Kelvin transform of $b_{x_3}(x_1, x_2, h)$ on $\mathbb{S}$

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1 Reflexion and Kelvin transform

Let R be the reflexion map from the half-space $\mathbb{R}^3_h = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3\,,\ x_3 \geq h\}$ to the unit ball $\mathbb{B}$. With $S = (0, 0, -1)$ the south pole of $\mathbb{B}$, it is given by:

$$R(x) = 2\,(1+h)\,\frac{x - S}{|x - S|^2} + S$$

$$= \begin{pmatrix} \dfrac{2\,(1+h)\,x_1}{x_1^2 + x_2^2 + (x_3+1)^2} \\[2ex] \dfrac{2\,(1+h)\,x_2}{x_1^2 + x_2^2 + (x_3+1)^2} \\[2ex] \dfrac{2\,(1+h)\,(x_3+1)}{x_1^2 + x_2^2 + (x_3+1)^2} - 1 \end{pmatrix} = \begin{pmatrix} \dfrac{2\,(1+h)\,x_1}{x_1^2 + x_2^2 + (x_3+1)^2} \\[2ex] \dfrac{2\,(1+h)\,x_2}{x_1^2 + x_2^2 + (x_3+1)^2} \\[2ex] -\dfrac{x_1^2 + x_2^2 + (x_3+1)(x_3+1-2\,(1+h))}{x_1^2 + x_2^2 + (x_3+1)^2} \end{pmatrix}. \tag{1}$$

The reflexion R maps the (boundary) plane $\mathbb{R}^2 \times \{h\} = \{x = (x_1, x_2, h)\}$ ($x_3 = h$) of $\mathbb{R}^3_h$ to the unit sphere $\mathbb{S} = \partial\mathbb{B}$: $R(x_1, x_2, h) \in \mathbb{S}$. It also maps ∞ (in any of the x_1, x_2 or x_3 directions) on the south pole S, as can be easily checked from the above definition. Conversely, because R^2 is equal to the identity, then $R^{-1} = R$ maps $\xi \in \mathbb{B}$ into $x = R(\xi) \in \mathbb{R}^3_h$, and $\mathbb{S}$ into the plane $x_3 = h$. For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{B}$:

$$R(\xi) = x = \begin{pmatrix} \dfrac{2\,(1+h)\,\xi_1}{\xi_1^2 + \xi_2^2 + (\xi_3+1)^2} \\[2ex] \dfrac{2\,(1+h)\,\xi_2}{\xi_1^2 + \xi_2^2 + (\xi_3+1)^2} \\[2ex] \dfrac{2\,(1+h)\,(\xi_3+1)}{\xi_1^2 + \xi_2^2 + (\xi_3+1)^2} - 1 \end{pmatrix} \in \mathbb{R}^3_h\,. \tag{2}$$

The associated Kelvin transform K^* applies to a function f defined in $\mathbb{R}^3_h$ and is a function $K^*[f]$ defined in $\mathbb{B}$ by:

$$K^*[f](\xi) = 2^{1/2} \frac{1}{|\xi - S|} f(R(\xi)) \,. \tag{3}$$

The Kelvin transform preserves harmonicity and is its own inverse: $K^*[K^*[f]] = f$ [1, Ch. 7].

2 Kelvin transform of the normal magnetic field

Let $Q \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ denote the support of the $\mathbb{R}^3$–valued magnetization $M = (M_1, M_2, M_3)$. Recall the expression of the measured x_3 (normal) component of the magnetic field b_{x_3} at $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_3 > 0$:

$$b_{x_3}(x) = -\iint_Q \frac{dt_1\, dt_2}{|x-t|^5} \times$$
$$\left[3\,(x_1 - t_1)\, x_3\, M_1(t) + 3\,(x_2 - t_2)\, x_3\, M_2(t) + \left(3\, x_3^2 - |x-t|^2\right)\, M_3(t)\right] \,,$$

where we put $t = (t_1, t_2, 0)$. In particular, at points $x = (x_1, x_2, h)$ in the measurement plane $x_3 = h$, we have:

$$b_{x_3}(x_1, x_2, h) = -\iint_Q \frac{dt_1\, dt_2}{|x-t|^5} \times \tag{4}$$
$$\left[3\,(x_1 - t_1)\, h\, M_1(t) + 3\,(x_2 - t_2)\, h\, M_2(t) + \left(3\, h^2 - |x-t|^2\right)\, M_3(t)\right]$$
$$= -\iint_Q \frac{dt_1\, dt_2}{|(x_1-t_1)^2 + (x_2-t_2)^2 + h^2|^{5/2}} \times$$
$$\left[3\,(x_1 - t_1)\, h\, M_1(t) + 3\,(x_2 - t_2)\, h\, M_2(t) - \left((x_1-t_1)^2 + (x_2-t_2)^2 - 2\, h^2\right)\, M_3(t)\right] \,.$$

