond and D. Ponomarev visited MIT, then VU;
2016, June 8 to June 22: D. Hardin, M. Northington and C. Villalobos visited Inria;
2016, June 13 to 17: E. Lima visited Inria;
2017, April 23 to 27: C. Borlina, E. Lima and D. Hardin visited Inria;
2017, November 1 to 10: S. Chevillard visited MIT, then VU;
2017, November 5 to 12: J. Leblond visited VU;
2018, May 15 to 21: E. Lima visited Inria;
2018, August 7 to 12: L. Baratchart visited VU;
2018, November 1 to 9: L. Baratchart, J. Leblond and K. Mavreas will visit MIT;
2018, December 13 to 20: D. Hardin, E. Saff and C. Villalobos plan to visit Inria.

1.2 Long Visits (duration $>$ 1 month)

2017, August to December: L. Baratchart spent a sabbatical semester at VU. Purpose and outcome: co-supervision of the thesis of C. Villalobos, and research that led to the preprint [7].

1.3 Seminars

March 2016. L. Baratchart, S. Chevillard, D. Ponomarev, J. Leblond and M. Northington all gave talks at the *Shanks Workshop on Mathematical Methods for Inverse Magnetization Problems Arising in Geoscience* (Nashville, USA).
May 2016. S. Chevillard. At the 5th *Approximation Day* in Lille (France).
June 2016. L. Baratchart. At the 25th *St. Petersburg Summer Meeting in Mathematical Analysis* (Russia).
June 2016. J. Leblond. At conference *PICOF (Problèmes Inverses, Contrôle et Optimisation de Formes)* in Autrans (France).
August 2016. J. Leblond. At the conference *WiSE (Waves in Science and Engineering)* in Queretaro (Mexico).
September 2016. L. Baratchart. At the conference *Quasilinear equations, inverse problems and their applications* in Moscow (Russia).
October-November 2016. L. Baratchart and J. Leblond both gave invited talks at the Workshop *Sigma (Signal, Image, Geometry, Modelling, Approximation)* in Luminy (France).
November 2016. D. Hardin. At the *Computational Mathematics Seminar*, Middle Tennessee State University, Murfreesboro (TN, USA).
May-June 2017. L. Baratchart. At the conference *AIP (Applied Inverse Problems)* in

Hangzhou (China).
July 2017. L. Baratchart. In the session *Harmonic Analysis and Inverse Problems* of the conference *MCA (Mathematical Congress of the Americas)*, Montréal (Canada).
September 2017. A poster (presented by J.-P. Marmorat, on the work we did together on the moment recovery problem, addressed by a bounded extremal problem) at the 18th *ISEM (International Symposium on Applied Electromagnetic and Mechanics)*, Chamonix (France).
September 2017. J. Leblond. At the *ERNSI (European Research Network on Systems Identification)* workshop in Lyon (France).
October 2017. J. Leblond. At the annual meeting of the GDR AFHP (Research Group on Functional and Harmonic Analysis, and Probabilities), Bordeaux (France).
November 2017. L. Baratchart and J. Leblond gave talks at the conference *Mathematics, Signal Processing and Linear Systems: New Problems and Directions*, Orange (California, USA).
May 2018. L. Baratchart, E. Lima, and K. Mavreas gave talks at the school / workshop *Inverse problems and approximation techniques in planetary sciences* that we organized at Inria.
May 2018. L. Baratchart. At the conference *IPMS 2018 (Inverse Problems, Modeling & Simulation)*, Malta.
November 2018. L. Baratchart is an invited speaker at the *SMAI-SIGMA* workshop, Paris.

1.4 Joint Workshops

The partners from VU organized a 3-days workshop (Shanks workshop "Mathematical methods for inverse magnetization problems arising in geosciences") at VU on March 2-4, 2016. The audience was composed of the partners of IMPINGE, of local students and a few guest researchers.

