On a spectral problem for the truncated Poisson operator

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December 1, 2013

1 Problem formulation and symmetries

Consider the eigenvalue problem with the Poisson kernel

$$T[f](x) := (P_h \star f)(x) = \frac{h}{\pi}\int_{-a}^{a} \frac{f(\tilde{x})}{(x-\tilde{x})^2 + h^2} d\tilde{x} = \lambda f(x), \quad x \in [-a, a], \tag{1}$$

$$D(T) := \left\{ f \in L^2(\mathbb{R}) : \text{supp } f \subset [-a, a] \right\},$$

where a, $h > 0$.

Since the Poisson kernel is real and symmetric, T is self-adjoint and compact (integration is over a finite range, and P_h is non-singular), which ensures us that there exist countably many eigenvalues $\lambda \in \mathbb{R}$ accumulating to 0 and corresponding eigenfunctions f are mutually orthogonal and complete in $L^2(-a, a)$.

We start by noting symmetries of the operator with respect to reflection and complex conjugation. First of all, the operator T preserves parity: if R is the sign-inversion operator (i.e. $Rf(x) = f(-x)$), then we check that the operators R and T commute:

$$R[T[f]](x) = \frac{h}{\pi}\int_{-a}^{a} \frac{f(\tilde{x})}{(-x-\tilde{x})^2 + h^2} d\tilde{x} = \frac{h}{\pi}\int_{-a}^{a} \frac{f(-\tilde{x})}{(x-\tilde{x})^2 + h^2} d\tilde{x} = T[R[f]](x).$$

But the vanishing commutator $\{R, T\} = RT - TR = 0$ implies that the spectral problem (1) can be splitted into two, for odd and even eigenfunctions, so we consider them separately. Similarly, since the kernel is real, it is evident that if $f(x)$ is an eigenfunction, so is $\text{Re} f(x)$ and hence it is sufficient to consider only real-valued functions.

2 Going to the Fourier domain...

First of all, since $\hat{\chi}_{[-a,a]}(k) = \int_{-a}^{a} e^{-2\pi i k x} dx = \dfrac{\sin 2\pi a k}{\pi k}$ and $\hat{P}_h(k) = e^{-2\pi h |k|}$, in the Fourier domain, the problem recasts as

$$\hat{T}\left[\hat{f}\right](k) := \int_{\mathbb{R}} \frac{\sin 2\pi a \left(k - \tilde{k}\right)}{\pi \left(k - \tilde{k}\right)} e^{-2\pi h |\tilde{k}|} \hat{f}\left(\tilde{k}\right) d\tilde{k} = \lambda \hat{f}(k), \tag{2}$$

where $\hat{f} \in PW^a := \left\{ g \in H(\mathbb{C}) \cap L^2(\mathbb{R}) : \quad |g(k)| \le Ce^{2\pi a |\mathrm{Im} k|} \text{ for some } C > 0 \right\}$.

Next, we notice that the identity $f(x) = \chi_{[-a,a]}(x) f(x)$ implies[1]

$$\hat{f}(k) = \mathcal{F}\left[\chi_{[-a,a]} f\right](k) = \left(\hat{\chi}_{[-a,a]} \star \hat{f}\right)(k) = \int_{\mathbb{R}} \mathcal{K}\left(k, \tilde{k}\right) \hat{f}\left(\tilde{k}\right) d\tilde{k}, \tag{3}$$

$$\mathcal{K}\left(k, \tilde{k}\right) := 2a \,\mathrm{sinc} 2a \left(k - \tilde{k}\right).$$

Another observation is that since Fourier transform is an isometry on $L^2(\mathbb{R})$, and $\left\{ \frac{1}{\sqrt{2a}} e^{i\pi n x/a} \chi_{[-a,a]}(x) \right\}_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2(-a, a)$, the functions $\left\{ \sqrt{2a}\,\mathrm{sinc}(2ka - n) \right\}_{n \in \mathbb{Z}}$ constitute orthonormal basis in PW^a. Moreover, due to (3), if $\hat{f}(k) = \sum_{n=-\infty}^{\infty} f_n \sqrt{2a}\,\mathrm{sinc}(2ka - n)$, then $f_n = \frac{1}{\sqrt{2a}} \hat{f}\left(\frac{n}{2a}\right)$, that is, the expansion coefficients of a function in are essentially its values on a uniform grid.

As it was mentioned, eigenfunctions of the operator T are either odd or even. We now focus on finding class of odd eigenfunctions so assume that $f(x) = -f(-x)$ and it is real-valued. Then the Fourier transform $\hat{f}(k) := \mathcal{F}[f]$ is an odd purely imaginary function: $\hat{f}(k) = -\hat{f}(-k) = -\overline{\hat{f}(k)}$ for $k \in \mathbb{R}$. Then the problem (2) transforms into

$$\hat{T}_o\left[\hat{f}\right](k) := \int_0^\infty \mathcal{K}_o\left(k, \tilde{k}\right) e^{-H\tilde{k}} \hat{f}\left(\tilde{k}\right) d\tilde{k} = \lambda \hat{f}(k), \tag{4}$$

$$\mathcal{K}_o\left(k, \tilde{k}\right) := 2a\left[\mathrm{sinc} 2a\left(k - \tilde{k}\right) - \mathrm{sinc} 2a\left(k + \tilde{k}\right)\right], \quad H := 2\pi h,$$

and $-\mathcal{K}_o\left(k, \tilde{k}\right)$ is a reproducing kernel for the odd Paley-Wiener space

$$PW_o^a = \left\{ g \in PW^a : \quad g(k) = -g(-k) = -\overline{g(k)} \text{ for } k \in \mathbb{R} \right\}$$

with respect to the standard $L^2(0, \infty)$ inner product:

$$\left\langle -\mathcal{K}_o(k, \cdot), \hat{f} \right\rangle_{L^2(\mathbb{R}_+)} = \int_0^\infty \mathcal{K}_o\left(k, \tilde{k}\right)\left(-\overline{\hat{f}\left(\tilde{k}\right)}\right) d\tilde{k} = \hat{f}(k).$$

