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Inverse source problem for electromagnetic fields, with physical applications

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Abstract

The considered problem consists in retrieving the magnetization distribution from the measurements of the external magnetic field which is produced by this magnetization and mapped by the scanning SQUID microscope. This inverse source problem is ill-posed. It appears in many domain and paleomagnetism is one of those.

We state the problem in mathematical terms. We solve the formulated minimization problem by approximating it by Maclaurin series. In order to examine the quality of the obtained solution we perform the recovering of the magnetization from the simulated measurements of the magnetic field. We retrieve the magnetization from synthetic measurements by solving the least square problem and show that for the computed position the recovered magnetization resembles the simulated one. We verify that the displacement of the plane where we recover magnetization, from the middle of the sample to the new position, improves the numerical results.

Keywords: inverse source problem, thin plate magnetization, SQUID microscope, Maclaurin series, least square method.

Contents

1 Introduction

The property of some rocks to hold the remanent magnetization is known for a long time. The rocks acquire this magnetic property due to presence of various minerals, for example, iron monoxide(FeO), ferric oxide (Fe_2O_3), siderite (the mineral composed of iron carbonate ($FeCO_3$)), iron chlorites, etc. There are several types of remanent magnetization which depend on the mechanism of rocks formation. For instance, the thermoremanent magnetization is acquired when the igneous rocks cool from a temperature above the Curie point. As well phase change or chemical action during the formation of magnetic oxides at low temperatures give rise to the crystallization (chemical) remanent magnetization.

The magnetization which is locked-in the rocks is called fossil magnetism. Paleomagnetism is the study of the records of the fossil magnetization in rocks. These records contain information about the direction and intensity of the geomagnetic field at that moment when rocks were formed. Paleomagnetic data is used to verify the theories of continental drift and plate tectonics, to determine the age of the igneous rocks, to estimate the age sites bearing fossils and hominid remains, to investigate the behavior of Earth's magnetic field. For more information about paleomagnetism we refer to [6, 7].

A scanning SQUID (superconducting quantum interference device) microscope is a super-sensitive magnetometer, it is capable of measuring very weak magnetic field about 10^{-13}T. It represents a superconducting ring and two Josephson tunnel junctions. The principle of operation is based on the presence of the wave properties of the electron. It is a very effective tool and is used in medicine (magnetoencephalography), experimental physics (search for the electron electric dipole moment), geophysical survey [3, 8]. The SQUID microscope is used to image the magnetic field produced by the magnetized geological sample.

The microscope measures the vertical component of the magnetic field in a uniform rectangular grid pattern at a certain height above the specimen placed on a horizontal plane by moving SQUID sensor across an area. However information about the distribution of the magnetization within the sample has a greater interest for research. Therefore it is required to solve the appeared ill-posed problem of recovering the magnetization from the measurements of the vertical components of the field it produced. The nonuniqueness of this inverse problem arises because different magnetization distributions can give rise to the same observed magnetic field. The use of supplementary information enables to restrict the set of possible solutions and choose among all magnetization distributions which produce the observed magnetic field that one which has the physical sense. For instance, one of these assumptions can be the unidirectionality of the magnetization it means that it has fixed direction but variable nonnegative magnitude [1]. The unidirectional character of the magnetization often occurs in nature so it is logical assumption. When rocks are formed from the rapid cooling of basaltic lava in presence of the external magnetic field (Earth magnetic field) they magnetize in the direction of the magnetic field lines.

The geological samples which are used in paleomagnetic analysis are polished rocks with tiny thickness from several hundred micrometers to several micrometers that is much smaller than the horizontal dimensions. The magnetization is measured at a distance several times greater than the thickness of the sample. In that scale it is possible to suppose that the sample has no thickness in the vertical direction and to represent the sample as the thin plate.

The case of the two-dimensional magnetization distribution with compact support was analysed in [1]. It involves applying harmonic analysis and characterizing the kernel of the magnetization operators. A necessary and sufficient condition for a magnetization to be silent source is determined. More precisely, a thin plate magnetization is silent when its normal component is zero and the tangential component is divergence-free. The authors prove that any planar magnetization distribution is equivalent to a unidimensional one. The magnetization is called unidimentional if $m = Qu$ for some fixed $u \in \mathbb{R}^3$ and some scalar valued distribution Q [?]. Also it is shown that for unidirectional magnetization with compact support it is possible to guarantee that the solution of the inverse problem is unique assuming that there are no sources outside a considered bounded region.

An inversion technique based on the classical technique in the Fourier domain to retrieve thin plate unidirectional magnetization distribution from the data of the normal component of the magnetic field is presented in the article [5]. The authors test their approach both on real geological samples and a synthetic samples. They compare results with previous obtained ones by using spatial-domain inversion techniques in the spatial domain [8].

As already indicated above values of the sample's thickness can be around several hundred micrometers. For such size it is not correct to neglect the thickness of the sample. We explore how thin plate magnetization distribution are varied depending on the placement this plate on the z axis. And in this work we investigate for which position the reconstruction of the magnetization is more successful by approximating the emerged minimization problem.

2 Volumetric magnetization

2.1 Derivation of the potential's formula

Consider the Maxwell-Ampère equation which states:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{1}$$

where $\mathbf{B}$ is the magnetic-field vector (magnetic induction vector), $\mu_0 = 4\pi \times 10^{-7} Hm^{-1}$ is the magnetic constant, $\mathbf{J}$ is the total current density and $\nabla \times$ is the curl operator.

We consider that the free current density is absent so the total current density is just the bound current density. The latter corresponds to the current which arises due to movements of electric and magnetic dipole moments per unit volume.