1.5 Submission of Joint Projects

1.5.1 On the French Side

The Inria team got a funding of 8,000 € from the C4PO project (Center for Planetary Origins: https://www.oca.eu/fr/c4po) from UCA^JEDI Idex in order to organize a Spring School on inverse problems and approximation techniques in planetary sciences at Inria (see Section 1.6). This amount covered organizational costs and part of the travel expenses for the participants.

1.5.2 On the European Commission Side N/A

1.5.3 On the Partner's Country Side

The participants from Inria and MIT obtained a grant from the MIT-France seed funding collaborative research program ("Development of Ultra-high Sensitivity Magnetometry for Analyzing Ancient Rock Magnetism"). It was awarded $29,500 for travel expenses to facilitate collaboration between the MIT and Inria teams on the period 2014-2017.

The US partners got the NSF grant DMS-1521749 ("Collaborative Research: Computational methods for ultra-high sensitivity magnetometry of geological samples") where the Inria partners were listed as external collaborators. The award for this grant was $197,000 on the period 2015-2018. Most of this amount covered operating costs of the SQUID microscope, preparation and measurement of samples and salaries for the US permanent researchers involved and grants for students. It covered parts of the expenses of the visits on both sides.

The US partners, with our support, plan to apply to a new NSF program early 2019.

1.6 Co-organization of Scientific Events

We organized a 3-days Spring School / Workshop at Inria on 2018, May 16-18 on the topic *Inverse problems and approximation techniques in planetary sciences*, see the website of the event http://www-sop.inria.fr/apics/IPAPS18/.

J. Leblond and L. Baratchart organized a mini-symposium on *Inverse Source Problems With Applications to Planetary Sciences and Medical Imaging* at the IPMS 2018 conference, on May 21-25 in Malta. See http://www.ipms-conference.org/ipms2018/index.php/m20.

1.7 Students Co-supervision

C. Borlina, PhD student at MIT, is co-supervised by E. Lima and B. Weiss. The defense is expected by Spring 2011. He works on constraining planetary dynamos and evolution with high-resolution magnetic microscopy.
K. Mavreas, PhD student at Inria, is co-supervised by S. Chevillard and J. Leblond. The defense is expected by October 2019. He works on inverse source problems in planetary sciences (dipole localization in Moon rocks from sparse magnetic data).
M. Northington was a PhD student at VU, supervised by D. Hardin. He defended in May 2016. He was working on sparse reconstruction using techniques inspired by compressed sensing.
D. Ponomarev was a PhD student at Inria, co-supervised by L. Baratchart and J. Leblond. He defended in June 2016 (see [5]). He was working on inverse problems with partial data.
C. Villalobos, PhD student at VU, is co-supervised by L. Baratchart and D. Hardin. The defense is expected by December 2018. He is working on inverse potential problems regularized with total variation.

1.8 Research Internships for Master and PhD students

X. Deng, an undergraduate student in space physics at University of Science and Technology of China, did an internship supervised by E. Lima at MIT (July 16 - August 31, 2018). He has been working on the systematic study of the practical behavior of net moment recovery through asymptotic formulas (the method described in [6]). He presented his results at the MIT Paleomagnetism Meetings, the seminar of the EAPS Department.

2 Scientific Achievements

During the first period of Impinge, moment recovery appeared as an interesting sub-problem, easier than full recovery of the magnetization, because in contrast to the latter, it is injective. Therefore, our initial investigations have focused on that sub-problem, in the hope of: (i) obtaining faster and more reliable algorithms than can be expected for full magnetization recovery, (ii) provide a valuable input for Fourier-based methods which are computationally attractive but can be imprecise, in particular to check for unidirectional samples [2], (iii) generalize the method to estimate the moment of sub-regions of the sample (recovering the moment of very small regions would essentially allow to recover the magnetization). In later years, the issue of full magnetization recovery has been approached as well, introducing assumptions of sparsity to cope with non-injectivity of the forward map. Such assumptions are new in the present, infinite-dimensional context. Hereafter we describe these contributions in greater detail.