2.1 Some differential operators

It would be desirable to construct a differential operator $\mathcal{D} : PW_o^a \to PW_o^a$ such that[2]

$$\int_0^\infty \mathcal{K}_o\left(k, \tilde{k}\right) e^{-H\tilde{k}} \mathcal{D}\left[\hat{f}\left(\tilde{k}\right)\right] d\tilde{k} = \int_0^\infty \mathcal{D}_k\left[\mathcal{K}_o\left(k, \tilde{k}\right)\right] e^{-H\tilde{k}} \hat{f}\left(\tilde{k}\right) d\tilde{k},$$

since then the problem would reduce to solving a differential equation $\mathcal{D}\left[\hat{f}\right] = \lambda \hat{f}$. We have not succeeded with that, yet observed an interesting differential property.

[1] We use the definition $\mathrm{sinc}(k) := \frac{\sin \pi k}{\pi k}$.

[2] The index k in $\mathcal{D}_k$ stands for the variable on which $\mathcal{D}$ operates.

Lemma 2.1. *Define* $\mathcal{D}_1 : PW_o^a \to PW_o^a$, $\mathcal{D}_1[f] := k^2 f'' + 2kf' + A^2k^2 f = \left(k^2 f'\right)' + (Ak)^2 f$, $A := 2\pi a$. *Then*

$$\int_0^\infty e^{-H\tilde{k}} \mathcal{D}_{1,\tilde{k}} \left[\mathcal{K}_o\left(k,\tilde{k}\right)\right] \hat{f}\left(\tilde{k}\right) d\tilde{k} = \lambda \mathcal{D}_1\left[\hat{f}(k)\right]. \tag{5}$$

Proof. We observe that $g_0(k) := \operatorname{sinc}(2ak)$ is a scaled zeroth spherical harmonic and hence is a solution to the following Bessel equation

$$k^2 g_0'' + 2kg_0' + A^2k^2 g_0 = 0. \tag{6}$$

Set $g(k) = 2ag_0(k)$ so $\mathcal{K}_o\left(k,\tilde{k}\right) = g\left(k-\tilde{k}\right) - g\left(k+\tilde{k}\right)$. By linearity, $g(k)$ satisfies the same equation (6), hence $g'(k) = -\frac{1}{2}k\left[g''(k) + A^2 g(k)\right]$.

We compute

$$\begin{aligned}
\partial_k \mathcal{K}_o\left(k,\tilde{k}\right) &= -\frac{1}{2}\left(k-\tilde{k}\right)\left[g''\left(k-\tilde{k}\right) + A^2 g\left(k-\tilde{k}\right)\right] \\
&\quad +\frac{1}{2}\left(k+\tilde{k}\right)\left[g''\left(k+\tilde{k}\right) + A^2 g\left(k+\tilde{k}\right)\right] \\
&= -\frac{1}{2}k\left[\partial_k^2 \mathcal{K}_o\left(k,\tilde{k}\right) + A^2\mathcal{K}_o\left(k,\tilde{k}\right)\right] + U_1\left(k,\tilde{k}\right),
\end{aligned}$$

where $U_1\left(k,\tilde{k}\right) := \frac{1}{2}\tilde{k}\left(\frac{d^2}{dk^2} + A^2\right)\left[g\left(k-\tilde{k}\right) + g\left(k+\tilde{k}\right)\right]$. Similarly

$$\partial_{\tilde{k}} \mathcal{K}_o\left(k,\tilde{k}\right) = -\frac{1}{2}\tilde{k}\left[\partial_{\tilde{k}}^2 \mathcal{K}_o\left(k,\tilde{k}\right) + A^2\mathcal{K}_o\left(k,\tilde{k}\right)\right] + U_2\left(k,\tilde{k}\right),$$

with $U_2\left(k,\tilde{k}\right) := \frac{1}{2}k\left(\frac{d^2}{d\tilde{k}^2} + A^2\right)\left[g\left(k-\tilde{k}\right) + g\left(k+\tilde{k}\right)\right]$. Since $kU_1\left(k,\tilde{k}\right) = \tilde{k}U_2\left(k,\tilde{k}\right)$, we obtain the relation

$$k\partial_k \mathcal{K}_o\left(k,\tilde{k}\right) - \tilde{k}\partial_{\tilde{k}}\mathcal{K}_o\left(k,\tilde{k}\right) = \frac{1}{2}\left(\tilde{k}^2 - k^2\right)\left[\partial_k^2\mathcal{K}_o\left(k,\tilde{k}\right) + A^2\mathcal{K}_o\left(k,\tilde{k}\right)\right]. \tag{7}$$

