Inria International program
Associate Team final report

Associate Team acronym: Impinge

Period of activity: 2013-2015

Principal investigator (Inria): L. Baratchart, APICS project team, INRIA Sophia-Antipolis.

Principal investigator (partner): B.P. Weiss, Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology (MIT), Cambridge MA, USA.

Other participants: Center for Constructive Approximation (CCA), Department of Mathematics, Vanderbilt University, Nashville TN, USA.

1 Overview of the activities

The associate team was composed of:

Inria: L. Baratchart (confirmed researcher), S. Chevillard (young researcher), J. Leblond (confirmed researcher) and D. Ponomarev (PhD student).

MIT: E. A. Lima (young researcher) and B. P. Weiss (confirmed researcher). I. Sanders (Master student) joined the associate team during the last year.

Vanderbilt University (VU): D. P. Hardin (confirmed researcher) and E. B. Saff (confirmed researcher). M. Northington (PhD student) joined the associate team for the last two years.

We started the collaboration on the associate team by a 3-days workshop at MIT, that all participants except D. Ponomarev attended. It gave us the opportunity to discuss the results and code that some members of the team had already produced prior to the associate team and to assign work to everyone. We subsequently had regular visits from either side to the other, favoring longer stays for PhD students. Additionally, we organized regular teleconferences to keep updates on everyone's work. The following table sums up the visits.

	2013	2014	2015
L. Baratchart	one week to VU and one week to MIT	two weeks to VU and one week to MIT	one week to MIT
S. Chevillard	one week to MIT	ten days to VU and one week to MIT	one week to MIT
J. Leblond	one week to MIT	one week to MIT	two weeks to MIT
D. Ponomarev		two weeks to MIT	two weeks to MIT
E. A. Lima		one week	one week
B. P. Weiss			
I. Sanders	×	×	one week
D. P. Hardin	ten days	one week	
M. Northington	×	two weeks	ten days
E. B. Saff	one month		

2 Scientific achievements

The first task has been to model thin-plate magnetizations theoretically and to understand the non-uniqueness phenomena inherent to this inverse potential problem in divergence form. The initial article [2] by members of the associate team sets up the mathematical framework and settles the issue of non-uniqueness by describing all magnetizations equivalent to a given one. In particular, magnetizations with bounded support, which is the case for rock samples, are equivalent from above if and only if they are equivalent from below. However, this is no longer true for unbounded supports. In fact, even for unidirectional magnetizations, which are of special interest to geophysicists, uniqueness of a magnetization generating a given field depends on the boundedness of the support, since *any* magnetization is equivalent from above to a unidirectional one (with infinite support in general). This has helped explaining why methods in the Fourier domain (which essentially loose track of the support information) undergo some difficulties as was confirmed in [3]. There the computational advantage of Fourier techniques is balanced against undesirable aliasing phenomena.

Next, we geared our efforts towards heuristic procedures to recover regularly spaced dipolar magnetizations. The latter seem general enough a model class to account for commonly encountered samples. We used a discrete least square criterion for computational simplicity, thereby reducing the problem to a singular value decomposition of the magnetization-to-field operator. Ill-posedness naturally led us to develop a regularization technique. The one we used initially has been based on iteratively truncating the support using a threshold on the intensity of the selected dipoles at each step (see [A]). Later, minimization under ℓ^1 constraints using the Salsa algorithm[1] were developed more systematically. Results with either methods turned out to be quite comparable.

Subsequently, we performed more systematic experiments on real data (namely Allende chondrules and Hawaiian basalt) provided by the SQUID scanning microscope at MIT lab. The shrinking of the support proved efficient for samples with localized support embedded in the slab (*e.g.*, chondrules). On the other hand, when the support is spread out (*e.g.*, Hawaiian basalt), the reduction of the support was not significant. This observation was a first achievement of the project in that it draws a clear cut between the recovery of magnetizations with localized support and that of magnetizations whose support is spread out over the sample.

From this point on, special emphasis has bear on estimating the net moment, which is an important piece of information, *e.g.*, for unidirectional recovery. Our first concern has been to check whether magnetizations recovered by our cropping support techniques yield a correct moment, even though they may be quite different from the original ones. This required to focus on samples for which independent estimates of the moment are available (*e.g.*, strong enough to be measured by conventional magnetometers instead of SQUID microscopes). In all experiments we performed, this was apparently the case, even though the quality of the estimates would depend on how small the support of the magnetization is, compared to the measurement area. However, we were able to single out magnetizations with compact support which are fairly silent from above but not from below. Such objects have received no satisfactory description so far, and they cast doubt on moment recovery via full inversion of the magnetization, in general. Considering, in addition, that magnetization recovery is computationally costly, especially when the support is large, we turned to dedicated ways of computing the net moment that should be easier a problem than to recover the full magnetization.

