Let C be the rectangle $[-\delta_1\,,\,\delta_1]\times[-\delta_2\,,\,\delta_2]$ and, for $(x_1,x_2)\in\mathbb{R}^2$,

$$\psi(x_1,x_2)=\left(1-\frac{|x_1|}{\delta_1}\right)\,\left(1-\frac{|x_2|}{\delta_2}\right)\,\chi_C(x_1,x_2)=\frac{1}{\delta_1\,\delta_2}\,(\delta_1-|x_1|)\,(\delta_2-|x_2|)\,\chi_C(x_1,x_2)$$
$$=\frac{1}{\delta_1\,\delta_2}\sum_{\epsilon_1\in\{-1,1\}}\,\sum_{\epsilon_2\in\{-1,1\}}(\delta_1-\epsilon_1\,x_1)\,(\delta_2-\epsilon_2\,x_2)\,\chi_C(x_1,x_2)\,,$$

with $\epsilon_i=\pm1$ depending on the sign of x_i for $i=1,2$ (to distinguish between the 4 quadrants in C). For $(t_1,t_2)\in\mathbb{R}^2$, let

$$\Lambda(t_1,t_2)=\Lambda[\psi](t_1,t_2)=\iint_{\mathbb{R}^2}\frac{\psi(x_1,x_2)\,dx_1\,dx_2}{[(x_1-t_1)^2+(x_2-t_2)^2+h^2]^{3/2}}$$
$$=\iint_C\frac{\psi(x_1,x_2)\,dx_1\,dx_2}{[(x_1-t_1)^2+(x_2-t_2)^2+h^2]^{3/2}}\,.$$

By symmetry and using the parity properties of the functions $F_{k,l}$, $k,l=0,1$, we obtain that, for $(t_1,t_2)\in\mathbb{R}^2$,

$$\Lambda(t_1,t_2)=\frac{4}{\delta_1\,\delta_2}\sum_{\epsilon_1\in\{-1,0,1\}}\,\sum_{\epsilon_2\in\{-1,0,1\}}\left(\frac{-1}{2}\right)^{|\epsilon_1|+|\epsilon_2|}\times$$
$$\begin{aligned}[(t_1+\epsilon_1\,\delta_1)\,(t_2+\epsilon_2\,\delta_2)\,&F_{0,0}\,(t_1+\epsilon_1\,\delta_1\,,\,t_2+\epsilon_2\,\delta_2)\\ &+F_{1,1}\,(t_1+\epsilon_1\,\delta_1\,,\,t_2+\epsilon_2\,\delta_2)\\ &-(t_1+\epsilon_1\,\delta_1)\,F_{0,1}\,(t_1+\epsilon_1\,\delta_1\,,\,t_2+\epsilon_2\,\delta_2)\\ &-(t_2+\epsilon_2\,\delta_2)\,F_{1,0}\,(t_1+\epsilon_1\,\delta_1\,,\,t_2+\epsilon_2\,\delta_2)]\,,\end{aligned}$$

where

$$\begin{cases}F_{0,0}(\tau_1,\tau_2)=\dfrac{1}{h}\arctan\dfrac{\tau_1\,\tau_2}{h\,d_h(\tau_1,\tau_2)}\,,\\[2ex] F_{0,1}(\tau_1,\tau_2)=-\operatorname{arcsinh}\dfrac{\tau_2}{(\tau_1^2+h^2)^{1/2}}\,,\;\;F_{1,0}(\tau_1,\tau_2)=F_{0,1}(\tau_2,\tau_1)\,,\\[2ex] F_{1,1}(\tau_1,\tau_2)=-d_h(\tau_1,\tau_2)\,,\end{cases}$$

and[1] $d_h(\tau_1,\tau_2)=(\tau_1^2+\tau_2^2+h^2)^{1/2}$.
For $(t_1,t_2)\in S$ (see [2]), $b_3^*[\psi]$ is then given (on S) by

$$b_3^*[\psi](t_1,t_2)=\frac{\mu_0}{4\,\pi}\begin{bmatrix}\partial_{t_1}\\ \partial_{t_2}\\ -\partial_h\end{bmatrix}(h\,\Lambda(t_1,t_2))\,.$$