Since $\partial_k^2\mathcal{K}_o\left(k,\tilde{k}\right) = \partial_{\tilde{k}}^2\mathcal{K}_o\left(k,\tilde{k}\right)$, we can combine

$$\int_0^\infty e^{-H\tilde{k}} k\partial_k \mathcal{K}_o\left(k,\tilde{k}\right)\hat{f}\left(\tilde{k}\right) d\tilde{k} = \lambda k\hat{f}'(k), \quad \int_0^\infty e^{-H\tilde{k}} \partial_{\tilde{k}}^2 \mathcal{K}_o\left(k,\tilde{k}\right)\hat{f}\left(\tilde{k}\right) d\tilde{k} = \lambda \hat{f}''(k)$$

with (7) to yield the property (5). □

Remark 2.2. Repetetive integration by parts of (5) leads to

$$\int_0^\infty e^{-H\tilde{k}} \mathcal{K}_o\left(k,\tilde{k}\right)\mathcal{D}_1\left[\hat{f}\right]\left(\tilde{k}\right) d\tilde{k} = \lambda\mathcal{D}_1\left[\hat{f}\right](k) + \int_0^\infty e^{-H\tilde{k}}\mathcal{K}_o\left(k,\tilde{k}\right)\mathcal{D}_2\left[\hat{f}\right]\left(\tilde{k}\right) d\tilde{k}, \tag{8}$$

where $\mathcal{D}_2[f](k) := Hk[2f(k) - Hkf(k) + kf'(k)]$. In other words, we have the following commutation property

$$\left(\hat{T}\mathcal{D}_1 - \mathcal{D}_1\hat{T}\right)\left[\hat{f}\right] = \left(\hat{T}\mathcal{D}_2\right)\left[\hat{f}\right]. \tag{9}$$

2.2 False solution by nonlinear Laplace transformation

Now moving to a further idea, we slightly modify the domain $D(T)$: let us additionally assume super smoothness, that is $f \in C_c^{\infty}(-a, a)$, and hence T becomes a densely defined operator. It then follows that f is in the Schwartz class $\mathcal{S}$, and thus so is $\hat{f}$ (since $\mathcal{F} : \mathcal{S} \to \mathcal{S}$). Therefore, $\hat{f}(k)k^n \in L^2(\mathbb{R})$ for any $n \in \mathbb{N}_0$ and we observe that $\hat{f}(k)k^{2n}$ and $\hat{f}'(k)k^{2n+1}$ are in PW_o^a. Taking inner product of (4) with each of these functions and using the reproducing kernel property, we obtain, respectively

$$\int_0^{\infty} e^{-H\tilde{k}}\hat{f}^2\left(\tilde{k}\right)\tilde{k}^{2n}d\tilde{k} = \lambda\int_0^{\infty}\hat{f}^2(k)k^{2n}dk, \tag{10}$$

$$\int_0^{\infty} e^{-H\tilde{k}}\hat{f}\left(\tilde{k}\right)\hat{f}'\left(\tilde{k}\right)\tilde{k}^{2n+1}d\tilde{k} = \lambda\int_0^{\infty}\hat{f}(k)\hat{f}'(k)k^{2n+1}dk. \tag{11}$$

Denote the Laplace transform $F(s) := \mathcal{L}\left[\hat{f}^2\right](s)$ and employing the following properties[3]

$$\mathcal{L}\left[\left(\hat{f}^2\right)'\right](s) = s\mathcal{L}\left[\hat{f}^2\right](s) - \hat{f}^2(0) = sF(s),$$

$$\mathcal{L}\left[k^n\hat{f}^2(k)\right](s) = (-1)^n\frac{d^n}{ds^n}\mathcal{L}\left[\hat{f}^2\right](s) = (-1)^nF^{(n)}(s),$$

$$(sF(s))^{(2n+1)} = sF^{(2n+1)}(s) + (2n+1)F^{(2n)}(s), \quad n \in \mathbb{N}_0,$$

we realize that (10)-(11) are nothing but the following relations between derivatives

$$\begin{cases} F^{(2n)}(H) & = \lambda F^{(2n)}(0), \\ HF^{(2n+1)}(H) + (2n+1)F^{(2n)}(H) & = \lambda(2n+1)F^{(2n)}(0). \end{cases} \tag{12}$$

This simplifies to

$$\begin{cases} F^{(2n)}(H) & = \lambda F^{(2n)}(0), \\ F^{(2n+1)}(H) & = 0. \end{cases} \tag{13}$$

Since $F(s)$ is analytic in the right half-plane of $\mathbb{C}$ (as a Laplace transform), we can expand, in particular, about $s = 1$ and such expansion must be valid at least up to the point $s = 0$, that is in the disk $|s - H| \leq H$. Taking into

[3]Since $\hat{f}(k)$ vanishes at the origin as it is odd and holomorphic. But in fact, the same would also remain true for the case of even eigenfunctions as it is stated in Lemma 3.1.

account the second equation in (13), we have

$$F(s) = \sum_{m=0}^{\infty} F^{(2m)}(H) \frac{(s-H)^{2m}}{(2m)!}.$$

Since

$$F^{(2k)}(s) = \sum_{m=k}^{\infty} F^{(2m)}(H) \frac{(s-H)^{2(m-k)}}{[2(m-k)]!},$$

the first equation in (13) yields

$$\sum_{m=k}^{\infty} c_m \frac{H^{2(m-k)}}{[2(m-k)]!} = \frac{1}{\lambda} c_k,$$

that is

$$c_m = \frac{\lambda}{1-\lambda} \sum_{k=1}^{\infty} c_{m+k} \frac{H^{2k}}{(2k)!}, \tag{14}$$

where we introduced $c_m := F^{(2m)}(H)$.