Moment estimation, asymptotic results. We developed computationally efficient asymptotic formulas for moment recovery using linear estimation, from a single component of the field on a measurement slab Q, when Q gets large. These were obtained over the past years, see in particular [5]. This approach works for thin as well as for thick samples, and was expounded in [6]. We recently tested their use on real data, measured with the SQUID microscope at MIT. This has raised issues concerning the impact of the electronic noise (drift of the measured field) on the quality of the estimates. Modeling this fairly deterministic drift as an affine function of the space variables, and then canceling it out by appropriate operations, has produced promising results on some experimental data (chondrules), see the report [I] on the website of Impinge. Regression techniques on the coefficients of the asymptotic expansion for the moment, with respect to the size of the measurement set (which plays the role of a regularization parameter), appeared to be helpful. Systematic experiments have been performed on several kinds of samples available at MIT (see the presentation [K] on the website). This allowed us to better understand in which situations the method gives less reliable results. In short, it performs very well when the signal-to-noise ratio is not too bad and the size of the measurement area is large enough compared to the sample (at least roughly four times bigger).

Moment estimation via bounded extremal problems. The linear estimators used in the method described above are rather simple functions (namely, affine functions). In the case when the magnetization is supported on a thin planar slab parallel to the measurements, we generalized this idea and sought for more accurate linear estimators (see [3, 4]). The method yet consists in deriving linear forms on a functional space in which we embed the data, whose effect on the latter yields an approximation to the components of the net moment. Under smoothness assumptions on the magnetization, such forms may be obtained by solving a family of best constrained approximation problems (or bounded extremal problems: BEP, regularized by the constraint). The regularity of the solutions with respect to space variables and parameters of the problem (*e.g.*, level of constraint), has been analyzed. The critical point equation leads to an elliptic integro-differential PDE in dimension 2. We implemented a resolution algorithm using a finite elements method. Results on synthetic data are good and confirm the validity of the approach on non-noisy data. The addition of a synthetic noise however reveals a sensitivity to a poor signal-to-noise ratio, in particular to the edges of the measurement slab where the estimator oscillates heavily. Such oscillations are the price to pay for an estimation procedure which uses data on a measurement set not much bigger than the sample. This is an interesting feature of the method, and further analysis is needed to offset the noise effect.

Moment estimation, multipolar approximation. Yet another method to recover net moments of (not necessarily thin) magnetic samples has been based on multipolar approximation (spherical harmonic expansion). The initial idea is that classical magnetometers moment measurements rest on some kind of dipolar approximation of the source which is valid at some distance from the sample or for some specific sample shapes carrying uniform magnetization (e.g., spherical). When SQUID microscopes are used to measure weak fields, measurements take place much closer to the sample and the dipolar approximation is often no longer valid. However, approximation by a truncated multipole expansion of sufficiently high order could improve the net moment estimate in this case, at the cost of solving a non-linear optimization problem to determine an appropriate location for the origin of the expansion. This has been experimented somewhat systematically with various samples at MIT, see the report [J] on the website of Impinge. The results have been promising, but regularization has to be introduced for measurements of non-dipolar samples taken at very close proximity due to non-uniqueness issues particularly for the origin depth.

Magnetization recovery Concerning full inversion of thin samples and magnetization recovery, after preliminary experiments on regularization via discretization followed by l^1-constrained residue minimization after discretization (a heavy trend in discrete linear inverse problems today, that favors sparse solutions), we started studying magnetizations modeled by signed measures. For thin slabs of rocks, modeled as a planar samples, silent magnetizations were described in [1]. We extended this characterization in [7] to so-called slender samples, which are still thin but may be irregular, not necessarily planar. A loop decomposition of silent sources in this case, along with a description of magnetizations of minimum total variation among equivalent ones, has been obtained in certain situations. These are connected with new notions of sparsity: either the support is scattered in the measure theoretic sense that it contains no loop (pure 1-unrectifiability), either the magnetization is finite-valued direction-wise (for instance unidirectional). An interesting property of such magnetization is that minimizing residuals while penalizing the total variation norm produces convergence of the estimates, in a fairly strong sense (narrow convergence of the total variation measure), as the regularization parameter goes to zero. An implementation using the FISTA algorithm is currently being set up with promising results. Still, a deeper understanding of how to adjust the parameters of the method is required.

