Inverse magnetization problem in 2D

Laurent Baratchart, Juliette Leblond, Dmitry Ponomarev

1 Formulation of the problem

It is of physical interest to recover magnetization $\mathbf{M}$ of a planar sample from the measurements of normal components of magnetic-flux density available above and below it [3, 4]. We consider a toy version of this is a problem on a plane where the sample is placed on a line.

Introducing scalar magnetic potential Φ such that the magnetic field is $\mathbf{H} = -\nabla\Phi$ (which is possible due to $\nabla \times \mathbf{H} = 0$). With the choice of system of units where the vacuum permeability constant is normalized to one, the magnetic flux density then reads $\mathbf{B} = -\nabla\Phi + \mathbf{M}$. Due to the Maxwell equation $\nabla \cdot \mathbf{B} = 0$, we thus arrive at

$$\Delta\Phi = \nabla \cdot \mathbf{M}. \tag{1.1}$$

Denote the radius vector $\mathbf{r} = (x, y)$, where the coordinate system is chosen such that the sample is in the middle of the line $y = 0$, that is,

$$\mathbf{M}(\mathbf{r}) = \mathbf{m}(x) \otimes \delta_0(y), \quad \mathbf{m} = (m_x, m_y),$$

where δ_0 is the Dirac delta function supported at 0, the notation that should not be confused with further instances of use of δ symbol, say δ_F denoting variation of a function F.

The Poisson equation (1.1) on the plane has solution in terms of the logarithmic potential $\frac{1}{2\pi}\log r$ (see, for instance, [6]), where $r = |\mathbf{r}|$,

$$\Phi(\mathbf{r}) = \frac{1}{2\pi}\int\int_{\mathbb{R}^2} \nabla \cdot \mathbf{M}(\mathbf{r}')\log|\mathbf{r} - \mathbf{r}'|\, d^2\mathbf{r}',$$

which upon integration by parts (under the formal decay assumption $\lim_{r\to\infty}|\mathbf{M}(\mathbf{r})|\log r = 0$) becomes

$$\Phi(\mathbf{r}) = \frac{1}{2\pi}\int\int_{\mathbb{R}^2} \mathbf{M}(\mathbf{r}') \cdot \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} d^2\mathbf{r}' = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{m_x(x')(x - x')}{(x - x')^2 + y^2} dx' + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{m_y(x')\, y}{(x - x')^2 + y^2} dx'.$$

Introducing the Poisson kernel $P_y(x) := P(x, y) = \dfrac{1}{\pi}\dfrac{y}{x^2 + y^2}$ and the conjugate Poisson kernel $Q_y(x) := Q(x, y) = \dfrac{1}{\pi}\dfrac{x}{x^2 + y^2}$, we can write

$$\Phi(x, y) = \begin{cases} \frac{1}{2}(Q_y \star m_x + P_y \star m_y), & y > 0, \\ \frac{1}{2}(Q_y \star m_x - P_{-y} \star m_y), & y < 0, \end{cases} \tag{1.2}$$

employing $\star$ as notation for convolution in x variable.

Next, since $P_y(x)$ is an approximate identity for $y>0$ (e.g. [7]), we realize that

$$\lim_{y\to 0^\pm}\Phi(x,y)=\frac{1}{2}\mathcal{H}[m_x]\pm\frac{1}{2}m_y,$$

where $\mathcal{H}$ denotes the Hilbert transform defined as $\mathcal{H}[f]:=\lim_{y\to 0}(Q_y\star f)$.