To sum up, we are searching for the set $\{c_m\}_{m=0}^{\infty} \subset \mathbb{C}$ satisfying, for some $\lambda \in \mathbb{R}$, (14) which is an infinite-dimensional system with upper-triangular Toeplitz matrix with zero diagonal elements. However, the solution is not unique for a given λ, for if $\{c_m\}_{m=0}^{\infty}$ is a solution, then so are $\{c_{m+n}\}_{m=0}^{\infty}$ for any $n \in \mathbb{N}$ (this corresponds to the fact that if $F(s)$ is a solution, then $F^{(2n)}(s)$, $n \in \mathbb{N}$ are solutions as well). This is in contrast with the original formulation of the problem which does not allow infinite multiplicities for each given eigenvalue $0 < \lambda < 1$ (as suggested by the general spectral theory of compact self-adjoint operators; the fact that eigenvalues are positive is seen, for instance, from (10)). The issue arises from the nonlinear transformation $\hat{f} \mapsto \mathcal{L}\left[\hat{f}^2\right]$ which breaks equivalence of the problems. Therefore, we have to formulate a criteria to weed out those solutions to (14) which are not Laplace transforms of the square of a function from the space PW_o^a. The Bernstein theorem (see Th.12b in [4]) which characterizes Laplace transforms of non-negative functions can serve as one such filter. Namely, if $\tilde{F}(s)$ is a Laplace transform of a non-negative function, then it must be completely monotonic, meaning that, for all $k \in \mathbb{N}_0$, $(-1)^k \tilde{F}^{(k)}(s) \geq 0$, $s \in \mathbb{R}_+$. In our case, since $\hat{f}$ is purely imaginary, $F(s)$ is Laplace transform of a non-positive function, and hence must satisfy, for all $k \in \mathbb{N}_0$,

$$(-1)^{k+1} F^{(k)}(s) \geq 0, \quad s \in \mathbb{R}_+. \tag{15}$$

We can see that even such condition is already restrictive enough to make simplest solutions to (14) fail. Indeed, suppose the solution is in the form $c_m = c_0 a_0^m$ for some c_0, $a_0 \in \mathbb{R}$. Then (14) leads to

$$1 = \frac{\lambda}{1-\lambda} \sum_{k=1}^{\infty} \frac{H^{2k} a_0^k}{(2k)!} \quad \Rightarrow \quad \lambda = \frac{1}{\cosh\left(H\sqrt{a_0}\right)},$$

$$F(s) = c_0 \sum_{m=0}^{\infty} a_0^m \frac{(s-H)^{2m}}{(2m)!} = c_0 \cosh\left[(s-H)\sqrt{a_0}\right], \tag{16}$$

which also comprise cosine functions when the parameter a_0 is negative. However, in either case, the derivatives fail to have a constant sign on $\mathbb{R}_+$ and therefore, such family does not qualify to yield appropriate solutions by the criterion (15). In fact, the general solution does not differ much from this. It is due to the mentioned shift invariance property that solutions must be of the form $c_m = c_0 a_0^m$, but in the general case it should be $a_0 \in \mathbb{C}$, which would give, for any $\beta_n \in \mathbb{C}$,

$$F(s) = \sum_{n=-\infty}^{\infty} \beta_n \cosh\left[\frac{1}{H}\left(\operatorname{arccosh}(1/\lambda) + 2\pi i n\right)(s-H)\right]. \tag{17}$$

However, because of the fact that Laplace transform of a real function must be real on the real axis, the restriction to the case $a_0 \in \mathbb{R}$ was appropriate and we have not reduced generality in (16) where parametric representation immediately allowed to sort out meaningless solutions. The same result would be obtained (17) should one impose simple conditions on derivatives: real-valuedness of $F(s)$ and $F'(s)$ for $s \in \mathbb{R}_+$ already implies that $\beta_0 \in \mathbb{R}$ and $\beta_n = 0$ for all $n \neq 0$.

This approach fails due to a fundamental issue which can pinned down to the following problem. Taking $n = 0$ in (11), we have

$$\int_0^\infty e^{-Hk} k \left(\hat{f}^2(k)\right)' dk = \lambda \int_0^\infty k \left(\hat{f}^2(k)\right)' dk,$$

which (under automatically fulfilled condition $k\hat{f}^2(k) \underset{k\to\infty}{\to} 0$) integrates by parts to yield

$$\int_0^\infty e^{-Hk} \hat{f}^2(k)\, dk - H \int_0^\infty e^{-Hk} k \hat{f}^2(k)\, dk = \lambda \int_0^\infty \hat{f}^2(k)\, dk.$$

But, on the other hand, the first term here exactly equals to the right-hand side due to (10) for $n = 0$. Therefore,

$$\int_0^\infty e^{-Hk} k \hat{f}^2(k)\, dk = 0 \quad \Rightarrow \quad \hat{f}(k) \equiv 0.$$

This nonsense stems from the assumption $k\hat{f}'(k) \in L^2(\mathbb{R})$ which means that we cannot assume more regularity than L^2-integrability of the extended by zero function in the original $D(T)$. We thus conclude with the following

Remark 2.3. An (odd) eigenfunction of T, $f(x)$, being smooth inside the interval $(-a, a)$ (as the convolution with a smooth kernel), must tend to a non-zero constant when approaching the boundary of the interval so the distributional derivative of its extension by zero outside $(-a, a)$ would result in a Dirac delta function there, and hence the eigenfunction would fail to be in the Sobolev space $W^{1,2}(\mathbb{R})$ (which is not a surprise realizing that T will not be self-adjoint when defined on such space).

2.3 Remedy attempts with linear Laplace transform

2.3.1 General strategy, Mittag-Leffler functions, etc

Even though we arrived to a wrong solution by making a wrong assumption, we don't want to give up the previous approach since it is proven to be constructive. But instead of "projecting" the integral equation (4) on $k^{2n}\hat{f}(k)$ and $k^{2n+1}\hat{f}'(k)$ as done in (10)-(11), we now would like to find a family of suitable functions $g(k)$ meaning that they are from the space PW_o^a and generate sufficient amount of information to reconstruct a single function in Laplace domain which is directly related to $\hat{f}(k)$. However, finding functions in PW_o^a may already be a challenge not to mention having any nice connection between their Laplace transforms. Ideal candidate would be a PW_o^a-function decaying on $\mathbb{R}$ faster than any polynomial (so we can generate infinitely many moments), such functions can be obtained as Fourier transform of infinitely smooth bump functions supported on $[-a,a]$, for instance, $g(k)=\mathcal{F}\left[\chi_{[-a,a]}(x)\exp\left(-\frac{1}{a^2-x^2}\right)\right]$, however it is not clear if we can obtain any explicit form of them.