The first approach proceeded in the Fourier domain, using a dipolar field at infinity to account for missing data (see [D,E]). This has resulted in asymptotic formulas for the net moments in terms of weighted integrals of the field over larger and larger portions of the measurement plane. These were later recovered and refined by different methods (see [F]). Recent experiments indicate that such formulas are easy to evaluate numerically and rather efficient on localized samples. For spread out samples, though, the portion of the measurement plane over which data are available seems not to be large enough for asymptotic behavior to take place.

This led us to take up a second approach. It dwells on constructing linear estimators for the net moment or even for local moments. Indeed, one can show that the net moment may in principle be computed, up to prescribed accuracy, by taking the scalar product of the normal component of the field, on the measurement slab, with a carefully chosen test function. The latter can be designed by solving a so-called bounded extremal problem, which is a best approximation problem of a vector field by the gradient of a harmonic function on a section of the plane (see [G]). Solving such a problem is tantamount to solve an elliptic integro-differential PDE in dimension 2. Now that theoretical results have been obtained, current research deals with algorithmic and experimental studies which still remain to be carried out. This line of approach offers further perspectives to compute local moments, *i.e.* moments of sub-parts of the sample.

[1] `http://cascais.lx.it.pt/~mafonso/salsa.html`

Finally, back to the inversion issue, we are studying a spectral problem for a simplified two-dimensional operator, defined by convolution with the Poisson kernel of the upper half-plane (see [C]). This should be helpful in order to design efficient basis vectors to expand the magnetization, thereby giving theoretical grounds to the above mentioned heuristics based on truncation of the support. Besides, a preliminary extension of the data, that should help the inversion procedure, is suggested by harmonic extension properties of the magnetic potential. A preliminary study of this issue is under course, again in the two dimensional situation, using that the potential is linked to the magnetization by Riesz transforms (see [B]).

3 Production

The topic was presented to general public through a post on the blog "Mαthématiques de la planète Terre" [1].

Joint publications by the partners:

[1] S. Chevillard. Une dynamo au centre de la Terre. Brèves de maths. Mathématiques de la planète Terre, 2013. `http://www.breves-de-maths.fr/une-dynamo-au-centre-de-la-terre/`. A translation in English is available at `http://www-sop.inria.fr/apics/IMPINGE/mpt2013_en.html`.

[2] E. A. Lima, L. Baratchart, D. P. Hardin, E. B. Saff, and B. P. Weiss. Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions. *Inverse Problems*, 29(1), 2013.

[3] E. A. Lima, B. P. Weiss, L. Baratchart, D. P. Hardin, and E. B. Saff. Fast inversion of magnetic field maps of unidirectional planar geological magnetization. *Journal of Geophysical Research: Solid Earth*, 118(6):2723–2752, 2013.

Manuscripts:

The manuscripts listed below are working notes and are all available from `http://www-sop.inria.fr/apics/IMPINGE/`. The last two manuscripts are articles reaching completion and will be submitted soon.

[A] Construction of a synthetic example and results obtained on this example with prototype algorithms, 2013.

[B] Study of a simplified 2D version of the magnetization recovery problem, 2013.

[C] About the spectrum of the truncated 2D Poisson operator, 2013.

[D] How to use the Kelvin transform in order to compute the moment of a sample by integral methods using measurements of the field, 2014.

[E] Using dipolar asymptotics to extend the data known only on a finite rectangle to the whole plane, and recovering the moment of a sample (2014, updated in 2015).

[F] Asymptotic recovery of the net moment from integrals of the vertical component of the field on a plane, 2015.

[G] Net moments estimation and bounded extremal problem: recovering the net moment of a magnetization from data on a finite square up to any given accuracy, 2015.

Software development:

For the purpose of numerical tests and illustration, we wrote around 3,000 lines of code in Matlab. This code includes in particular the implementation of the cropping support algorithm (see [A]), of Salsa algorithm, and moment recovery strategies (see [D,E]). It is still experimental and not yet publicly available, but it aims at being eventually transformed into a released software package.

