Moment estimation of very weak magnetizations

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Abstract

We estimate the net magnetic moment of a thin rock sample from the only measures of the vertical component of the weak magnetic field induced near the sample. This problem comes from paleomagnetism, where planetary scientists try to understand how rocks record the Earth's magnetic field history. This comes to be a ill-posed inverse problem, which is solved via a bounded extremal problem regularization. We present here numerical results.

1. SQUID measures: vertical component of magnetic field

Remanent magnetization $\vec{m}$ inside a thin plane rock sample S induces in its neighbourhood a low level magnetic field $\vec{B}$. Only vertical component B_3 of $\vec{B}$ can be measured by a SQUID in a parallel plane subset Q at distance h, very close to the sample S.

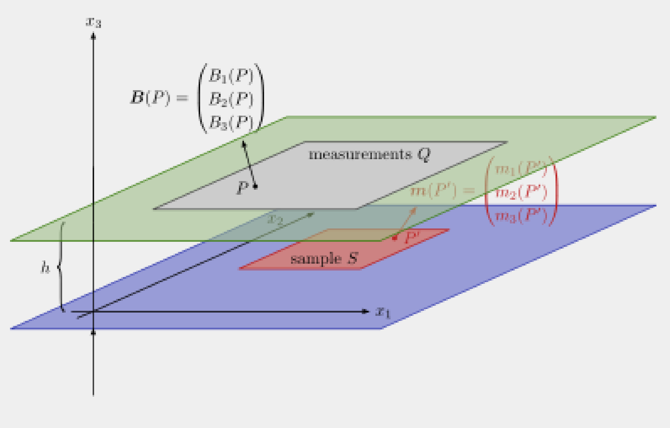

Figure: Sample S, measures set Q.

2. Moment estimation problem

- Given a magnetization vector field $\vec{m}(s)$, $s \in S$, for $q \in Q$ let $\vec{r} = q - s$, then magnetic field $\vec{B}$ on Q is given by:

$$\vec{B}(q) = \frac{\mu_0}{4\pi} \int_S \frac{3(\vec{m}.\vec{r})\vec{r} - |\vec{r}|^2 \vec{m}}{|\vec{r}|^5} ds \qquad (1)$$

- From sole measure of the vertical component B_3 of $\vec{B}$ on subset Q, estimation of the whole vector field $\vec{m}$ on S is a difficult ill-posed inverse problem, which suffers non-uniqueness due to the existence of *silent sources* [1].
- We focus to an easier problem: estimation of the total sum of $\vec{m}$, or net magnetic moment:

$$\langle \vec{m} \rangle = \int_S \vec{m}(s)\, ds \ \in R^3 \qquad (2)$$

As shown in [2], uniqueness is now granted, but problem is still ill-posed.

3. Mathematical setting

- **Squid geometry**: given three dimensions d_Q, d_S, h, then in the x_1, x_2, x_3 reference space, domains Q and S are the subsets:

$$S = (|x_1| \le d_S,\ |x_2| \le d_S,\ x_3 = 0) \qquad Q = (|x_1| \le d_Q,\ |x_2| \le d_Q,\ x_3 = h)$$

- **Functional spaces**: $L^2(Q)$ (resp. $L^2(S)$) is the classical Hilbert space of square-integrable real-valued (x_1, x_2)-functions on Q (resp. S). Measurements on Q are elements of $W^{1,2}(Q)$, functions in $L^2(Q)$, with gradient in $[L^2(Q)]^2$, Magnetizations $\vec{m}$ on S are elements of $[L^2(S)]^3$. Three constant functions in this space, with values the three unit-vectors of the R^3 basis, are noted $\vec{e}_1, \vec{e}_2, \vec{e}_3$ so we have $\langle m_i \rangle = \int_S \vec{e}_i.\vec{m}\, ds$
- **Operators**: $b_3 : [L^2(S)]^3 \to L^2(Q)$ associates to a magnetization m on S the vertical component B_3 of the induced magnetic field on Q in equation(1). $b_3^* : L^2(Q) \to [L^2(S)]^3$ is the adjoint of b_3. $2D$ gradient and laplacien operators on Q are noted ∇ and Δ.
- **Linear estimation**: we look for three functions $\Phi_i \in W^{1,2}(Q)$, $i = 1, 2, 3$ acting linearly on $B_3 = b_3[\vec{m}]$ to give an estimation of $\langle \vec{m} \rangle$:

$$\langle m_i \rangle = \int_Q \Phi_i(q) B_3(q) dq \qquad (3)$$

- **Bounded extremal problem**: functions $\Phi_i \in W^{1,2}(Q)$ have to minimize

$$\sup_{|m| \le 1} \left| \langle b_3[m], \Phi_i \rangle_{L^2(Q)} - \langle m_i \rangle \right| = |b_3^*[\Phi] - \vec{e}_i|_{[L^2(S)]^3}$$

among such functions Φ with gradient norm bounded by some constant M. The norm constraint is required for this issue to be well-posed (regularized).

- **Critical point equation, variational form**: given bound M, there exist a positive Lagrange multiplier λ, functions Φ_i are solutions of equation $\forall v \in W^{1,2}(Q)$:

$$\left(b_3^*[\Phi_i],\ b_3^*[v]\right)_{[L^2(S)]^3} + \lambda \left(\nabla \Phi_i,\ \nabla v\right)_{[L^2(Q)]^2} = \left(b_3[\vec{e}_i],\ v\right)_{L^2(Q)} \qquad (4)$$

4. Numerical implementation

Simulations are performed under `matlab 2017a`. Geometry of the squid is fixed to $d_Q \simeq 0.00255$, $d_S \simeq 0.00197$, $h \simeq 0.00027$, which gives $d_Q/d_S \simeq 1.2944 > 1$, quite a realistic situation. Square domains Q and S are discretized with regular grids, both of size 100×100. Critical point equation(4) is approximated on a Q_1-finite element basis. Corresponding $10^4 \times 10^4$ linear system is solved by direct inversion.