The magnetization current density $\mathbf{J}_M$ can be expressed in terms of the magnetization vector $\mathbf{M}$ by formula $\mathbf{J}_M = \nabla \times \mathbf{M}$. In consequence the Maxwell-Ampere equation can be reformulated as follows:

$$\nabla \times \mathbf{H} = 0, \tag{2}$$

where $\mathbf{H}$ is given by the following expression:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \tag{3}$$

This function $\mathbf{H}$ is called the magnetic intensity (cf. [4]).

We conclude from Helmholtz's theorem and the obtained equation (2) that the magnetic intensity $\mathbf{H}$ can be defined by a function ϕ as follows:

$$\mathbf{H} = -\nabla \phi. \tag{4}$$

The function ϕ is called the magnetic scalar potential.

We obtain the Poisson equation taking the divergence of the equation (3), substituting the expression (4) for $\mathbf{H}$ and using the Gauss's law which states that $\nabla \cdot \mathbf{B} = 0$:

$$\triangle \phi = \nabla \cdot \mathbf{M}, \tag{5}$$

where $\triangle$ is the Laplacian in Cartesian coordinates.

Recalling that the fundamental solution of the Laplacian in $\mathbb{R}^3$ is $G_L(r) = -\frac{1}{4\pi\|r\|}$ that is a Green's function. Hence the solution of the Poisson equation is given by:

$$\phi(r) = (G_L * \nabla \cdot \mathbf{M})(r) = -\frac{1}{4\pi} \iiint\limits_{\mathbb{R}^3} \frac{1}{\|r - r'\|} (\nabla \cdot \mathbf{M})(r')\, dr', \tag{6}$$

where $r \in \mathbb{R}^3$ is a position vector and $*$ is the three-dimensional convolution operator.

We apply the product rule for the scalar function G_L and the vector field $\mathbf{M}$ as follows:

$$\nabla \cdot (G_L \mathbf{M}) = \mathbf{M} \cdot \nabla G_L + G_L \nabla \cdot \mathbf{M}$$

Noting that $\nabla \frac{1}{\|r\|} = -\frac{r}{\|r\|^3}$ and considering that $\mathbf{M}$ is compactly supported we obtain for all $r \notin \operatorname{supp}(\mathbf{M})$ the formula for the magnetic scalar potential:

$$\phi(r) = \frac{1}{4\pi} \iiint \frac{\mathbf{M}(r') \cdot (r - r')}{\|r - r'\|^3} \, dr'. \tag{7}$$

2.2 Formulation of the problem

In fact the SQUID microscope maps the vertical component of the magnetic field B_z. However the vertical component of the magnetic field is the partial derivative with respect to z of the potential:

$$B_z(r) = \mu_0 \frac{\partial}{\partial z} \phi(r) = \frac{\mu_0}{4\pi} \iiint\limits_{\operatorname{supp}(\mathbf{M})} \frac{3\mathbf{M}(r') \cdot (r - r')}{\|r - r'\|^5} (r - r') - \frac{\mathbf{M}(r')}{\|r - r'\|^3} \, dr' \tag{8}$$

It is not significant which of quantities B_z or ϕ we explore. We formulate the problem for the scalar magnetic potential.

The sample is thin and so the magnetization distribution undergoes marginal changes within the sample. To be more exact, we assume the direction magnetization is uniform in depth and the intensity differs by a constant. We define a function $\varphi(z) : [-\beta, \beta] \to \mathbb{R}$ describes the variation of the magnetization intensity depending on the depth in the sample. A priori we do not know any information about the form of the function $\varphi(z)$. At the beginning we suppose that it is enough differentiable. More precisely, we assume that the magnetization has the following form:

$$\mathbf{M}(x, z) = \varphi(z)(m_T(x), m_3(x)). \tag{9}$$

We denoted the tangential and normal components of the magnetization by $m_T(x) = (m_1(x), m_2(x))$ and $m_3(x)$ respectively. In the most general framework the components of $\mathbf{M}$ lie in $\mathcal{D}'(\mathbb{R}^3)$. Thus the potential at the position $r = (x, h)$ for a certain h and $x \in \mathbb{R}^2$ is computed by formula (7) as follows:

$$V_\beta(x, h) = \frac{1}{4\pi} \iint\limits_{\operatorname{supp}(\mathbf{M})} \int_{-\beta}^{\beta} \varphi(z') \frac{m_T(x') \cdot (x - x') + m_3(x')(h - z')}{(\|x - x'\|^2 + (h - z')^2)^{\frac{3}{2}}} \, dz' \, x', \tag{10}$$

where β is a half of the thickness of the sample. Further this potential will be called volumetric.

As mentioned above the proportions of the sample allow of regarding it as a thin plate. In that case we consider that the support of the magnetization is distributed on the finite section of the x-y plane at a certain height z'. The magnetization can be expressed as a tensor product distribution in $\mathcal{D}'(\mathbb{R}^3)$ of the form:

$$\overline{\mathbf{M}}(x,z) = (\overline{m}_T(x), \overline{m}_3(x)) * \delta_{z'}(z), \tag{11}$$

Then the thin plate's potential which is measured at the height h is defined as:

$$\Gamma_{z'}(x,h) = \frac{1}{4\pi} \iint\limits_{\mathrm{supp}(\overline{\mathbf{M}})} \frac{\overline{m}_T(x')\cdot(x-x') + \overline{m}_3(x')(h-z')}{(\|x-x'\|^2 + (h-z')^2)^{\frac{3}{2}}}\, dx'. \tag{12}$$