A naive guess would be $g_{n,m}(k)=k^{2n+1}\left(\operatorname{sinc}\frac{2ak}{m}\right)^m$ for any $m\in\mathbb{N}_+$, $n\in\mathbb{N}_0$ such that $2n+1\leq m$ which, due to loss of integrability at infinity, for a given m, limits the number of moments we can obtain whereas, on the other hand, varying m makes it difficult to establish any neat relation between the functions.

More intelligent constructions can be made from the Mittag-Leffler function $E_{\alpha,\beta}(z):=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$ which is known to be entire and of order $1/\alpha$. Namely, taking odd part, scaling and rotating by $\pi/2$ in the $\mathbb{C}$-plane to achieve the decay at infinity on the real axis, we construct

$$g_\beta(k)=\sum_{n=0}^{\infty}\frac{(-1)^n(Ak)^{2n+1}}{\Gamma(2n+1+\beta)}, \tag{18}$$

where as before we denoted $A:=2\pi a$. We see that for $\beta=1$ this gives $g_1(k)=\sin(Ak)$ which is not L^2-summable whereas $g_2(k)=\frac{1}{Ak}[\cos(Ak)-1]$ is already suitable, so perhaps $g_\beta(k)$ for any $\beta>3/2$ is good enough. From the integral representation of the Mittag-Leffler function, we can also obtain $g_\beta(k)=\frac{A}{2\pi i}k\int_C\frac{z^{1-\beta}e^z}{z^2+(2\pi ak)^2}dz$, where the contour C starts and ends at $-\infty$ going counterclockwise encompassing all three singular points $z=0,\,\pm iAk$.

2.3.2 Bessel functions

One can also consider different generalized versions of the Mittag-Leffler function, while we notice that Bessel functions $J_n(Ak)$ might serve as suitable candidates as well: they are entire, have even/odd parity according to the index n, have properly restricted exponential growth along the imaginary axis, but they lack L^2-integrability due to insufficient asymptotic decay $J_n(Ak)\sim\sqrt{\frac{2}{\pi Ak}}\cos(Ak-n\pi/2-\pi/4)$ as $k\to+\infty$. This can be remedied by improving power of denominator which is possible to do without adding singularities due to the fact that J_n have zero at the origin of order n. Finally, let us also introduce an additional scaling degree of freedom $0<\gamma\leq A$ and hence consider function families $\frac{1}{k^{2m-1}}J_{2n}(\gamma k)$ and $\frac{1}{k^{2m}}J_{2n+1}(\gamma k)$ for $n,\,m\in\mathbb{N}_+$, $m\leq n$, which are by construction in PW_o^a but suffer from the same problem as sinc-family functions defined above: for a given order n, only limited number of (negative) moments are possible to construct[4], however there are number of recurrence formulas which might be useful to establish a link between functions in those two families with different values n

$$2J_n'(z) = J_{n-1}(z) - J_{n+1}(z), \quad \frac{2n}{z} J_n(z) = J_{n+1}(z) + J_{n-1}(z), \quad \frac{d}{dz}\left(z^n J_n(z)\right) = z^n J_{n-1}(z).$$

However, now we want to employ other relations for Bessel functions which can be obtained from their generating function, namely

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\left[\sin(2n+1)\theta\right] \tag{19}$$

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos(2n\theta), \quad \cos(z\cos\theta) = J_0(z) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(z)\cos(2n\theta)$$

$$\Rightarrow \quad \cos(z\sin\theta) - \cos(z\cos\theta) = 4\sum_{n=0}^{\infty} J_{4n+2}(z)\cos\left[(2n+1)2\theta\right] \tag{20}$$

We are going to work with the both families of the mentioned functions fixing $m = 1$. As before, for the sake of brevity, denote $H := 2\pi h$. We therefore have

$$\int_0^\infty e^{-Hk}\frac{J_{2n}(\gamma k)}{k}\hat{f}(k)\,dk = \lambda\int_0^\infty \frac{J_{2n}(\gamma k)}{k}\hat{f}(k)\,dk,$$

$$\int_0^\infty e^{-Hk}\frac{J_{2n+1}(\gamma k)}{k^2}\hat{f}(k)\,dk = \lambda\int_0^\infty \frac{J_{2n+1}(\gamma k)}{k^2}\hat{f}(k)\,dk.$$

Premultiplying this with the respective cosine or sine factors and summing over n according to the right-hand side of (20), we obtain

$$\int_0^\infty e^{-Hk}\left[\cos(k\gamma\sin\theta) - \cos(k\gamma\cos\theta)\right]\frac{\hat{f}(k)}{k}dk = \lambda\int_0^\infty \left[\cos(k\gamma\sin\theta) - \cos(k\gamma\cos\theta)\right]\frac{\hat{f}(k)}{k}dk, \tag{21}$$

$$\int_0^\infty e^{-Hk}\left[\sin(k\gamma\sin\theta) - 2J_1(k\gamma\sin\theta)\right]\frac{\hat{f}(k)}{k^2}dk = \lambda\int_0^\infty \left[\sin(k\gamma\sin\theta) - 2J_1(k\gamma\sin\theta)\right]\frac{\hat{f}(k)}{k^2}dk. \tag{22}$$

We are going to proceed with (21) denoting $F(s) := \mathcal{L}\left[\hat{f}(k)/k\right](s)$. Then, we are facing a problem of finding analytic in $\mathrm{Re}s > 0$ function satisfying the functional equation

$$\begin{aligned} F(H + i\gamma\sin\theta) + F(H - i\gamma\sin\theta) - F(H + i\gamma\cos\theta) - F(H - i\gamma\cos\theta) &= \\ \lambda\left[F(i\gamma\sin\theta) + F(-i\gamma\sin\theta) - F(i\gamma\cos\theta) - F(-i\gamma\cos\theta)\right]. \end{aligned} \tag{23}$$

We note that the substitution $\theta \to \theta + \pi/2$ does not change the equation so the solution must be $\frac{\pi}{2}$-periodic in

[4] Again the same problem is encountered when one tries to work with family of spherical Bessel functions: j_{2n-1}, $n \in \mathbb{N}_+$.