By uniqueness of harmonic extension upward/downward from the boundary, we can then rewrite (1.2) as

$$\Phi(x,y)=\frac{1}{2}P_{|y|}\star(\mathcal{H}[m_x]\pm m_y),\quad y\gtrless 0, \tag{1.3}$$

and hence the normal component of magnetic flux density is given by

$$B_y(x,y)=\begin{cases}-\frac{1}{2}\partial_y P_y\star(\mathcal{H}[m_x]+m_y) & , \quad y>0,\\ -\frac{1}{2}\partial_y P_{-y}\star(\mathcal{H}[m_x]-m_y) & , \quad y<0,\end{cases} \tag{1.4}$$

where notation $\partial_y P_{-y}$ should be understood in a sense that $\partial_y P_a:=(\partial_y P_y)|_{y=a}=\frac{1}{\pi}\frac{x^2-a^2}{(x^2+a^2)^2}$, the convention we adopt and will use further.

Advantage of such representation is that we can use the property of Poisson kernels $P_{y_1}\star\partial_y P_{y_2}=\partial_y P_{y_1+y_2}$, which is similar to the known semigroup property $P_{y_1}\star P_{y_2}=P_{y_1+y_2}$ and is evident on the Fourier transform side, given that $\mathcal{F}[P_y](\kappa)=e^{-2\pi y|\kappa|}$, and combine measurements of magnetic flux density on both sides of the sample to separate unknown magnetization components m_x and m_y. Namely, suppose $\mathbf{m}$ is supported on a set $S\subseteq\mathbb{R}$ and $B_y(x,h_1)$, $B_y(x,h_2)$ are known for $x\in I\subseteq\mathbb{R}$, where $h_1>0$, $h_2<0$, $|h_2|\geq h_1$.

Let $S=(-a,a)$, $I=(-b,b)$ for a, $b>0$. For arbitrary $0<h_0\leq h_1$, $\gamma\in\mathbb{R}$, consider

$$u_{\gamma,h_0}(x,y):=P_{\gamma-h_1-h_2}\star B_y(x,y+h_1-h_0)-P_\gamma\star B_y(x,-y+h_2+h_0), \tag{1.5}$$

$$v_{\gamma,h_0}(x,y):=P_{\gamma-h_1-h_2}\star B_y(x,y+h_1-h_0)+P_\gamma\star B_y(x,-y+h_2+h_0). \tag{1.6}$$

Then, if the measurement area is much wider than the sample size, that is, $a\ll b$, the convolution integrals can be truncated and the quantities

$$u_{\gamma,h_0}(x,h_0)=P_{\gamma-h_1-h_2}\star B_y(x,h_1)-P_\gamma\star B_y(x,h_2),$$

$$v_{\gamma,h_0}(x,h_0)=P_{\gamma-h_1-h_2}\star B_y(x,h_1)+P_\gamma\star B_y(x,h_2)$$

are considered to be approximately known for $x\in S$.

On the other hand, we note that in the expressions (1.5)-(1.6), the second arguments in B_y in the both terms are of constant and opposite signs when $y>0$, so due to (1.4) and the mentioned property of Poisson kernels, we obtain

$$u_{\gamma,h_0}(x,y):=-\partial_y P_{y-h_2-h_0+\gamma}\star m_y,\quad v_{\gamma,h_0}(x,y):=-\partial_y P_{y-h_2-h_0+\gamma}\star\mathcal{H}[m_x],$$

which are harmonic functions in the upper halfplane providing that $\gamma \geq h_0 + h_2$. The choice $\gamma = h_0 + h_2$ leads to

$$u(x,y) := u_{h_0}(x,y) = -\partial_y \phi(x,y), \quad v(x,y) := v_{h_0}(x,y) = -\partial_y \tilde{\psi}(x,y), \tag{1.7}$$

where ϕ, $\tilde{\psi}$ are harmonic functions in the upper half-plane such that $\phi(x,0) = m_y$, $\tilde{\psi}(x,0) = \mathcal{H}[m_x]$.

Alternatively, we may consider a more realistic physical setup corresponding to $I = S$ where, however, we shall restrict to the situation with $h_1 = -h_2$. In this case, we set $h_0 = h_1$, $\gamma = 0$ to simply work with the expressions

$$u(x,y) = B_y(x,y) - B_y(x,-y), \quad v(x,y) = B_y(x,y) + B_y(x,-y),$$

which are exactly known for $x \in S$, $y = h_0$ and, on the other hand, can be represented in terms of ϕ, $\tilde{\psi}$ precisely as in (1.7).