Hence the volumetric potential can be expressed in terms of the thin plate's potential as follows:

$$V_\beta(x,h) = \int\limits_{-\beta}^{\beta} \varphi(z')\Gamma_{z'}(x,h)\, dz'. \tag{13}$$

We search for the layer within the sample which makes a rather more contribution to the total magnetization. In other words we want to find a point on the z axis such that the magnetization of the thin plate at this height describes the obtained measurements of the magnetic field the best way. In mathematical terms the problem can be represented as the minimization with respect to z_0 of the square of the $L^2(\mathbb{R}^2)$ - norm of the difference between the potentials:

$$S_{z_0}(x) = \alpha(\beta)\Gamma_{z_0}(x,h) - V_\beta(x,h). \tag{14}$$

subject to the equality of net-moments:

$$\alpha(\beta) \iint\limits_{\mathbb{R}^2} \begin{pmatrix} m_T(x') \\ m_3(x') \end{pmatrix} dx' = \iint\limits_{\mathbb{R}^2} \int\limits_{-\beta}^{\beta} \varphi(z') \begin{pmatrix} m_T(x') \\ m_3(x') \end{pmatrix} dz'\, dx'.$$

The function $\alpha(\beta)$ plays a role of the weighting coefficient in the equation (14), generally speaking it is the mean value of the function $\varphi(z)$ on the closed interval $[-\beta,\beta]$.

This expression also equivalents to the following:

$$\alpha(\beta) = \int\limits_{-\beta}^{\beta} \varphi(z')\, dz'. \tag{15}$$

In general the problem can be formulated for $L^2(\mathbb{R}^2$, but since the magnetic field is measured on the finite region we can consider the problem on a square $[-a,a]^2$ and value of a depends on the experimental setup. Thus we minimize the square of the norm on $L^2([-a,a]^2)$:

$$\min_{z_0\in[-\beta,\beta]} \|S_{z_0}(x)\|^2_{L(\mathbb{R}^2)} \tag{16}$$

Since the function $\varphi(z)$ is continuous on the interval $[-\beta, \beta]$ and by taking the equality of net-moments (15) into account we obtain:

$$\alpha(\beta) = 2\varphi(0)\beta + o(\beta^2). \tag{17}$$

2.3 Approximation of the problem

The value of the thickness is small in comparison with other involved quantities such that its width and length as well the sensor-to-sample distance therefore we can approximate the potentials by a Taylor expansion centered at zero in order to find the minimum of the difference. That sort of the series is called Maclaurin series. Also in order to compare the order of magnitude of the potentials we identically assume that β tends to zero and analyse how the solution z_0 of the minimization problem varies.

The general formula of the Maclaurin series is given by:

$$F(x) = \sum_{k=0}^{n} \frac{x^k F^{(k)}(0)}{k!} + R_n(x),$$

where $R_n(x)$ is a remainder of the series and a function $F(x)$ is at least $n+1$ times differentiable on the interval between 0 and x. The remainder can be expressed in Lagrange form $R_n(x) = \frac{x^{(n+1)}F^{(n+1)}(\theta)}{(n+1)!}$, for some real number $\theta \in (0, x)$.

Let us consider the second-order approximation. So as to avoid cumbersome expressions we omit h in what follows since we always measure the external magnetic field on the same plane (i.e. h is fixed) and we introduce some notations:

$$\begin{aligned}
\gamma_0(x) &= \Gamma_0(x, h), \\
\gamma_1(x) &= \frac{\partial}{\partial z}\Gamma_z(x, h)\bigg|_{z=0}, \\
\gamma_2(x) &= \frac{\partial^2}{\partial z^2}\Gamma_z(x, h)\bigg|_{z=0}.
\end{aligned}$$

The Maclaurin series of the potential $\Gamma_z(x)$ with respect to the variable z with the remainder $Q_2(z, x)$ is given by:

$$\Gamma_z(x) = \gamma_0(x) + z\gamma_1(x) + \frac{z^2}{2}\gamma_2(x) + Q_2(z, x). \tag{18}$$

The general formula of the Maclaurin series of the potential $V_\beta(x)$ as a function of the variable β with the remainder $P_n(\beta, x)$ is given by:

$$V_\beta(x) = \sum_{k=0}^{n} \frac{\partial^k}{\partial \beta^k} V_\beta(x)\bigg|_{\beta=0} \frac{\beta^k}{k!} + P_n(\beta, x).$$

The function $V_\beta(x)$ is a odd function of the variable β so the graph of it has rotational symmetry with respect to the origin. The derivative of an odd function is an even function and vice-versa. Therefore the values of the even order partial derivatives of this function at the point zero equal zero. So the Maclaurin series takes the form:

$$V_\beta(x) = \sum_{k=0}^{\frac{n-1}{2}} \frac{\partial^{2k+1}}{\partial\beta^{2k+1}} V_\beta(x)\bigg|_{\beta=0} \frac{\beta^{2k+1}}{(2k+1)!} + P_n(\beta, x).$$

We remind that for the time being we approximate the function to the second order therefore to expand the potential $V_\beta(x)$ into series we need compute the partial derivative:

$$\frac{\partial}{\partial\beta} V_\beta(x) = \frac{\partial}{\partial\beta} \int_{-\beta}^{\beta} \varphi(z')\Gamma_{z'}(x,h)\, dz'.$$