We see (e.g. from [2] for $F_{0,0}=k$ and $F_{0,1}=\ell$), that, with $d=d_h(\tau_1,\tau_2)$,

$$\begin{cases}\partial_{\tau_1}F_{0,0}(\tau_1,\tau_2)=\dfrac{\tau_2}{(\tau_1^2+h^2)\;d}\,,\\[2ex] \partial_{\tau_2}F_{0,0}(\tau_1,\tau_2)=\partial_{\tau_1}F_{0,0}(\tau_2,\tau_1)=\dfrac{\tau_1}{(\tau_2^2+h^2)\;d}\,,\\[2ex] \partial_hF_{0,0}(\tau_1,\tau_2)=-\dfrac{1}{h}\,\left(F_{0,0}(\tau_1,\tau_2)+\dfrac{\tau_1\,\tau_2\;(d^2+h^2)}{(\tau_1^2\,\tau_2^2+h^2\,d^2)\;d}\right)\,,\end{cases}$$

[1] Make use of the change of variables $\tau_i\rightsquigarrow\tau_i/h$?

whence

$$\partial_h \left(h\, F_{0,0}(\tau_1,\tau_2)\right) = -\frac{\tau_1\, \tau_2\, h\, \left(d^2+h^2\right)}{\left(\tau_1^2\, \tau_2^2 + h^2\, d^2\right)\, d}\,.$$

Also

$$\begin{cases} \partial_{\tau_1} F_{0,1}(\tau_1,\tau_2) = -\dfrac{1}{d}\,, \\ \\ \partial_{\tau_2} F_{0,1}(\tau_1,\tau_2) = \dfrac{\tau_1\, \tau_2}{\left(\tau_2^2+h^2\right)\, d} = \tau_2\, \partial_{\tau_2} F_{0,0}(\tau_1,\tau_2)\,, \\ \\ \partial_h F_{0,1}(\tau_1,\tau_2) = \dfrac{\tau_1\, h}{\left(\tau_2^2+h^2\right)\, d} = h\, \partial_{\tau_2} F_{0,0}(\tau_1,\tau_2)\,, \end{cases}$$

and

$$\begin{cases} \partial_{\tau_1} F_{1,1}(\tau_1,\tau_2) = -\dfrac{\tau_1}{d} = \tau_1\, \partial_{\tau_1} F_{0,1}(\tau_1,\tau_2)\,, \\ \\ \partial_{\tau_2} F_{1,1}(\tau_1,\tau_2) = -\dfrac{\tau_2}{d}\,, \\ \\ \partial_h F_{1,1}(\tau_1,\tau_2) = -\dfrac{h}{d}\,. \end{cases}$$

If we let $C_{p,q}$ be the rectangle $[x_{1,p}-\delta_1\,,\ x_{1,p}+\delta_1] \times [x_{2,q}-\delta_2\,,\ x_{2,q}+\delta_2]$ similar to C but shifted and centered at $(x_{1,p}, x_{2,q}) \in Q \subset \mathbb{R}^2$, and

$$\psi_{p,q}(x_1,x_2) = \psi(x_1 - x_{1,p}, x_2 - x_{2,q})\,,$$

the associated element following [1, Sec. 5] then, for $(t_1,t_2) \in \mathbb{R}^2$

$$\Lambda[\psi_{p,q}](t_1,t_2) = \Lambda[\psi](t_1 - x_{1,p}, t_2 - x_{2,q})\,.$$

Next, for $(t_1,t_2) \in S$, do we have?

$$b_3^*[\psi_{p,q}](t_1,t_2) = \frac{\mu_0}{4\,\pi} \begin{bmatrix} \partial_{t_1} \\ \partial_{t_2} \\ -\partial_h \end{bmatrix} \left(h\, \Lambda[\psi](t_1 - x_{1,p}, t_2 - x_{2,q})\right)?$$

Note that $b_3^*[\psi_{p,q}](t_1,t_2) \neq b_3^*[\psi](t_1 - x_{1,p}, t_2 - x_{2,q})$ because $b_3^*[\psi_{p,q}]$ is defined on S only... However, if we let $bs_3[\psi]$ be defined on $\mathbb{R}^2$ by

$$bs_3[\psi](t_1,t_2) = \frac{\mu_0}{4\,\pi} \begin{bmatrix} \partial_{t_1} \\ \partial_{t_2} \\ -\partial_h \end{bmatrix} \left(h\, \Lambda(t_1,t_2)\right)\ ,\ \text{for } (t_1,t_2) \in \mathbb{R}^2\,,$$

so that $b_3^*[\psi] = (bs_3[\psi])_{|S}$, then $bs_3[\psi_{p,q}](t_1,t_2) = bs_3[\psi](t_1 - x_{1,p}, t_2 - x_{2,q})$. Also, $b_3^*[\psi_{p,q}](t_1,t_2) = (bs_3[\psi](t_1 - x_{1,p}, t_2 - x_{2,q}))_{|(t_1,t_2)\in S}$.