Introduce, for $y > 0$,

$$f(z) := \phi(x,y) + i\tilde{\phi}(x,y), \quad g(z) := \psi(x,y) + i\tilde{\psi}(x,y), \tag{1.8}$$

where $\tilde{\phi}(x,y) := Q_y \star m_y$ and $\psi(x,y) := -Q_y \star \mathcal{H}[m_x]$. Then, the expressions in (1.8) define analytic functions in the upper half-plane. Moreover, $\lim_{y\to 0^+} \phi(x,y) = m_y$, $\lim_{y\to 0^+} \psi(x,y) = m_x$, and hence the real parts of $f(z)$ and $g(z)$ on $y = 0$ inherit the magnetization support S. If S is bounded, we can construct analytic continuations through $\{(x,0) : x \in \mathbb{R}\} \backslash S$ downwards defining

$$F(z) := \begin{cases} f(z), & y > 0, \\ -\overline{f(\bar{z})}, & y < 0, \end{cases} \quad G(z) := \begin{cases} g(z), & y > 0, \\ -\overline{g(\bar{z})}, & y < 0. \end{cases} \tag{1.9}$$

Without loss of generality, we proceed further working with F alone, situation with G is absolutely the same.

Note that here and onwards we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ and abuse notation referring to x or y as a real or imaginary part of the variable $z = x + iy \in \mathbb{C}$ or a coordinate of the point $(x,y) \in \mathbb{R}^2$.

Lemma 1. *For $m_y \in L^1_{\mathbb{R}}(\mathbb{R})$ such that $\operatorname{supp} m_y \subseteq S$, $|S| < \infty$, the integral*

$$F(z) = \frac{1}{\pi i} \int_S \frac{m_y(\xi)}{\xi - z} d\xi \tag{1.10}$$

defines an analytic function on $\mathbb{C}\backslash S$ such that $\lim_{y\to 0^\pm} F(x+iy) = \pm m_y + i\mathcal{H}[m_y]$.

Proof. Consider the rectangle $V = [-x_0, x_0] \times [-y_0, y_0]$ for some $x_0 > a$ and $y_0 > 0$. Since $S \times \{0\} \subset V$, for $z \in \mathbb{C}\backslash V$, by the Cauchy formula, we have

$$F(z) = -\frac{1}{2\pi i} \int_{\partial V} \frac{F(\xi)}{\xi - z} d\xi,$$

where the rectangular contour ∂V is traversed in the counterclockwise direction.

In order to obtain the representation (1.10), we would like to take limit as $y_0 \to 0^+$. First of all, contribution from the vertical segments $\xi \in [-x_0 - iy_0, -x_0 + iy_0]$, $[x_0 - iy_0, x_0 + iy_0]$ is negligible under such limit passage because of the mean-value theorem which applies due to analyticity of F across $\{(x,y) : y = 0\} \backslash S$. Therefore, employing (1.9), since m_y is real-valued, we obtain

$$\begin{aligned} F(z) &= -\frac{1}{2\pi i}\lim_{y_0\to 0^+}\left(-\int_{-x_0}^{x_0}\frac{\overline{f(iy_0+x')}}{-iy_0+x'-z}dx' + \int_{x_0}^{-x_0}\frac{f(iy_0+x')}{iy_0+x'-z}dx'\right) = \frac{1}{2\pi i}\lim_{y_0\to 0^+}\left(\int_{-x_0}^{x_0}\frac{2\phi(x',y_0)}{x'-z}dx'\right) \\ &= \frac{1}{\pi i}\int_S \frac{m_y(\xi)}{\xi - z}d\xi, \end{aligned}$$

where the limit passage under the integral sign is justified since $m_y \in L^1(S)$ and $z \notin S$. □