The most attractive direction at present seems to be recovery of unidirectional planar magnetizations using total variation regularization methods. These play for measures the role of l^1 constraints in finite-dimension. Our program has been twofold: on the one hand we developed the theoretical ground for such methods, using tools from geometric measure theory [7]; on the other hand we completed implementation of a FISTA like software and proceeded with experiments on real data which are known to be unidirectional because they were prepared at MIT so as to be magnetically saturated in some unidirectional ambient field.

The above achievements are the results of a joint effort by the US partner and Inria. Moment recovery methods using linear estimators as well as theoretical aspects of total variation methods were led by Inria. The experimental aspects, as well as the multipole approximation techniques, were led by MIT. The adaptation of a FISTA like software and convergence issues in the case of unidirectional magnetizations were led by VU.

3 Production & Impact

3.1 Joint Publications

Publications prior to the current associate team:

[1] E. A. Lima, L. Baratchart, D. P. Hardin, E. B. Saff, and B. P. Weiss. Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions. *Inverse Problems*, 29(1), 2013.

[2] E. A. Lima, B. P. Weiss, L. Baratchart, D. P. Hardin, and E. B. Saff. Fast inversion of magnetic field maps of unidirectional planar geological magnetization. *Journal of Geophysical Research: Solid Earth*, 118(6):2723–2752, 2013.

Publications and theses:

[3] L. Baratchart, S. Chevillard, D. P. Hardin, J. Leblond, E. A. Lima, and J.-P. Marmorat. Magnetic moment estimation and bounded extremal problems. Submitted to Inverse Problems and Imaging, 2017.

[4] L. Baratchart, S. Chevillard, and J. Leblond. Silent and equivalent magnetic distributions on thin plates. In *Harmonic Analysis, Function Theory, Operator Theory, and their Applications*, Theta Series in Advanced Mathematics. The Theta Foundation, 2017.

[5] D. Ponomarev. *Some inverse problems with partial data.* PhD thesis, Univ. Nice Sophia Antipolis, 2016.

Preprints:

[6] L. Baratchart, S. Chevillard, J. Leblond, E. A. Lima, and D. Ponomarev. Asymptotic method for estimating magnetic moments from field measurements on a planar grid. Preprint available at https://hal.inria.fr/hal-01421157, 2018.

[7] L. Baratchart, C. Villalobos Guillen, D. P. Hardin, M. C. Northington, and E. B. Saff. Inverse Potential Problems for Divergence of Measures with Total Variation Regularization. Preprint available at https://arxiv.org/abs/1809.08334, 2018.

Manuscripts, slides, notes:

The documents listed below are all available from the website of Impinge. The letters within brackets to denote them correspond to their reference therein.

[I] Technical notes reporting practical experimentation using the asymptotic formulas to recover the moment of physical data (2017).

[J] Informal presentation of the multipole fitting approach to recover the net moment of samples with reduced support (2017).

[K] Slides of a presentation at MIT Paleomagnetism Meetings, about results obtained with the asymptotic method for synthetic and experimental sources (2018).

[L] Slides of the lecture by E. Lima at the Spring School / Workshop *Inverse problems and approximation techniques in planetary sciences* (2018).

[M] Notes on explicit formulas to compute the adjoint of the forward map operator (2018).

3.2 Software

During the two successive associate teams, we produced Matlab scripts and functions, for the purpose of numerical tests and illustration. This represents roughly 15,000 lines of code on the Inria side, 46,000 lines from the VU partners, and 5,000 from the MIT partners. This includes code to test ideas that have not been fruitful, code to produce synthetic examples, and of course implementation of the methods that we set up (see, *e.g.*, [2], [3], [6], [J]). It is not yet publicly available. Our goal is to keep the methods that work the best on true data, in order to make a software package that would be used at MIT to operate the SQUID, and that could be of possible interest for other geologists teams doing magnetic microscopy.