Lemma 1. *At $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{S}$, we have*

$$K^*\left[b_{x_3}\right](\xi) = -(1+\xi_3) \iint_Q \frac{d\,t_1\; d\,t_2}{\left[(1+\xi_3)\,(t_1^2 + t_2^2 + 1) - 2\,(1+h)\,(t_1\,\xi_1 + t_2\,\xi_2 + \xi_3 - h)\right]^{5/2}} \times$$
$$[3\, h\, \left((1+h)\,\xi_1 - (1+\xi_3)\, t_1\right)\, M_1(t) + 3\, h\, \left((1+h)\,\xi_2 - (1+\xi_3) t_2\right)\, M_2(t)$$
$$+ \left((1+\xi_3)\,(t_1^2 + t_2^2 + 1 - 3\, h^2) - 2\,(1+h)\,(t_1\,\xi_1 + t_2\,\xi_2 + \xi_3 - h)\right)\, M_3(t)] \,.$$

Proof. Let $R(\xi)_i$, $i = 1, 2, 3$ denote the components of $R(\xi)$. Rewriting the expression (2) of $R(\xi)$ for $\xi \in \mathbb{S}$, using that $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$ there, we obtain:

$$R(\xi)_1 = \frac{(1+h)\,\xi_1}{1+\xi_3}\,,\ R(\xi)_2 = \frac{(1+h)\,\xi_2}{1+\xi_3}\,,\ R(\xi)_3 = h\,.$$

From the expression (4) of b_{x_3} and the definition (3) of the Kelvin transform, we get that, at $\xi = (\xi_1, \xi_2, \xi_3) = R(x_1, x_2, h) \in \mathbb{S}$,

$$K^*\left[b_{x_3}\right](\xi) = \frac{2^{1/2}\, b_{x_3}(R(\xi))}{|\xi - S|} = -\frac{1}{(1+\xi_3)^{1/2}} \iint_Q \frac{dt_1\, dt_2}{|R(\xi) - t|^5} \times$$
$$\left[3\,h \left(\frac{(1+h)\,\xi_1}{1+\xi_3} - t_1\right) M_1(t) + 3\,h \left(\frac{(1+h)\,\xi_2}{1+\xi_3} - t_2\right) M_2(t) \right.$$
$$\left. + \left(3\,h^2 - |R(\xi) - t|^2\right) M_3(t)\right]\,.$$

For the expression involved in the above denominator of $K^*\left[b_{x_3}\right]$, we get for $t = (t_1, t_2, 0)$:

$$|R(\xi) - t|^2 = |R(\xi)|^2 - 2\langle t, R(\xi)\rangle + |t|^2$$
$$= \frac{1}{1+\xi_3}\,\left[(1+\xi_3)\,(t_1^2 + t_2^2 + 1) - 2\,(1+h)\,(t_1\,\xi_1 + t_2\,\xi_2 + \xi_3 - h)\right]$$

Alternatively, note that $|R(\xi) - t|^2 = (R(\xi)_1 - t_1)^2 + (R(\xi)_2 - t_2)^2 + h^2$. Next, for the factor of M_3 in the numerator of $K^*\left[b_{x_3}\right]$, we use that, for $\xi \in \mathbb{S}$:

$$|R(\xi) - t|^2 - 3\,h^2$$
$$= \frac{1}{1+\xi_3}\,\left[(1+\xi_3)\,(t_1^2 + t_2^2 + 1 - 3\,h^2) - 2\,(1+h)\,(t_1\,\xi_1 + t_2\,\xi_2 + \xi_3 - h)\right]\,.$$

□

An expression of $K^*\left[b_{x_3}\right]$ in terms of $\tau = (\tau_1, \tau_2, \tau_3) = R(t_1, t_2, 0) \in R(Q) \subset S_0$, the sphere of center $(0, 0, h)$ and radius $1+h$, is given by the next result. Let $d\,\sigma_0(\tau)$ be the (non normalized) Lebesgue measure on S_0 (note that S_0 is contained in $\mathbb{R}^3 \setminus \mathbb{B}$ has also S as its south pole) and $N_i(\tau) = M_i(R(\tau)) = M_i(t)$, for $i = 1, 2, 3$.