Lemma 1 shows that magnetization components m_y and m_x uniquely define the functions F and G, respectively, which are analytic in $\mathbb{C}\backslash S$, and in the upper half-plane in particular. Now, because of the Cauchy-Riemann equations,

$$\partial_x\tilde{\phi} = -\partial_y\phi, \quad \partial_x\psi = \partial_y\tilde{\psi},$$

and since the quantities u, v are known on $I_0 := S \times \{h_0\}$ from the measurement data as described above, we can compute

$$W_{\tilde{\phi}}(x) := \int_{-a}^{x} u(x',h_0)\,dx', \quad W_{\psi}(x) := -\int_{-a}^{x} v(x',h_0)\,dx', \quad x \in (-a,a), \tag{1.11}$$

the quantities which, up to constants, give estimates on I_0 for $\tilde{\phi}$, ψ, respectively. Based on this, we attempt to recover analytic functions F, G, and thus magnetization components m_y, m_x, respectively, by casting the following bounded extremal problems

$$\min_{F\in\mathcal{B}_{M_f}}\left\|\mathrm{Im}F - W_{\tilde{\phi}}\right\|_{L^2(I_0)}, \qquad \min_{\substack{G\in\mathcal{B}_{M_g}\\ C_0\in\mathbb{R}}}\left\|\mathrm{Re}G - W_{\psi} + C_0\right\|_{L^2(I_0)}, \tag{1.12}$$

where

$$\mathcal{B}_M := \left\{F \in H^2_+ : \mathrm{supp}\left(\mathrm{Re}F|_{y=0}\right) \subseteq S,\ \|\mathrm{Re}F\|_{L^2(S)} \le M\right\}, \tag{1.13}$$

$$H^2_+ := \left\{F \text{ analytic for } y>0 : \sup_{y>0}\int_{\mathbb{R}}|F(x+iy)|^2\,dx < \infty\right\}, \tag{1.14}$$

M_f, M_g denote positive constants which *a priori* bound L^2-norms of m_y, m_x, accordingly, and C_0 is an integration constant from (1.11) that cannot be absorbed by the definition of $\mathcal{B}_{M_g}$.

We shortly discuss motivation to formulate such bounded extremal problems. Without loss of generality, let us focus on the first one in (1.12). We have seen in Lemma 1 that given m_y supported on S extends uniquely to an analytic function F. But an analytic function can also be uniquely represented from a subset of a line by means of the Carleman's formula for a half-plane (Ch.1, Th.5.1 in [1]). Applying this to known data on I_0 (real part is zero, imaginary part is an available function from measurements processing), we obtain an analytic function above h_0 which must coincide with restriction of the original function F produced by m_y, and hence the obtained function is, in fact, analytic all the way down to $y = 0$ where its real part must be exactly m_y. However, not every square-integrable on I_0 function is the restriction of an analytic function (see, for instance, Th.11.2 in [5]), and in practice it is never the case since measurement and numerical processing of data are necessarily prone to errors. Therefore, assuming that the available data $W_{\tilde{\phi}}$ are only $L^2(I_0)$, the minimum is never zero, and as it is approaching, we expect growth on other segments (namely, on $J_0 := (\mathbb{R}\times\{y_0\})\backslash I_0$) that would blow-up

H^2-norm in the spirit of [2]. This growth is, nevertheless, controlled by the imposed requirement in $\mathcal{B}_{M_f}$. Indeed, one can show (as a consequence of Lemma on p.122 and the top of p.127 in [8]) that $\|\mathrm{Im}F\|_{L^2(\mathbb{R})} = \|\mathrm{Re}F\|_{L^2(S)} \leq M$, and thus $\|F\|_{L^2(J_0)} \leq \|F\|_{H^2_+} := \|F\|_{L^2(\mathbb{R})} \leq 2M$.