1 Integrals

In this section we give results on integrals involved in the evalutation of $b_3^\star[P]$ where P is a polynomial in the variables x_1 and x_2.
We introduce the following notations: we usually denote with i an arbitrary index $i \in \{1,2\}$ and with j the other element of $\{1,2\}$. We define $d = (x_1^2 + x_2^2 + h^2)^{1/2}$, $A = \arctan(x_1\, x_2/(h\, d))$, $a_i = x_i^2 + h^2$ and $L_i = \operatorname{arctanh}(x_i/d)$. We focus first on various indefinite integrals with respect to the variable x_i. Whenever a function is even with respect to variable x_i, we select amongst the possible values for the indefinite integral the one that is odd. In all other cases, the integral is defined up to an additive constant that we do not specify.

1. Obviously,

$$\int \frac{1}{d}\,\mathrm{d}x_i = \int \frac{1}{\sqrt{x_i^2 + a_j}}\,\mathrm{d}x_i = \int \frac{1}{\sqrt{a_j}} \cdot \frac{1}{\sqrt{\left(\frac{x_i}{\sqrt{a_j}}\right)^2 + 1}}\,\mathrm{d}x_i = \operatorname{arcsinh}\left(\frac{x_i}{\sqrt{a_j}}\right) = L_i.$$

The last equality is obtained by remembering that the inverse hyperbolic functions can be expressed with respect to ln, namely, on $(-1,1)$, $\operatorname{arctanh}(u) = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$ and, on $\mathbb{R}$,

$$\operatorname{arcsinh}(u) = \ln(u + \sqrt{1+u^2}) = \frac{1}{2}\ln\left(\frac{u + \sqrt{1+u^2}}{-u + \sqrt{1+u^2}}\right).$$

2. If p_i is an integer greater or equal to 1, [3, Sec. 1.2.43, Eq. 6] explicitly gives an expression for the integral of $x_i^{2p_i}/d$ as:

$$\frac{d}{2p_i}\left[\sum_{k=0}^{p_i-1}\left(\frac{-a_j}{2}\right)^k\left(\prod_{s=1}^{k}\frac{2p_i-2s+1}{p_i-s}\right)x_i^{2p_i-2k-1}\right] + \left(\frac{-a_j}{2}\right)^{p_i}\frac{\prod_{s=1}^{p_i}(2p_i-2s+1)}{p_i!}\,L_i.$$

This (somehow heavy expression) can be elegantly summed up as

$$\int \frac{x_i^{2p_i}}{d}\,\mathrm{d}x_i = \beta_{-1}\,L_i + \left(\sum_{k=1}^{p_i}\beta_{2k-1}\,x_i^{2k-1}\right)d,$$

where $\beta_{2p_i-1} = \frac{1}{2p_i}$, $\beta_{2k-1} = -a_j\,\frac{2k+1}{2k}\,\beta_{2k+1}$ for $k \in [\![1,\, p_i-1]\!]$ and $\beta_{-1} = -a_j\,\beta_1$.

3. Now, if $p_i \geq 0$ and $n \in \mathbb{Z}$, [3, Sec. 1.2.43, Eq. 8] expresses the integral of $x_i^{2p_i+1}/d^{2n+1}$ as

$$\sum_{k=0}^{p_i}(-1)^{k+p_i+1}\binom{p_i}{k}\frac{a_j^{p_i-k}d^{2k}}{(2n-2k-1)d^{2n-1}}.$$

Since $d^2 = x_i^2 + a_j$, we immediately see that it takes a form reminiscent of the previous case, namely, $\left(\sum_{k=0}^{p_i}\beta_{2k}\,x_i^{2k}\right)d$, although the coefficients β_{2k} are not as easily expressed as before.