θ and singular when either $\theta = \pi/4$ or $\gamma = 0$ as the equation degenerates. In other words, instead of matching discrete moments as in the previous approach, the left- and right-hand sides of (23) must match for continuous set of parameters $\theta \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $\gamma \in (0, A]$.

Moreover, since $F(s)$ is purely imaginary on the real line, we have $F(H - i\gamma\sin\theta) = F\left(\overline{H + i\gamma\sin\theta}\right) = -\overline{F(H + i\gamma\sin\theta)}$, so $F(H + i\gamma\sin\theta) + F(H - i\gamma\sin\theta) = 2\mathrm{Im}F(H + i\gamma\sin\theta)$ and (23) becomes

$$v(H, \gamma\sin\theta) - v(H, \gamma\cos\theta) = \lambda\left[v(0, \gamma\sin\theta) - v(0, \gamma\cos\theta)\right], \tag{24}$$

where $v(x, y) := \mathrm{Im}F(x + iy)$. Since $F(s)$ is analytic in $\mathrm{Re}s > 0$, we have $\Delta v = 0$ in the right half-plane $\{(x, y) : x > 0\} \subset \mathbb{R}^2$.

Denote $g(y) := v(0, y)$, $y \in \mathbb{R}$, then $v(x, y) = (P_x \star g)(x, y)$ and (24) reads

$$\int_{-\infty}^{\infty} \frac{g(y)}{(y - \gamma\sin\theta)^2 + H^2} dy - \int_{-\infty}^{\infty} \frac{g(y)}{(y - \gamma\cos\theta)^2 + H^2} dy = \frac{\pi}{H}\lambda\left[g(\gamma\sin\theta) - g(\gamma\cos\theta)\right],$$

that is

$$\int_{-\infty}^{\infty} \frac{g(y + \gamma\sin\theta) - g(y + \gamma\cos\theta)}{y^2 + H^2} dy = \frac{\pi}{H}\lambda\left[g(\gamma\sin\theta) - g(\gamma\cos\theta)\right], \quad \gamma \in [0, A], \theta \in \left[0, \frac{\pi}{2}\right]. \tag{25}$$

We now note the connection with the original eigenfunctions f:

$$g(y) = v(0, y) = \mathrm{Im}F(iy) = \int_0^{\infty} \cos(yk) \frac{\hat{f}(k)}{ik} dk$$

$$\Rightarrow \quad g'(y) = -\int_0^{\infty} \sin(yk) \frac{\hat{f}(k)}{i} dk = -\frac{1}{2}\mathrm{Im}\mathcal{F}^{-1}\left[\hat{f}/i\right](y/2\pi) = \frac{1}{2} f(y/2\pi).$$

In particular, due to the support of f, this implies that $g(y) \equiv C$ for $|y| > A$ and, by continuity (as a result of integration) and parity, it follows that $C = g(A) = g(-A)$.

Therefore, (25) becomes

$$\left[\frac{H}{\pi}\int_{-A}^{A} \frac{g(y)}{H^2 + (y - \gamma\cos\theta)^2} dy - \lambda g(\gamma\cos\theta)\right] - \left[\frac{H}{\pi}\int_{-A}^{A} \frac{g(y)}{H^2 + (y - \gamma\sin\theta)^2} dy - \lambda g(\gamma\sin\theta)\right] = R(\gamma, \theta), \tag{26}$$

where

$$\begin{aligned} R(\gamma, \theta) : &= 2C + \frac{C}{\pi}\Bigg[\arctan\left(\frac{\gamma}{H}\sin\theta - \frac{A}{H}\right) - \arctan\left(\frac{\gamma}{H}\sin\theta + \frac{A}{H}\right) \\ &\qquad - \arctan\left(\frac{\gamma}{H}\cos\theta - \frac{A}{H}\right) + \arctan\left(\frac{\gamma}{H}\cos\theta + \frac{A}{H}\right)\Bigg]. \end{aligned}$$

By rescaling and trigonometric transformations, (26) can also be rewritten as the equation in $x \in [0, a]$ involving the original quantities h and a:

$$\left[\frac{h}{\pi}\int_{-a}^{a}\frac{g\left(2\pi\tilde{x}\right)}{\left(\tilde{x}-x\cos\theta\right)^{2}+h^{2}}d\tilde{x}-\lambda g\left(2\pi x\cos\theta\right)\right]-\left[\frac{h}{\pi}\int_{-a}^{a}\frac{g\left(2\pi\tilde{x}\right)}{\left(\tilde{x}-x\sin\theta\right)^{2}+h^{2}}d\tilde{x}-\lambda g\left(2\pi x\sin\theta\right)\right]=R_{0}\left(x,\theta\right)$$