2 Solution

Again, for the moment we focus on the first bounded extremal problem in (1.12). Omitting proof of existence and uniqueness (which hinges on demonstration that $\mathcal{B}_M$ is a closed convex subset of H^2_+), we skip to obtaining an integral equation for the magnetization component and further set up a fixed-point argument procedure for its solution.

Given $W_{\tilde{\phi}} \in L^2(I_0)$, we denote

$$\mathcal{J} := \left\|\mathrm{Im}F - W_{\tilde{\phi}}\right\|^2_{L^2(I_0)}, \tag{2.1}$$

we pursue the idea of Lagrange multipliers [9], and thus require, for $\lambda \in \mathbb{R}$,

$$\delta_{\mathcal{J}} = 2\lambda \left\langle \mathrm{Re}F, \mathrm{Re}\delta_F \right\rangle_{L^2(S)}, \quad \delta_{\mathcal{J}} = 2\left\langle \mathrm{Im}F - W_{\tilde{\phi}}, \mathrm{Im}\delta_F \right\rangle_{L^2(I_0)}, \tag{2.2}$$

where, by means of Lemma 1,

$$\delta_F = \frac{1}{\pi i}\int_S \frac{\delta_{m_y}(\xi)}{\xi - z} d\xi \quad \Rightarrow \quad \mathrm{Re}\delta_F = P_y \star \delta_{m_y}, \quad \mathrm{Im}\delta_F = Q_y \star \delta_{m_y}$$

for arbitrary $\delta_{m_y} \in L^2(S)$.

Therefore, (2.2) implies

$$\left\langle Q_{h_0} \star m_y - W_{\tilde{\phi}},\, Q_{h_0} \star \delta_{m_y} \right\rangle_{L^2(S)} = \lambda \left\langle m_y, \delta_{m_y} \right\rangle_{L^2(S)},$$

that is,

$$\frac{1}{\pi}\int_S \left(\frac{1}{\pi}\int_S \frac{x - x'}{(x - x')^2 + h_0^2} m_y(x')\, dx' - W_{\tilde{\phi}}(x) \right) \left(\int_S \frac{x - x''}{(x - x'')^2 + h_0^2} \delta_{m_y}(x'')\, dx'' \right) dx = \lambda \int_S m_y(x'')\, \delta_{m_y}(x'')\, dx''.$$

Using Fubini theorem to interchange the order of integration in x and x'' in the left-hand side, we have

$$\int_S \left[\frac{1}{\pi}\int_S \frac{x - x''}{(x - x'')^2 + h_0^2} \left(Q_{h_0} \star m_y - W_{\tilde{\phi}} \right)(x)\, dx - \lambda m_y(x'') \right] \delta_{m_y}(x'')\, dx'' = 0,$$

or equivalently,

$$\left\langle -Q_{h_0} \star Q_{h_0} \star m_y + Q_{h_0} \star W_{\tilde{\phi}} - \lambda m_y,\, \delta_{m_y} \right\rangle_{L^2(S)} = 0,$$

and hence, by arbitrariness of δ_{m_y}, we conclude

$$-Q_{h_0} \star \left(\chi_S \left(Q_{h_0} \star m_y \right) \right) + Q_{h_0} \star \left(\chi_S W_{\tilde{\phi}} \right) = \lambda m_y, \tag{2.3}$$

where χ_S is a characteristic function of the set S.

We, therefore, have deduced the following

Theorem 1. *If solution to the bounded extremal problem (1.12) for F exists, $m_y = \lim_{y\to 0^+} ReF(x+iy)$ must satisfy (2.3), where the parameter λ has to be chosen such that the inequality constraint in $\mathcal{B}_{M_f}$ is fulfilled.*

We will now show that the Fredholm integral equation of the second kind arising in the problem has the unique solution which can be constructed by the Banach fixed-point theorem once we rewrite (2.3) in the operator form $m_y = T_\lambda[m_y]$ and prove

Proposition 1. *The operator*

$$\begin{aligned} T_\lambda : L^2(S) &\to L^2(S) \\ f &\mapsto T_\lambda[f] := \frac{1}{\lambda}\left(Q_{h_0} \star (\chi_S (Q_{h_0} \star f)) + Q_{h_0} \star \left(\chi_S W_{\tilde{\phi}}\right)\right) \end{aligned}$$

is a contraction for $|\lambda| > \left[\frac{1}{2\pi}\log\left(1+\frac{4a^2}{h_0^2}\right)\right]^2$.