In order to obtain the explicit expression for the partial derivative of $V_\beta(x)$ we use the formula of the differentiation under the integral sign, generally it can be stated as follows:

$$\frac{\partial}{\partial t} \int_{\xi(t)}^{\psi(t)} f(x,t)\, dt = f(t,\psi(t))\psi'(t) - f(t,\xi(t))\xi'(t) + \int_{\xi(t)}^{\psi(t)} \frac{\partial}{\partial t} f(x,t)\, dx, \tag{19}$$

where both $f(x,t)$ and $\frac{\partial}{\partial t} f(x,t)$ are continuous functions in x and t, where $\xi(t) \le x \le \psi(t)$. Observing that the integrand in expression (19) does not depend on the integration variable and setting $\xi = -\beta$ and $\psi = \beta$ we apply the formula (19) and we obtain that the partial derivative of $V_\beta(x)$ is of the form:

$$\frac{\partial}{\partial\beta} V_\beta(x) = \varphi(\beta)\Gamma_\beta(x) + \varphi(-\beta)\Gamma_{-\beta}(x). \tag{20}$$

The series expansion of the potential $V_\beta(x)$ of the volumetric sample is given by:

$$V_\beta(x) = 2\varphi(0)\beta\gamma_0(x) + P_2(\beta, x). \tag{21}$$

Replacing in the expression (14) the potentials by their expansion into Maclaurin series and $\alpha(\beta)$ by its explicit formula (17) we observe that terms multiplied by γ_0 disappear and we obtain the following expression for $S_{z_0}(x)$:

$$2\varphi(0)\beta\left(z_0\gamma_1(x) + \frac{z_0^2}{2}\gamma_2(x) + Q_2(z_0,x)\right) + P_2(\beta,x) \tag{22}$$

This expression has the minimum with respect to z_0 at the point zero. It corresponds to the model considered in the previous works [1,8] for which the volumetric sample is regarded as the thin-plate $z = 0$. The same . In order to improve this result let us consider next terms of the series of the function $V_\beta(x)$.

The next term of the series is multiplied by the third partial derivative. We compute the third partial derivative of $V_\beta(x)$ by using the general Leibniz rule for the nth derivative of a product of two functions. The Leibniz rule is formulated as follows:

$$(f(x)\cdot g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{n}(x),$$

where $\binom{n}{k}$ are the binomial coefficients for all $k = 1, \dots, n$.

According to this formula and taking into account the fact that the even order partial derivatives vanish at the point zero, the general formula of the nth derivative of $V_\beta(x)$ with respect to β can be computed for all integer numbers $n \geq 0$ as follows:

$$\frac{\partial^{n+1}}{\partial \beta^{n+1}} V_\beta(x)\bigg|_{\beta=0} = \begin{cases} 2\sum\limits_{k=0}^{n} \binom{n}{k} \varphi^{(n-k)}(0) \, \frac{\partial^k}{\partial \beta^k} \Gamma_\beta(x,h)\Big|_{\beta=0}, & \text{if } n+1 is odd; \\ 0, & \text{if } n+1 is even. \end{cases} \tag{23}$$

We recall that the derivative of order zero is the function itself. Thus as a particular case we have the formula of the third-order partial derivative of $V_\beta(x)$ at the point zero:

$$\frac{\partial^3}{\partial \beta^3} V_\beta(x)\bigg|_{\beta=0} = 2\varphi''(0)\gamma_0(x) + 4\varphi'(0)\gamma_1(x) + 2\varphi(0)\gamma_2(x). \tag{24}$$

As a result we obtain the Maclaurin series of $V_\beta(x)$ in the following form:

$$V_\beta(x) = 2\varphi(0)\beta\gamma_0(x) + \frac{\beta^3}{3}\left(\varphi''(0)\gamma_0(x) + 2\varphi'(0)\gamma_1(x) + \varphi(0)\gamma_2(x)\right) + P_4(\beta, x). \tag{25}$$

We try to find the optimal position z_0 in the z axis such that the planar magnetization distribution at height z_0 is the most similar to the whole magnetization of the sample as seen on the measured potential at the height h. For this purpose we compute the difference between the thin plate and the volumetric potentials. We expand into series the function $\alpha(\beta)$:

$$\alpha(\beta) = 2\varphi(0)\beta + \frac{\beta^3}{3}\varphi''(0) + o(\beta^4). \tag{26}$$

At this point the difference between potentials can be approximated as follows:

$$\begin{aligned} S_{z_0}(x) &= \alpha(\beta)\Gamma_{z_0}(x) - V_\beta(x) = \alpha(\beta)\left(\gamma_0(x) + z_0\gamma_1(x) + \frac{z_0^2}{2}\gamma_2(x) + Q_2(z_0,x)\right) - \\ &- \left(2\varphi(0)\beta\gamma_0(x) + \frac{\beta^3}{3}\left(\varphi''(0)\gamma_0(x) + 2\varphi'(0)\gamma_1(x) + \varphi(0)\gamma_2(x)\right) + P_4(\beta,x)\right) = \\ &= \underline{\alpha(\beta)\gamma_0(x)} + \alpha(\beta)\left(z_0\gamma_1(x) + \frac{z_0^2}{2}\gamma_2(x)\right) + \alpha(\beta)Q_2(z_0,x) - \\ &- \underline{\left(2\varphi(0)\beta + \frac{\beta^3}{3}\varphi''(0)\right)\gamma_0(x)} - \frac{\beta^3}{3}\left(2\varphi'(0)\gamma_1(x) + \varphi(0)\gamma_2(x)\right) + P_4(\beta,x). \end{aligned}$$

The underlined terms with γ_0 vanish by taking into account the obtained formula of the weighted parameter.