with

$$\begin{aligned}R_{0}\left(x,\theta\right)/C &:= 2+\frac{1}{\pi}\arctan\left(\frac{2ax^{2}\left(\cos^{2}\theta-\sin^{2}\theta\right)}{4a^{2}h+h^{3}\left(1+\frac{a^{2}-x^{2}\sin^{2}\theta}{h^{2}}\right)\left(1+\frac{a^{2}-x^{2}\cos^{2}\theta}{h^{2}}\right)}\right)\\ &= 2+\frac{1}{\pi}\arctan\left(\frac{2ax^{2}\cos 2\theta}{4a^{2}h+h^{3}\left[\left(1+\frac{2a^{2}-x^{2}}{2h^{2}}\right)^{2}-\frac{x^{4}\cos^{2}2\theta}{4h^{4}}\right]}\right),\end{aligned}$$

moreover, by parity of g (or by noting that the change of variable $\theta\to\pi+\theta$, $x\to-x$ leaves the equation invariant), the range of validity of this equation extends to $x\in[-a,a]$. In other words, the original eigenvalue problem (1) reformulates into finding odd functions $w\left(x\right):=g\left(2\pi x\right)\in L^{2}\left(-a,a\right)$ such that

$$T\left[w\right]\left(x\cos\theta\right)-T\left[w\right]\left(x\sin\theta\right)=\lambda\left[w\left(x\cos\theta\right)-w\left(x\sin\theta\right)\right]+R_{0}\left(x,\theta\right).$$

Since $R_{0}\left(x,\theta\right)$ is a given term (up to a normalization constant), it can be computed for any value θ. In particular, when θ is close to $\frac{\pi}{4}$, all the differences entering the equation are small, but must be comparable in some sense. Perhaps, at this point, tools like Stieltjes inversion formula or secondary measure may be useful after an appropriate change of variable.

2.3.3 Power series approach after "regularization"

We recall the discussion that the bump functions are ideal candidates to project the equation since they act as regularizing multipliers allowing us to work with infinitely many moments exactly as it was done before when the nonlinear transformation was considered. In this subsection, we would like to pursue this approach but now paying more respect to functional spaces.

For the sake of determinicity, put

$$g_{e}\left(k\right):=\mathcal{F}\left[\chi_{[-a,a]}\left(x\right)\exp\left(-\frac{1}{a^{2}-x^{2}}\right)\right]\left(k\right),\tag{27}$$

however, we are not going to use the specific form of this function rather than just general properties of a Fourier transformed infinitely smooth even bump function. Namely, if we denote $g_{o}\left(k\right):=g_{e}'\left(k\right)$, then the functions $g_{n}\left(k\right):=k^{2n+1}g_{e}\left(k\right)$ and $\tilde{g}_{n}\left(k\right):=k^{2n}g_{o}\left(k\right)$, $n\in\mathbb{N}_{0}$ are all in the space PW_{o}^{a}. Let us introduce the following "regularized" Laplace transforms

$$F(s) := \mathcal{L}\left[\hat{f}(k)\, g_e(k)\right](s), \quad G(s) := \mathcal{L}\left[\hat{f}(k)\, g_o(k)\right](s), \quad Q(s) := \mathcal{L}\left[\hat{f}'(k)\, g_e(k)\right](s).$$

Because of vanishing at the origin of $\hat{f}(k)\, g_e(k)$, we have the connection formulas

$$G(s) = sF(s) - Q(s) \quad \Rightarrow \quad G^{(2n)}(s) = sF^{(2n)}(s) + 2nF^{(2n-1)}(s) - Q^{(2n)}(s), \quad n \in \mathbb{N}_+. \tag{28}$$

Analogously to (10)-(11), we have

$$\int_0^\infty e^{-H\tilde{k}} \hat{f}\left(\tilde{k}\right) g_e\left(\tilde{k}\right) \tilde{k}^{2n+1} d\tilde{k} = \lambda \int_0^\infty \hat{f}(k)\, g_e(k)\, k^{2n+1} dk, \tag{29}$$

$$\int_0^\infty e^{-H\tilde{k}} \hat{f}\left(\tilde{k}\right) g_o\left(\tilde{k}\right) \tilde{k}^{2n} d\tilde{k} = \lambda \int_0^\infty \hat{f}(k)\, g_o(k)\, k^{2n} dk. \tag{30}$$

These give

$$\begin{cases} F^{(2n+1)}(H) & = \lambda F^{(2n+1)}(0), \\ HF^{(2n)}(H) + 2nF^{(2n-1)}(H) - Q^{(2n)}(H) & = \lambda\left(2nF^{(2n-1)}(0) - Q^{(2n)}(0)\right), \end{cases}$$

$$\Rightarrow \quad \begin{cases} F^{(2n+1)}(H) & = \lambda F^{(2n+1)}(0), \\ HF^{(2n)}(H) - Q^{(2n)}(H) & = -\lambda Q^{(2n)}(0), \end{cases} \quad n \in \mathbb{N}_0. \tag{31}$$

$F(s)$ is analytic in the right half-plane $\mathrm{Re}s > 0$, so

$$F(s) = \sum_{m=0}^\infty F^{(m)}(H) \frac{(s-H)^m}{m!} \quad \Rightarrow \quad F^{(l)}(s) = \sum_{m=l}^\infty F^{(m)}(H) \frac{(s-H)^{m-l}}{(m-l)!}.$$

Denoting $c_m := F^{(m)}(H)$, due to the validity of the expansion in the disk $|s-H| < H$ up to the origin, the system (31) reads

$$c_{2n} = \frac{1}{H}\left(Q^{(2n)}(H) - \lambda Q^{(2n)}(0)\right),$$

$$c_{2n+1} = \lambda \sum_{m=2n+1}^\infty c_m \frac{(-H)^{m-2n-1}}{(m-2n-1)!}.$$

If we relabel the coefficients $a_n := c_{2n+1}$, then, for $n \in \mathbb{N}_0$,

$$\frac{1}{\lambda} a_n - \sum_{k=n}^\infty a_k \frac{H^{2(k-n)+1}}{[2(k-n)]!} = b_n,$$

$$b_n := -\sum_{k=n+1}^\infty \left(Q^{(2k)}(H) - \lambda Q^{(2k)}(0)\right) \frac{H^{2(k-n)-1}}{[2(k-n)-1]!},$$

which are infinite systems with solvable matrices (see [1, 2, 3]). Nevertheless, such a decoupling does not seem to be useful since the coefficients a_n and b_n are certainly not independent, as $F(s)$ and $Q(s)$ are connected with each other by means of $\hat{f}$ according to (28). Establishing this connection requires if not explicit form but at least more precise characterization of (27) or similar functions. Alternatively, we can try to fill this gap working out the projections of (8).