Proof. Since T_λ is affine, we need to show that

$$\left\| \frac{1}{\lambda}\left(Q_{h_0} \star (\chi_S (Q_{h_0} \star f))\right)\right\|_{L^2(S)} \leq q\, \|f\|_{L^2(S)} \tag{2.4}$$

for some $0 < q < 1$.

First, writing $Q_{h_0} = Q_{h_0}^{1/2} Q_{h_0}^{1/2}$, we employ the Cauchy-Schwarz inequality to estimate

$$\begin{aligned} \left|(Q_{h_0} \star f)(x)\right| &\leq \frac{1}{\pi}\left(\int_S \frac{x-x'}{(x-x')^2+h_0^2}\left|f(x')\right|^2 dx'\right)^{1/2}\left(\int_S \frac{x-x'}{(x-x')^2+h_0^2} dx'\right)^{1/2} \\ &\leq \frac{r^{1/2}}{\pi}\left(\int_S \frac{x-x'}{(x-x')^2+h_0^2}\left|f(x')\right|^2 dx'\right)^{1/2}, \end{aligned}$$

where

$$r := \sup_{x\in S}\left(\int_S \frac{x-x'}{(x-x')^2+h_0^2} dx'\right) = \frac{1}{2}\sup_{x\in(-a,a)}\left[\log\frac{(x-a)^2+h_0^2}{(x+a)^2+h_0^2}\right] = \frac{1}{2}\log\left(1+\frac{4a^2}{h_0^2}\right)$$

since $\log\left(1-\frac{4ax}{(x+a)^2+h_0^2}\right)$ is a monotonically decreasing function on $x\in[-a,a]$.

Therefore

$$\left\|P_{h_0} \star f\right\|_{L^2(S)} \leq \frac{r}{\pi}\|f\|_{L^2(S)} \leq \frac{1}{2\pi}\log\left(1+\frac{4a^2}{h_0^2}\right)\|f\|_{L^2(S)},$$

and hence (2.4) holds with $q = \frac{1}{|\lambda|}\left[\frac{1}{2\pi}\log\left(1+\frac{4a^2}{h_0^2}\right)\right]^2$ for $|\lambda| > \left[\frac{1}{2\pi}\log\left(1+\frac{4a^2}{h_0^2}\right)\right]^2$. □

References

[1] L. Aizenberg, “Carleman’s Formulas in Complex Analysis”, Kluwer Academic Publishers, 1993.

[2] L. Baratchart, J. Leblond, “Hardy approximation to L^p functions on subsets of the circle with $1 \leq p < \infty$”, Constructive Approximation, 14, 41-56, 1998.

[3] L. Baratchart, D. P. Hardin, E. A. Lima, E. B. Saff and B. P. Weiss, “Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions”, Inverse Problems, 29(1), 2013.

[4] L. Baratchart, D. P. Hardin, E. A. Lima, E. B. Saff and B. P. Weiss, “Fast Inversion of Unidirectional Planar Magnetization Distributions in Geological Samples”, to appear, 2013.

[5] P. L. Duren, “Theory of H^p Spaces”, Academic Press, 1970.

[6] G. B. Folland, “Introduction to Partial Differential Equation”, Princeton University Press, 1995.

[7] J. B. Garnett, “Bounded Analytic Functions”, Academic Press, 1981.

[8] P. Koosis, “Introduction to H^p spaces”, Cambridge University Press, 2008.

[9] P. Varaia, “Notes on Optimization”, Van Nostrand Reinhold, New York, 1972.