So we conclude that the problem consists in minimization of the following expression:

$$\|S_{z_0}(x)\|^2_{L^2([-a,a]^2)} = \|\alpha(\beta)\left(\frac{z_0^2}{2}\gamma_2(x) + z_0\gamma_1(x) + Q_2(z_0,x)\right) - \\ - \frac{\beta^3}{3}\left(2\varphi'(0)\gamma_1(x) + \varphi(0)\gamma_2(x)\right) + P_4(\beta,x)\|^2_{L^2([-a,a]^2)}.$$

Let us denote $\hat{\gamma}(x) = 2\varphi'(0)\gamma_1(x) + \varphi(0)\gamma_2(x)$ for simplification.

Compute the square of the L^2-norm of this expression:

$$\|S_{z_0}(x)\|^2 = \|\alpha(\beta)\left(\frac{z_0^2}{2}\gamma_2(x) + z_0\gamma_1(x)\right) - \frac{\beta^3}{3}\hat{\gamma}(x) + \alpha(\beta)Q_2(z_0,x) + P_4(\beta,x)\|^2 = \\ = \underbrace{\|\alpha(\beta)\left(\frac{z_0^2}{2}\gamma_2(x) + z_0\gamma_1(x)\right) - \frac{\beta^3}{3}\hat{\gamma}(x)\|^2}_{(1)} + \underbrace{\|\alpha(\beta)Q_2(z_0,x) + P_4(\beta,x)\|^2}_{(2)} \\ + \underbrace{\left\langle \alpha(\beta)\left(z_0^2\gamma_2(x) + 2z_0\gamma_1(x)\right) - \frac{2\beta^3}{3}\hat{\gamma}(x), \alpha(\beta)Q_2(z_0,x) + P_4(\beta,x)\right\rangle}_{(3)}. \quad (27)$$

We compute each terms individually and analyse the order of their summands taking into account that $\alpha(\beta) \in \mathcal{O}(\beta)$. Since we consider approximation of the problem we have already neglected some information i.e. we should compare the components of the equation regard to the fact that some terms can be smaller than remainders.

The first term of the equation (27) equals:

$$\|\alpha(\beta)\left(\frac{z_0^2}{2}\gamma_2(x) + z_0\gamma_1(x)\right) - \frac{\beta^3}{3}\hat{\gamma}(x)\|^2 = \\ = \alpha^2(\beta)\left(\frac{z_0^4}{4}\|\gamma_2(x)\|^2 + z_0^3\langle\gamma_2(x),\gamma_1(x)\rangle + z_0^2\|\gamma_1(x)\|^2\right) - \\ - \alpha(\beta)\left(\frac{\beta^3 z_0^2}{3}\langle\gamma_2(x),\hat{\gamma}(x)\rangle - \frac{2\beta^3 z_0}{3}\langle\gamma_1(x),\hat{\gamma}(x)\rangle\right) + \frac{\beta^6}{9}\|\hat{\gamma}(x)\|^2. \quad (27.1)$$

The second term of the equation (27) equals:

$$\|\alpha(\beta)Q_2(z_0,x) + P_4(\beta,x)\|^2 = \\ = \alpha^2(\beta)\|Q_2(z_0,x)\|^2 + \alpha(\beta)\langle Q_2(z_0,x), P_4(\beta,x)\rangle + \|P_4(\beta,x)\|^2 \in \\ \in o(\beta^2 z_0^4) + o(\beta^5 z_0^2) + o(\beta^8). \quad (27.2)$$

It should be noted that $P_4(\beta,x)$ and $Q_2(z_0,x)$ are square-integrable functions with respect to variable x so they do not affect the order with respect to z_0.

The last term of the equation (27) equals the following:

$$\begin{aligned}&\big\langle \alpha(\beta)\left(z_0^2\gamma_2(x)+2z_0\gamma_1(x)\right)-\frac{2\beta^3}{3}\hat\gamma(x),\alpha(\beta)Q_2(z_0,x)+P_4(\beta,x)\big\rangle=\\&=\alpha^2(\beta)\left(z_0^2\big\langle\gamma_2(x),Q_2(z_0,x)\big\rangle+2z_0\big\langle\gamma_1(x),Q_2(z_0,x)\big\rangle\right)-\frac{2\alpha(\beta)\beta^3}{3}\big\langle\hat\gamma(x),Q_2(z_0,x)\big\rangle\\&+\alpha(\beta)z_0^2\big\langle\gamma_2(x),P_4(\beta,x)\big\rangle+2\alpha(\beta)z_0\big\langle\gamma_1(x),P_4(\beta,x)\big\rangle-\frac{2\beta^3}{3}\big\langle\hat\gamma(x),P_4(\beta,x)\big\rangle.\qquad(27.3)\end{aligned}$$

Let us analyse the order of magnitude of the summands of this expression:

$$\begin{aligned}&\alpha^2(\beta)z_0^2\big\langle\gamma_2(x),Q_2(z_0,x)\big\rangle\in o(\beta^2z_0^4),\\&2\alpha^2(\beta)z_0\big\langle\gamma_1(x),Q_2(z_0,x)\big\rangle\in o(\beta^2z_0^3),\\&\frac{2\alpha(\beta)\beta^3}{3}\big\langle\hat\gamma(x),Q_2(z_0,x)\big\rangle\in o(\beta^4z_0^2),\\&\alpha(\beta)z_0^2\big\langle\gamma_2(x),P_4(\beta,x)\big\rangle\in o(\beta^5z_0^2),\\&2\alpha(\beta)z_0\big\langle\gamma_1(x),P_4(\beta,x)\big\rangle\in o(\beta^5z_0),\\&\frac{2\beta^3}{3}\big\langle\hat\gamma(x),P_4(\beta,x)\big\rangle\in o(\beta^7).\end{aligned}$$

We neglect the element $o(\beta^2z_0^4)$ in the second and third expressions because it is smaller according to z_0 than the element $o(\beta^2z_0^3)$ of the third expression. By the same consideration the elements of order $o(\beta^5z_0^2)$ vanish by comparing it with $o(\beta^5z_0)$. Since $o(\beta^8)\subset o(\beta^7)$ we remove $o(\beta^8)$. The other elements of the second and third expressions are not comparable with each other.