3 Meanwhile in the original domain...

3.1 Use of the Green's identity

There is an observation about the Fourier transform of the solution to (1) which can be obtained without transferring the problem to the Fourier domain. Perhaps the same approach may yield other useful results.

Lemma 3.1. *Regardless of parity of f, we have $\hat{f}(0) := \mathcal{F}[f](0) = 0$.*

Proof. Suppose that f solves (1) for some fixed $\lambda \neq 0$. For $x \in [-a, a]$, $y \in [0, h]$, consider

$$u(x, y) := P_y \star \left(\chi_{[-a,a]} f\right) = \frac{1}{\pi} \int_{-a}^{a} \frac{f(\tilde{x})\, y}{(x - \tilde{x})^2 + y^2} d\tilde{x}. \tag{32}$$

By the property of the Poisson kernel, this defines a harmonic function in the rectangle $\mathcal{R} := [-a, a] \times [0, h]$ such that $\lim_{y \to 0^+} u(x, y) = \chi_{[-a,a]} f(x)$ pointwise. In other words, $u(x, y)$ solves the following boundary value problem

$$\begin{cases} \Delta u(x, y) = 0, & (x, y) \in \mathcal{R}, \\ u(-a, y) = \frac{1}{\pi} \int_{-a}^{a} \dfrac{f(x)\, y}{(a + x)^2 + y^2} dx, & \\ u(a, y) = \frac{1}{\pi} \int_{-a}^{a} \dfrac{f(x)\, y}{(a - x)^2 + y^2} dx, & y \in [0, h], \\ u(x, 0) = f(x), & \\ u(x, h) = \lambda f(x), & x \in [-a, a]. \end{cases}$$

□

Proof. Let us now apply the Green's identity

$$\int_{\mathcal{R}} (u \Delta v + \nabla u \cdot \nabla v)\, dx dy = \oint_{\partial \mathcal{R}} u \nabla v \cdot \boldsymbol{dS}$$

choosing $v(x, y) = y$ (so $\Delta v = 0$ and $\partial_x v \equiv 0$, hence contribution from the vertical boundaries in the right-hand side is zero) and $u\left(x,y\right)$ as in (32). Then, using the boundary conditions, we have

$$\int_{-a}^{a}\int_{0}^{h}\partial_{y}u dx dy=-\left(1+\lambda\right)\int_{-a}^{a}f\left(x\right)dx \quad \Rightarrow \quad \left(\lambda-1\right)\int_{-a}^{a}f\left(x\right)dx=-\left(1+\lambda\right)\int_{-a}^{a}f\left(x\right)dx,$$

and therefore, since $\lambda\neq 0$, we conclude that $\int_{-a}^{a}f\left(x\right)dx=0=\int_{-\infty}^{\infty}f\left(x\right)dx=\hat{f}\left(0\right)$. □

3.2 Reformulation with the Plemelj-Sokhotski formulas

By partial fraction expansion, the equation (1) can be rewritten as

$$\frac{i}{2\pi}\left(\int_{-a}^{a}\frac{f\left(\tilde{x}\right)}{\tilde{x}-x+ih}d\tilde{x}-\int_{-a}^{a}\frac{f\left(\tilde{x}\right)}{\tilde{x}-x-ih}d\tilde{x}\right)=\lambda f\left(x\right),\quad x\in\left[-a,a\right]. \tag{33}$$

Now consider $F\left(z\right):=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f\left(\tilde{x}\right)}{\tilde{x}-z}d\tilde{x}=\frac{1}{2\pi i}\int_{-a}^{a}\frac{f\left(\tilde{x}\right)}{\tilde{x}-z}d\tilde{x}$. It defines the functions $F_{+}\left(z\right)$ and $F_{-}\left(z\right)$ analytic in the horizontal half-planes $\mathrm{Im}z>0$ and $\mathrm{Im}z<0$, respectively. The Plemelj-Sokhotski formulas provide information of boundary behavior of these functions:

$$\begin{cases} F_{+}\left(x\right) & =\frac{1}{2}f\left(x\right)+F\left(x\right), \\ F_{-}\left(x\right) & =-\frac{1}{2}f\left(x\right)+F\left(x\right), \end{cases} \quad \Rightarrow \quad f\left(x\right)=F_{+}\left(x\right)-F_{-}\left(x\right),\quad x\in\mathbb{R}.$$

Then (33) reads

$$F_{+}\left(x+ih\right)-F_{-}\left(x-ih\right)=\lambda\left(F_{+}\left(x\right)-F_{-}\left(x\right)\right),\quad x\in\left[-a,a\right], \tag{34}$$

whereas $F_{+}\left(x\right)=F_{-}\left(x\right)$ for $x\in\mathbb{R}\backslash\left[-a,a\right]$. In other words, we are after finding an analytic function in the cut complex plane $\mathbb{C}\backslash\left[-a,a\right]$ with (34) characterizing a jump across the cut. At least for small h, this formulation of the problem reduces to some perhaps already available in literature results and thus makes our problem asymptotically solvable.

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