Actually, we can obtain this expression by another way, and with an explicit formula for the β_{2k} using [3, Sec. 1.2.43, Eq. 3]:

$$\int \frac{x_i^{2p_i+1}}{d^{2n+1}}\,\mathrm{d}x_i = \frac{x_i^{2p_i}}{(2p_i-2n+1)d^{n-1}} - \frac{2p_i\,a_j}{2p_i-2n+1}\int \frac{x_i^{2p_i-1}}{d^{2n+1}}\,\mathrm{d}x_i.$$

Unrolling this recurrence when $n=0$, and using the fact the $\int \frac{x_i}{d}\,\mathrm{d}x_i = d$, we see that $\beta_{2p_i} = \frac{1}{2p_i+1}$ and $\beta_{2k} = -a_j\,\frac{2k+2}{2k+1}\,\beta_{2k+2}$ for $k \in [\![0,\,p_i-1]\!]$.

4. Indeed, the results of both previous paragraphs can be summed up in a single formula: for $p_i \geq 1$,

$$\int \frac{x_i^{p_i}}{d}\,\mathrm{d}x_i = \beta_{-1}\,L_i + \Big(\sum_{k=0}^{p_i-1}\beta_k\,x_i^k\Big)\,d$$

where $\beta_{p_i-1} = \frac{1}{p_i}$, $\beta_{p_i-2} = 0$, $\beta_k = -a_j\,\frac{k+2}{k+1}\,\beta_{k+2}$ for $k \in [\![0,\,p_i-3]\!]$ and $\beta_{-1} = -a_j\,\beta_1$.

5. More generally, for $p_i \geq 2$,

$$\int \frac{\sum_{k=0}^{p_i}\gamma_k\,x_i^k}{d}\,\mathrm{d}x_i = \beta_{-1}\,L_i + \Big(\sum_{k=0}^{p_i-1}\beta_k\,x_i^k\Big)\,d,$$

where $\beta_{p_i-1} = \frac{\gamma_{p_i}}{p_i}$, $\beta_{p_i-2} = \frac{\gamma_{p_i-1}}{p_i-1}$, $\beta_k = \frac{\gamma_{k+1}-(k+2)\,a_j\,\beta_{k+2}}{k+1}$ for $k \in [\![0,\,p_i-3]\!]$ and $\beta_{-1} = \gamma_0 - a_j\,\beta_1$.
To prove it, let us simply differentiate the proposed expression with respect to x_i, rewriting once d as $d^2/d = (x_i^2+a_j)/d$ and grouping terms by powers of x_i:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x_i}\left[\beta_{-1}\,L_i + \Big(\sum_{k=0}^{p_i-1}\beta_k\,x_i^k\Big)\,d\right] &= \frac{\beta_{-1}}{d} + \Big(\sum_{k=1}^{p_i-1}k\,\beta_k\,x_i^{k-1}\,d\Big) + \Big(\sum_{k=0}^{p_i-1}\beta_k\,x_i^{k+1}\,\frac{1}{d}\Big)\\
&= \frac{\beta_{-1}}{d} + \Big(\sum_{k=1}^{p_i-1}k\,\beta_k\,x_i^{k-1}\,\frac{x_i^2+a_j}{d}\Big) + \Big(\sum_{k=0}^{p_i-1}\beta_k\,x_i^{k+1}\,\frac{1}{d}\Big)\\
&= \frac{1}{d}\sum_{k=0}^{p_i}\gamma_k\,x_i^k.
\end{aligned}$$

Notice that, even though p_i is required to be greater or equal to 2 for the recurrence formula to make sense, none of the γ_k is required to be non-zero. This means that the formula is valid for a polynomial of any degree: it must simply be completed with coefficients equal to 0 when the degree is smaller or equal to 1. For instance taking $p_i = 2$, $\gamma_0 = 1$, $\gamma_1 = \gamma_2 = 0$ leads to $\beta_1 = \beta_0 = 0$ and $\beta_{-1} = 1$, whence retrieving that the indefinite integral of $1/d$ is L_i.

6. We now turn to the indefinite integrals of a polynomial over d^3. We obviously have

$$\int \frac{x_i}{d^3}\,\mathrm{d}x_i = \frac{-1}{d}.$$

Hence,

$$\int \frac{x_i^3}{d^3}\,\mathrm{d}x_i = \int \frac{(d^2-a_j)x_i}{d^3}\,\mathrm{d}x_i = d + \frac{a_j}{d} = \frac{x_i^2+2a_j}{d}.$$

The integral of x_i^2/d^3 is performed by parts:

$$\int \frac{x_i^2}{d^3}\,\mathrm{d}x_i = \int x_i\cdot\frac{\mathrm{d}}{\mathrm{d}x_i}\big(-\frac{1}{d}\big)\,\mathrm{d}x_i = -\frac{x_i}{d} + L_i.$$

Finally, remarking that $1 = \frac{d^2-x_i^2}{a_j}$, we get

$$\int \frac{1}{d^3}\,\mathrm{d}x_i = \frac{1}{a_j}\int \frac{d^2-x_i^2}{d^3}\,\mathrm{d}x_i = \frac{x_i}{a_j\,d}.$$

Notice that all these integrals are explicitly listed in [3, Sec. 1.2.43, Eq. 17 to Eq. 20].