Hence we obtain the following approximate expression for $\|S_{z_0}(x)\|^2$:

$$\begin{aligned}&\alpha^2(\beta)\frac{z_0^4}{4}\|\gamma_2(x)\|^2+\alpha^2(\beta)z_0^3\big\langle\gamma_2(x),\gamma_1(x)\big\rangle+\alpha^2(\beta)z_0^2\|\gamma_1(x)\|^2-\\&\qquad-\frac{\alpha(\beta)\beta^3z_0^2}{3}\big\langle\gamma_2(x),\hat\gamma(x)\big\rangle-\frac{2\alpha(\beta)\beta^3z_0}{3}\big\langle\gamma_1(x),\hat\gamma(x)\big\rangle+\frac{\beta^6}{9}\|\hat\gamma(x)\|^2+\\&\qquad\qquad+o(\beta^2z_0^3)+o(\beta^4z_0^2)+o(\beta^5z_0)+o(\beta^7).\qquad(28)\end{aligned}$$

Notice that the terms $o(\beta^2z_0^3)$ and $o(\beta^4z_0^2)$ are subsets of $o(\beta^2z_0^2)$. Likewise the components $\alpha^2(\beta)z_0^3\big\langle\gamma_2(x),\gamma_1(x)\big\rangle$ as well as $\frac{\alpha(\beta)\beta^3z_0^2}{3}\big\langle\gamma_2(x),\hat\gamma(x)\big\rangle$ belong to $o(\beta^2z_0^2)$. In addition $\alpha^2(\beta)\frac{z_0^4}{4}\|\gamma_2(x)\|^2\in o(\beta^2z_0^3)$ is negligible quantity by comparison with $o(\beta^2z_0^2)$. Therefore we can replace all of them by the term $o(\beta^2z_0^2)$.

As a result we substitute the explicit formula for $\hat\gamma(x)$ and we obtain the following final equation of $\|S_{z_0}(x)\|^2$:

$$\begin{aligned}&z_0^2\alpha^2(\beta)\|\gamma_1(x)\|^2-\frac{2}{3}z_0\alpha(\beta)\beta^3\big\langle\gamma_1(x),2\varphi'(0)\gamma_1(x)+\varphi(0)\gamma_2(x)\big\rangle\\&\qquad+\frac{\beta^6}{9}\|2\varphi'(0)\gamma_1(x)+\varphi(0)\gamma_2(x)\|^2+o(\beta^2z_0^2)+o(\beta^5z_0)+o(\beta^7).\qquad(29)\end{aligned}$$

It is represented as the quadratic parabola with respect to variable z_0. The coefficient of the highest order term is positive therefore the minimum of this expression is at the vertex of the parabola and can be computed as:

$$z_0 = \frac{\frac{2}{3}\alpha(\beta)\beta^3\big\langle\gamma_1(x), 2\varphi'(0)\gamma_1(x) + \varphi(0)\gamma_2(x)\big\rangle}{2\alpha^2(\beta)\|\gamma_1(x)\|^2}. \tag{30}$$

We simplify this equation, substitute $\alpha(\beta)$ for the expression (26) and denoted by a_1 and a_2 respectively $\|\gamma_1(x)\|^2_{L^2([-a,a]^2)}$ and $\big\langle\gamma_1(x), \gamma_2(x)\big\rangle_{L^2([-a,a]^2)}$.

Eventually the minimum point given by:

$$z_0 = \frac{\beta^2\left(2\varphi'(0)a_1 + \varphi(0)a_2\right)}{a_1\left(6\varphi(0) + \beta^2\varphi''(0)\right)}, \tag{31}$$

or else

$$z_0 = \frac{2\varphi'(0)a_1 + \varphi(0)a_2}{6\varphi(0)a_1}\left(\beta^2 - \beta^4\frac{\varphi''(0)}{6\varphi(0)} + o(\beta^4)\right). \tag{32}$$

We introduce the obtained z_0 into the expression (28) and we observe that sum of little-o terms equal $o(\beta^7) + o(\beta^8) \subset o(\beta^7)$. According to the fact that $\alpha(\beta) \in \mathcal{O}(\beta)$ the first two summands and term $\frac{\alpha(\beta)\beta^3 z_0^2}{3}\big\langle\gamma_2(x), \hat{\gamma}(x)\big\rangle$ also belong to $o(\beta^7)$. Hence the approximation order of $\|S_{z_0}(x)\|^2$ is $o(\beta^7)$.

3 Numerical experiments

Commonly the experiments are performed for two actual geological samples: Hawaiian basalt (30-mm thin section of basalt from the Mauna Loa volcano) and Lonar Spherule (two 100mm diameter glass spherules from the Moon). We construct a synthetic example of the measurements of the magnetic field for carrying out numerical experiments. The simulation of the measurements data is justified that the true net-moment is known in this case. It makes it possible to estimate the quality of the numerical experiments by comparing the recovered net-moment with the true one. It is desired that the synthetic magnetization resembles approximately the magnetization of the true Lonar Spherule example which is presented in Figure 1.