Lemma 2. *At $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{S}$, we have*

$$K^*\left[b_{x_3}\right](\xi) = -(1+\xi_3) \iint_{R(Q)} \frac{d\sigma_0(\tau)}{(1+\tau_3)^{1/2}\,\left[(1 + h\,(1+\tau_3) - (\tau_1\,\xi_1 + \tau_2\,\xi_2 + \tau_3\,\xi_3)\right]^{5/2}} \times$$
$$\left[3\,h\,\left((1+h)\,(1+\tau_3)\,\xi_1 - (1+\xi_3)\,\tau_1\right)\,N_1(\tau) + 3\,h\,\left((1+h)\,(1+\tau_3)\,\xi_2 - (1+\xi_3)\tau_2\right)\,N_2(\tau)\right.$$
$$\left. + \left(3\,h^2\,(1+\xi_3)\,(1+\tau_3)\ - 2\,(1+h)\,(1 + h(1+\tau_3) - (\tau_1\,\xi_1 + \tau_2\,\xi_2 + \tau_3\,\xi_3))\right)\,N_3(\tau)\right]\,.$$

Proof. First, let us check that (see also [1, Ch. 7]):

$$dt_1\, dt_2 = \frac{d\,\sigma_0(\tau)}{(1+\tau_3)^2}\,.$$

Indeed, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3_h$, let $\partial_{x_i} R(x)$ denote the partial derivative of R w.r.t. x_i. We compute from the definition (1) of R that at such x:

$$\partial_{x_1} R(x) = \left[\frac{2\,(1+h)\,\left(-x_1^2 + x_2^2 + (x_3+1)^2\right)}{x_1^2 + x_2^2 + (x_3+1)^2} \,;\, \frac{-4\,(1+h)\,x_1\,x_2}{x_1^2+x_2^2+(x_3+1)^2} \,;\, \frac{-4\,(1+h)\,x_1\,(x_3+1)}{x_1^2+x_2^2+(x_3+1)^2} \right],$$

and

$$\partial_{x_2} R(x) = \left[\frac{-4\,(1+h)\,x_1\,x_2}{x_1^2+x_2^2+(x_3+1)^2} \,;\, \frac{2\,(1+h)\,\left(x_1^2 - x_2^2 + (x_3+1)^2\right)}{x_1^2 + x_2^2 + (x_3+1)^2} \,;\, \frac{-4\,(1+h)\,x_2\,(x_3+1)}{x_1^2+x_2^2+(x_3+1)^2} \right].$$

If $\wedge$ is the curl between to vectors in $\mathbb{R}^3$, we thus get (after computations) at $x = (x_1, x_2, x_3) \in \mathbb{R}^3_h$:

$$|\partial_{x_1} R(x) \wedge \partial_{x_2} R(x)| = \frac{4\,(1+h)^2}{\left(x_1^2 + x_2^2 + (x_3+1)^2\right)^2}\,.$$

In particular, at $x = t$, because $x_1 = t_1, x_2 = t_2, x_3 = 0$, we obtain (or directly from $\partial_{x_i} R(t)$):

$$|\partial_{x_1} R(t) \wedge \partial_{x_2} R(t)| = \frac{4\,(1+h)^2}{\left(t_1^2 + t_2^2 + 1\right)^2} = (\tau_3 + 1)^2\,,$$

at $\tau = (\tau_1, \tau_2, \tau_3) = R(t) = R(t_1, t_2, 0) \in R(Q) \subset S_0$, using the correspondance:

$$\tau_3 + 1 = \frac{2\,(1+h)}{t_1^2 + t_2^2 + 1}\,.$$

Together with the relation $d\sigma_0(\tau) = |\partial_{x_1} R(t) \wedge \partial_{x_2} R(t)|\; dt_1\, dt_2$, this establishes the above claim. Besides, for $\tau \in S_0$ we also have:

$$\tau_1^2 + \tau_2^2 + (\tau_3 - h)^2 = (1+h)^2 \Rightarrow \tau_1^2 + \tau_2^2 + (\tau_3+1)^2 = 2\,(1+h) \Rightarrow t_1 = \frac{\tau_1}{1+\tau_3}\,,\ t_2 = \frac{\tau_2}{1+\tau_3}\,.$$

Lemma 1 thus implies that

$$K^*\left[b_{x_3}\right](\xi) = -2^{5/2}\,(1+\xi_3) \iint_{R(Q)} \frac{d\,\sigma_0(\tau)}{(1+\tau_3)^{1/2}\,|\tau - \xi|^5} \times$$
$$\big[3\,h\,\left((1+h)\,(1+\tau_3)\,\xi_1 - (1+\xi_3)\,\tau_1\right)\,N_1(\tau) + 3\,h\,\left((1+h)\,(1+\tau_3)\,\xi_2 - (1+\xi_3)\tau_2\right)\,N_2(\tau)$$
$$+ \left(3\,h^2\,(1+\xi_3)\,(1+\tau_3)\, - (1+h)\,|\tau - \xi|^2\right)\,N_3(\tau)\big]\,.$$

Finally, we use that, for $\xi \in \mathbb{S}$ and $\tau \in S_0$:

$$|\tau - \xi|^2 = |\tau|^2 + 1 - 2\langle \tau, \xi \rangle = 2\,\left[1 + h\,(1+\tau_3) - (\tau_1\,\xi_1 + \tau_2\,\xi_2 + \tau_3\,\xi_3)\right]\,.$$

$\square$

References

[1] Axler Bourdon Ramey, Harmonic Function Theory, Springer-Verlag, 2001.