7. For $p_i \geq 2$, the integral of $x_i^{2p_i}/d^3$ is not given explicitly in [3] although it is possible to deduce an expression of it using, *e.g.*, [3, Sec. 1.2.43, Eq. 2 or Eq. 3]. On the other hand, as we have seen in paragraph 3, an expression of the integral of $x_i^{2p_i+1}/d^3$ is given by [3, Sec. 1.2.43, Eq. 8] and takes the form of $1/d$ times a polynomial in variable x_i^2, but with an obfuscated expression for the coefficients of the polynomial.
Similarily to what we did in paragraph 5, we show that, when $p_i \geq 4$:

$$\int \frac{\sum_{k=0}^{p_i}\gamma_k\,x_i^k}{d^3}\,\mathrm{d}x_i = \beta_{-1}\,L_i + \frac{\sum_{k=0}^{p_i-1}\beta_k\,x_i^k}{d},$$

where $\beta_{p_i-1} = \frac{\gamma_{p_i}}{p_i-2}$, $\beta_{p_i-2} = \frac{\gamma_{p_i-1}}{p_i-3}$, $\beta_k = \frac{\gamma_{k+1}-(k+2)\,a_j\,\beta_{k+2}}{k-1}$ for $k \in [\![0,\, p_i-3]\!]\backslash\{1\}$, $\beta_{-1} = \gamma_2 - 3a_j\beta_3$ and $\beta_1 = \frac{\gamma_0}{a_j} - \beta_{-1}$.
To prove it, let us simply differentiate the proposed expression with respect to x_i, rewriting $1/d$ as $d^2/d^3 = (x_i^2+a_j)/d^3$ and grouping terms by powers of x_i:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x_i}\left[\beta_{-1}\,L_i + \Big(\sum_{k=0}^{p_i-1}\beta_k\,\frac{x_i^k}{d}\Big)\right] &= \frac{\beta_{-1}}{d} + \Big(\sum_{k=1}^{p_i-1}k\,\beta_k\,\frac{x_i^{k-1}}{d}\Big) - \Big(\sum_{k=0}^{p_i-1}\beta_k\,x_i^{k+1}\,\frac{1}{d^3}\Big)\\
&= \frac{1}{d^3}\Big(\beta_{-1}(a_j+x_i^2) + \sum_{k=1}^{p_i-1}k\,\beta_k\,a_j\,x_i^{k-1} + \sum_{k=0}^{p_i-1}(k-1)\beta_k\,x_i^{k+1}\Big)\\
&= \frac{1}{d^3}\sum_{k=0}^{p_i}\gamma_k\,x_i^k.
\end{aligned}$$

As in paragraph 5, the formula is indeed valid for polynomials of any degree, provided that one completes them with zero coefficients in order to take $p_i \geq 4$.

8. For $p_i \geq 6$,

$$\int \frac{\sum_{k=0}^{p_i}\gamma_k\,x_i^k}{d^5}\,\mathrm{d}x_i = \beta_{-1}\,L_i + \frac{\sum_{k=0}^{p_i-1}\beta_k\,x_i^k}{d^3},$$

where $\beta_{p_i-1} = \frac{\gamma_{p_i}}{p_i-4}$, $\beta_{p_i-2} = \frac{\gamma_{p_i-1}}{p_i-5}$, $\beta_k = \frac{\gamma_{k+1}-(k+2)\,a_j\,\beta_{k+2}}{k-3}$, for $k \in [\![0,\, p_i-3]\!]\backslash\{1,3\}$, $\beta_{-1} = \gamma_4 - 5\beta_5 a_j$, $\beta_1 = \frac{\gamma_0}{a_j} - \beta_{-1}\,a_j$ and $\beta_3 = \frac{\gamma_2}{3a_j} + \frac{2\gamma_0}{3a_j^2} - \frac{4}{3}\,\beta_{-1}$.
The technique of the proof is as before and the same remark about the degree applies.

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