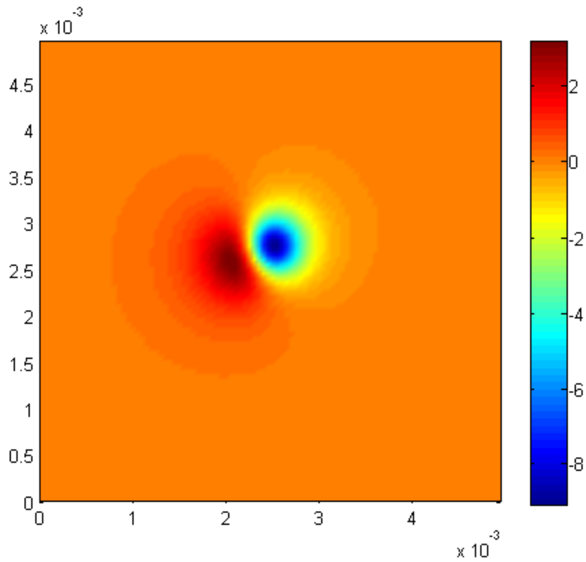

Figure 1: The normal component of the magnetic field of the Lonar Spherule example

The magnetization of the thin geological sample is measured at a distance three times greater than a half of the sample thickness. We set the thickness of the sample equals $2 \cdot 10^{-4} m$. We represent each magnetization source element by the single magnetic dipole. We choose a main direction and an amplitude of the dipoles in that way that the magnetization looks like the Lonar Spherule example. All dipoles almost points in that direction but there is small component in the plane orthogonal to this direction too. This orthogonal component varies smoothly for all dipoles. Three components of the synthetic magnetization are represented below.

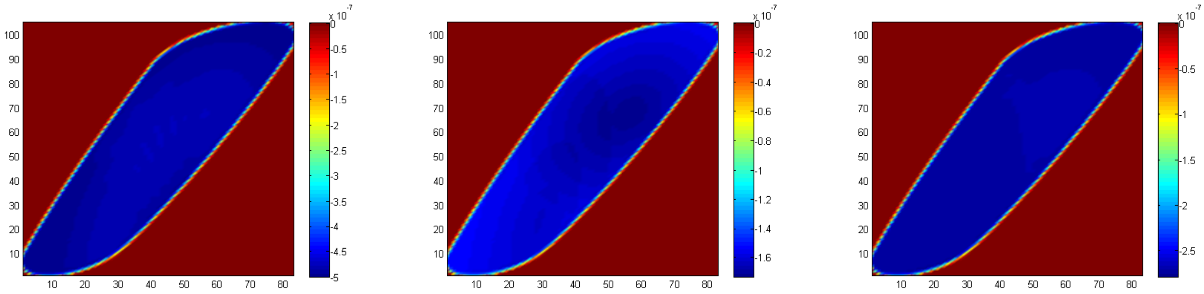

Figure 2: Three components of the synthetic magnetization

We use the grid with the sampling step is ten times smaller than the step of the measurement grid in order to model a continuous magnetization. The sampling step of the measurement grid equals $5 \cdot 10^{-5}$. We consider the sample as a set of thin layers from $-\beta$ to β. The magnetization from each layers multiplied by the function $\varphi(z)$ compose the volumetric magnetization. For the numerical experiments we suppose that the function φ is a constant since we do not have any information about this function. Introducing any physical specificity of the function $\varphi(z)$ is possible with some modification since all computations were done for a general case. We set the quantity of layers equal to 200. The magnetization is supported in a rectangle 83x105. Its shape is the dot of the 'i' of the Inria logo.

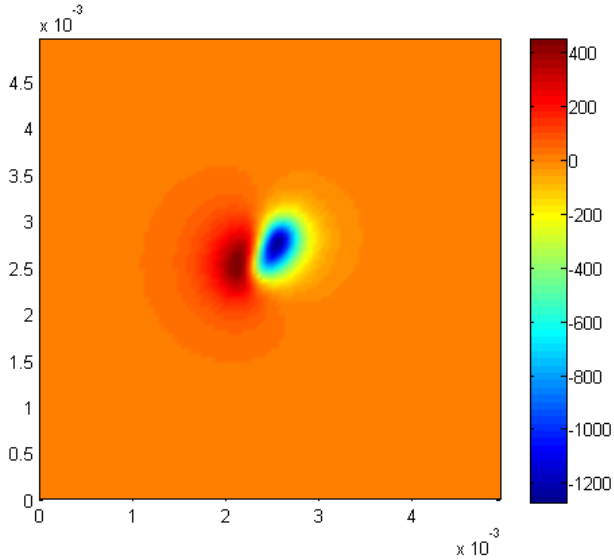

Figure 3: The normal component of the synthetic volumetric magnetization field B_z

We regard the synthetic magnetic field as measurements provided by the scanning microscope on the whole surface of the thin plate. We suppose that the information about the support of the magnetization is unknown. At first, we let z_0 equal zero. We use the technique described in [2] for retrieving the magnetization from the synthetic data. We reconstruct the magnetization on the grid with the sampling step in three times bigger then for the measurement grid. We solve the least squares problem which consists in seeking a vector X that minimizes the squared Euclidean norm $\|AX - B\|_2$, where B is an vector containing the measurements of the vertical component of the field and A is a discrete linear operator that maps the dipoles on the measurements. In order to find a minimum we equate to zero the derivative with respect to X of this norm:

$$\frac{\partial}{\partial X}\|AX - B\|_2^2 = -2A^T B + 2A^T AX = 0.$$

Hence we obtain the solution in form of $X = (A^T A)^{-1} A^T B$. Then we invert a matrix $M = A^T A$ which is symmetric positive. We do a singular value decomposition of this matrix $M = UDV^T$, matrices U and V are unitary, D is diagonal matrix. Then the solution is $X = VD^{-1}U^T A^T B$. As a result we obtain the vector X containing the three component of the magnetization m_1, m_2, m_3. The true magnetization has a finite support which is smaller that whole magnetization grid hence at most points it has very small values. Therefore we takes the maximal absolute value for each components of the magnetization and sort out values at least 10 % of the maximum. We shrink the support and then we perform the same method to the operator $\hat{A}$ that maps the dipoles disposed on the new support.