3.3 Patents N/A

3.4 Demos & Videos

A video is available on the website of Impinge. It illustrates how the solution of the bounded extremal problem described in [3] depends on the regularization parameter.

3.5 Current Position of Students & Postdocs Involved in the Associate Team

C. Borlina, K. Mavreas and C. Villalobos are still PhD students. M. Northington is now a visiting assistant professor at *Georgia Institute of Technology* (Atlanta, USA). D. Ponomarev is now in postdoc at ENSTA ParisTech (Saclay, France).

3.6 Other Forms of Impact N/A

4 Future of the Partnership

In view of the achievements reported in Section 2, the partners identified a number of issues that need to be addressed in order to fully match the potential of their methods.

Moment estimation. The method based on asymptotic formulas proved to work well but suffers from the important limitation that the measurement area must be fairly large compared to the sample. The method based on a bounded extremal problem (BEP) is the natural candidate to overcome this limitation, but it should be improved in two ways: first its formulation must be changed, so as to account for the noise (specifically, the drift of the measured field, due to the instrument). Second, it is currently limited to net moment recovery of thin samples, and should be modified in order to work with thick samples.

On the practical side, more precise and efficient implementations of the BEP resolution scheme may be set up; in particular, the adjoint of the magnetization-to-field map can be explicitly computed in a piecewise polynomial basis (we did it for polynomials of degree 1 in [6], and we recently generalized it to higher degree, see the notes [M] on the website). Another promising computational strategy is to represent the estimator on the basis of eigenfunctions of the Laplacian. In theory, it yields pointwise uniform convergence of the resolution scheme as the truncation order gets large, whence increasing the regularity of the estimator.

One valuable feature of the BEP approach is that it allows one in principle to estimate local moments, *i.e.*, individual moments of separated connected components of the magnetization. This is of special interest for some samples, and the algorithm should be adapted to this case.

We also started working on an intermediate method, which requires less computational burden than BEP and yet is expected to work in situations where the ratio of the sizes of the measurement set and the sample is moderate. The idea is to look for a piecewise affine linear estimator, with few pieces (the number of pieces being the regularization parameter here). This research is still at an early stage, which makes it difficult to assess, but the goal is appealing: to fill the gap between the two methods developed so far.

Finally, the multipolar approximation approach is still in progress. We still need to assess the best regularization strategy and devise efficient ways of selecting the regularization parameter. When combined with upward continuation of field maps, this technique may be used to address a large class of samples.

Full magnetization recovery. The total variation regularization technique must be systematically tested against real samples. The method is especially promising for piecewise unidirectional magnetizations, which was one initial goal of our collaborative research effort.

As regards full inversion in 3D, further theoretical work is needed, starting with the characterization of silent sources which is still much of an open issue in this context.

Expected funding: MIT-France, NSF, Associate Inria Team, ANR.

Do you request the Inria International Partner (IIP) label? Yes. As explained above, we intend to continue this collaboration to address the many remaining open issues.

5 Self-Assessment

Impinge has been a well-balanced association of computer scientists, mathematicians, physicists and experimentalists that generated progress understanding inverse magnetization problems, which are instances of inverse potential problems that recur in many fields. Most notably, the structure of silent sources in the thin plate case, the degree of ill-posedness and the role of the noise have been carried out to a fairly deep extend. IMPINGE granted our team, Apics then Factas, a unique occasion to work with colleagues at MIT providing actual measures from their SQUID and to deepen links with VU. We underestimated the difficulty of doing experiments with the SQUID. They were sometimes hard to produce, time-consuming, and could lead to unpredictable setbacks when the device needed maintenance.

Our partners agreed to pursue collaboration in the near future, so that our joint research project under the auspices of the associate team's program is certainly a success.

6 Feed-back on the associate team's programme

As mentioned above, the program was really helpful. A suggestion: Inria could provide a limited number of PhD grants, reserved to associate teams.