Theses:

- Master's theses:
  - O. Permiakova (2014), Master *Computational Biology and Biomedicine (CompBio)*, University of Nice Sophia Antipolis (UNSA), co-advised by S. Chevillard and J. Leblond, on how to recover the net moment of a magnetization whose thickness is small but not completely negligible, by looking for a thin-plate magnetization on a well-chosen plane.
  - K. Mavreas (2015), Master *CompBio*, UNSA, co-advised by S. Chevillard and J. Leblond, on dipole localization in Moon rocks from sparse magnetic data. This thesis concerns the topic "Lunar magnetism, field evolution and dynamo generation".
  - The Master thesis of I. Sanders at MIT, advised by E. Lima, is in progress on sensitivity of magnetization recovery.
- PhD's theses:
  - The ongoing PhD thesis of D. Ponomarev (UNSA, ED STIC, co-advised by L. Baratchart and J. Leblond) focuses on the spectral analysis of the 2D magnetization operator, as well as on moment recovery in 3D using Fourier techniques (defense expected by Spring 2016).
  - The starting PhD thesis of K. Mavreas (UNSA, ED STIC, co-advised by S. Chevillard and J. Leblond) concerns dipole fitting for Moon rocks from sparse magnetic data (beginning October 2015).
  - The ongoing PhD thesis of M. Northington (Vanderbilt University, co-advised by D. Hardin and A. Powell) focuses on sparse reconstruction using techniques inspired by compressed sensing.

Conferences:

- L. Baratchart was invited to present a communication at the conferences:
  - *Inverse Problems and Nonlinear Equations*, Centre de Mathématiques Appliquées, École Polytechnique, Palaiseau, May 2013, `http://www.cmap.polytechnique.fr/~novikov/ipne/titles.htm`.
  - *Constructive Functions 2014*, Nashville (TN, USA), June 2014 (plenary talk), `http://www.math.vanderbilt.edu/~constructive2014/`.
  - *ISAAC (International Society for Analysis, its Applications and Computations)*, Macao (China), August 2015, `http://www.fst.umac.mo/conference/isaac2015/`.

  He also gave a general presentation of the subject of Impinge at Inria Sophia Antipolis, October 2014, `http://project.inria.fr/mastic/category/cafein/`.

- S. Chevillard was invited to present a communication at the conferences:
  - *PICOF'14, Inverse Problems, Control and Shape Optimization*, Hammamet (Tunisia), May 2014.
  - *27th IFIP TC7 2015 on System Modelling and Optimization*, Sophia Antipolis, July 2015, `http://ifip2015.inria.fr/`.

4 Future of the partnership

This collaborative work on such an exciting and interdisciplinary subject was a key element to attract new PhD students to the team and is likely to generate further opportunities for our students after their PhD defense. Besides, our US partners introduced us to J. Gattacceca from CEREGE (UMR 7330, Aix-en-Provence) which led to our enrollment to the ANR Maglune project (2014-2017)[2] jointly with MIT and CEREGE.

Regarding the scientific topic taken up by the associate team IMPINGE, results obtained so far are quite promising but are not yet fully satisfactory in several respects:

- On the net moment recovering issue, we recently obtained asymptotic expansions, for which experiments on real samples with a fairly localized magnetization gave rather good results. But to go beyond localized samples, we still need to obviate the loss of accuracy due to the asymptotic character of these expansions when the support is spread out.
- Regarding the recovery of the magnetic distribution itself, our experience with IMPINGE taught us there is a big difference between localized magnetizations, whose recovery can be approached at reasonable cost in the space domain by cropping the support, and magnetizations with spread out support which are hardly amenable to the same computational techniques. Especially, it will be necessary to better understand the role of almost silent sources that we observed numerically previously, and to turn to Fourier domain techniques.
- Finally, most samples under study at MIT so far are planar, *i.e.*, consist of thin slabs for which a 2D-supported magnetization is a good model. However there is a growing interest by geophysicists for non-planar models. Some of our previous work should extend fairly easily to 3D magnetizations, but specific issues arise in 3D, especially when recovering the whole magnetization and not just its net moment.

We feel therefore half way to the initial objective, having broken a theoretical path to evaluate the net moment but still facing the need to confirm its practical value in experimental situations and to apply it to unidirectional magnetization recovery. Besides, parallel efforts on the recovery of sparse magnetizations via full inversion of a discrete model of the magnetization-to-field map are still to be pursued.

For all these reasons, we consider our collaboration with MIT and VU to be very valuable and we look forward to continue and to strengthen the latter.

APICS and the participants from MIT have obtained a MIT-France *seed fund* grant to help supporting scientific collaboration for the years 2015 and 2016. Moreover, our US partners have obtained a NSF grant (2015-2018, "Computational methods for ultra-high sensitivity magnetometry of geological samples") of which we appear to be external collaborators. It seems to us that renewing the associate team is a natural way of complementing these fundings and of securing the collaboration for long enough a period to bring our efforts to fruition, up to effective magnetization recovery of featureful though weak magnetic samples. Hence we applied this year for a renewal of IMPINGE to the *Équipe associée* program. We also intend to apply for a renewal of our MIT-France funding for two more years by 2016.

[2] `http://maglune.cerege.fr`