We compute the normal component B_z of the magnetic field which is produced by the recovered magnetization and the partial derivative of B_z with respect to the variable z by expressing the formula (8) as two-dimensional convolutions:

$$\begin{aligned} B_z(x, y, h) = \mu_0 \left(m_1 * \frac{3xh}{(x^2 + y^2 + h^2)^{\frac{5}{2}}} \right) + \mu_0 \left(m_2 * \frac{3yh}{(x^2 + y^2 + h^2)^{\frac{5}{2}}} \right) + \\ + \mu_0 \left(m_3 * \frac{3h^2}{(x^2 + y^2 + h^2)^{\frac{5}{2}}} - \frac{1}{(x^2 + y^2 + h^2)^{\frac{3}{2}}} \right) \end{aligned} \tag{33}$$

$$\begin{aligned} \frac{\partial}{\partial z} B_z(x, y, h) = \mu_0 \left(m_1 * \frac{15xh^2}{(x^2 + y^2 + h^2)^{\frac{7}{2}}} - \frac{3x}{(x^2 + y^2 + h^2)^{\frac{5}{2}}} \right) + \\ + \mu_0 \left(m_2 * \frac{15yh^2}{(x^2 + y^2 + h^2)^{\frac{7}{2}}} - \frac{3y}{(x^2 + y^2 + h^2)^{\frac{5}{2}}} \right) + \\ + \mu_0 \left(m_3 * \frac{15h^3}{(x^2 + y^2 + h^2)^{\frac{7}{2}}} - \frac{9h}{(x^2 + y^2 + h^2)^{\frac{5}{2}}} \right) \end{aligned} \tag{34}$$

Then we compute the point z_0 according to the formula obtained above:

$$z_0 = \frac{(2\varphi'(0)a_1 + \varphi(0)a_2)\,\beta^2}{6\varphi(0)a_1}, \tag{35}$$

where a_1 and a_2 denote respectively $\|B_z(x,y,h)\|^2_{L^2([-a,a]^2)}$ and $\left\langle B_z(x,y,h), \frac{\partial}{\partial z} B_z(x,y,h) \right\rangle_{L^2([-a,a]^2)}$. Assuming that $\varphi(z)$ is a constant the point z_0 equate to $\frac{a_2 \beta^2}{6 a_1}$. We obtain that the value z_0 equals $9.38 \cdot 10^{-6}$.

Subsequently we again recover the magnetization at the new depth by using the same approach.

The magnetization retrieved by the described above method is quite similar to the synthetic magnetization. Comparing the net-moments the magnetization we can see that the difference between the directions is very small, equals 0,87 degrees and it is more accurate than for the recovered magnetization at the middle for which the difference is about 0,98 degrees. The quantity of the points for which the magnetization has significant amplitude is about 5 %. In the figure 4 the new supports are marked by a red color.

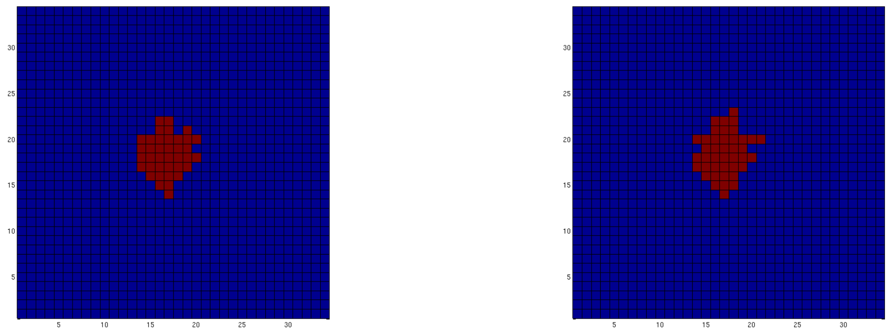

Figure 4: The left image: the reduced support of the magnetization $z = 0$ (3%) . The right image: the reduced support of the magnetization $z = z_0$ (4%).

4 Conclusion

Solving the minimization problem by Maclaurin series approximation we determined at which height one should dispose the thin plate in order to recover the magnetization distribution from the measurements of the magnetic field which is mapped above the sample at the distance h from the middle of the sample. We generated a synthetical example to test the obtained results numerically. The numerical experiments confirm that the location for the thin plate at the plane $z = 0$ is not optimal since the net-moment of the magnetization at the new position is improved.

Owing to the absence of the information about physical characteristics of the function $\varphi(z)$ we cannot suppose anything about the behavior or properties of it. The quality of the approximation as well the results depend on the form of this function. On the other hand, possessing supplementary information we can adjust computations according new constraints. For the future work it is possible to explore how the calculations are changed if we introduce the specific function $\varphi(z)$. In case when $\varphi(z)$ is nonuniform (significant value lie outside the interval $[-\beta^2, \beta^2]$) the second order approximation with respect to z is inappropriate. We can introduce the function $\varphi(\frac{z}{\beta})$ and obtain the solution for this function.

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