5. Errors on S and gradient norm on Q as λ varies

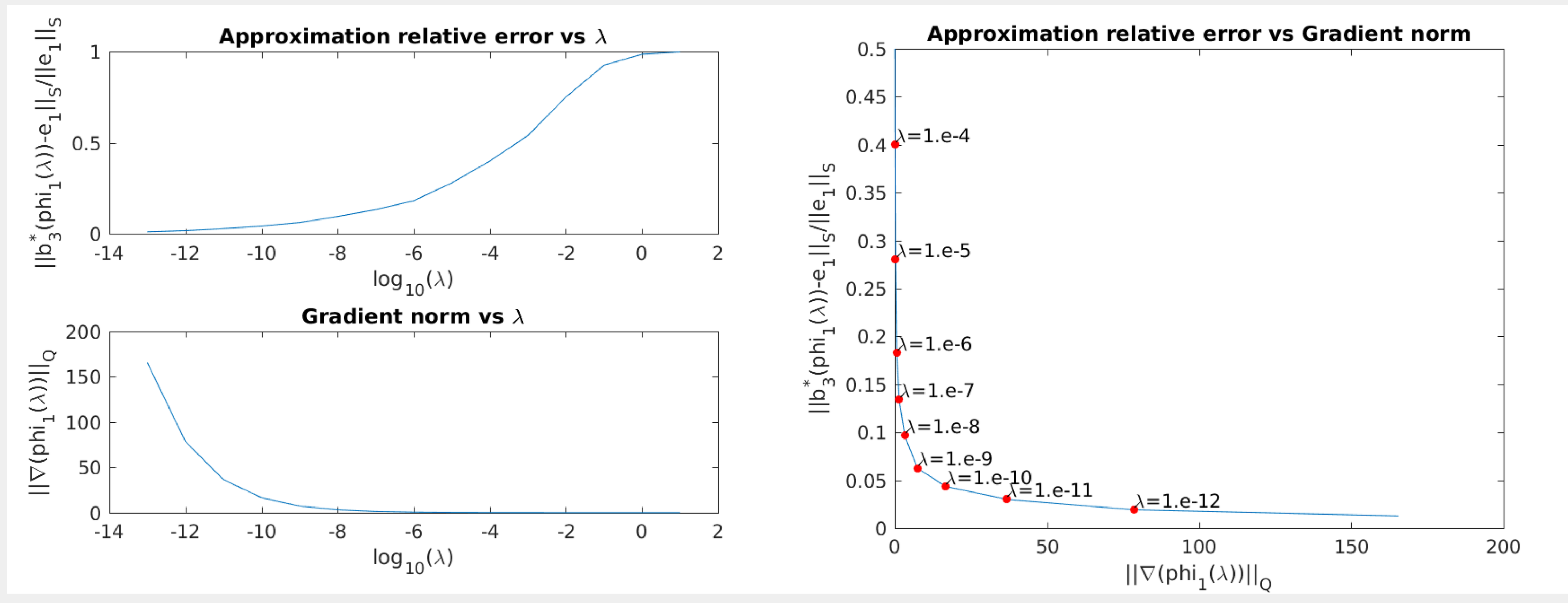

Figure: Error on S vs gradient norm on Q as λ varies

6. Build a magnetometer: choose λ

Choosing λ is a trade-off between error on S and gradient norm on Q. Reasonable choices lie near the maximum curvature zone of the L-curve [3].

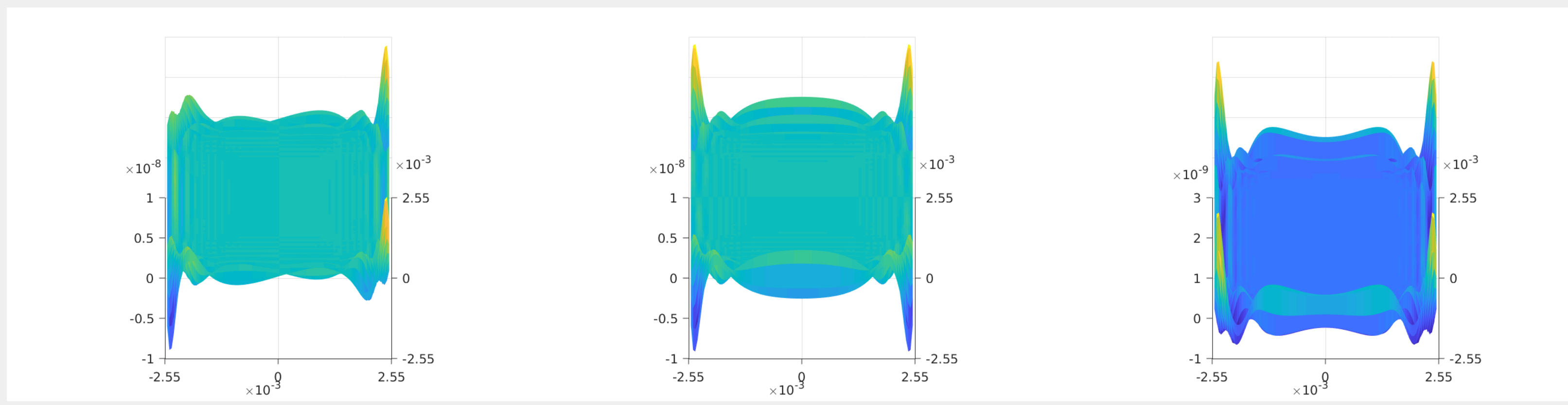

Figure: λ-magnetometer: $\Phi_1^\lambda, \Phi_2^\lambda, \Phi_3^\lambda$ on Q for $\lambda = 10^{-11}$

7. Illustration example: synthetic data

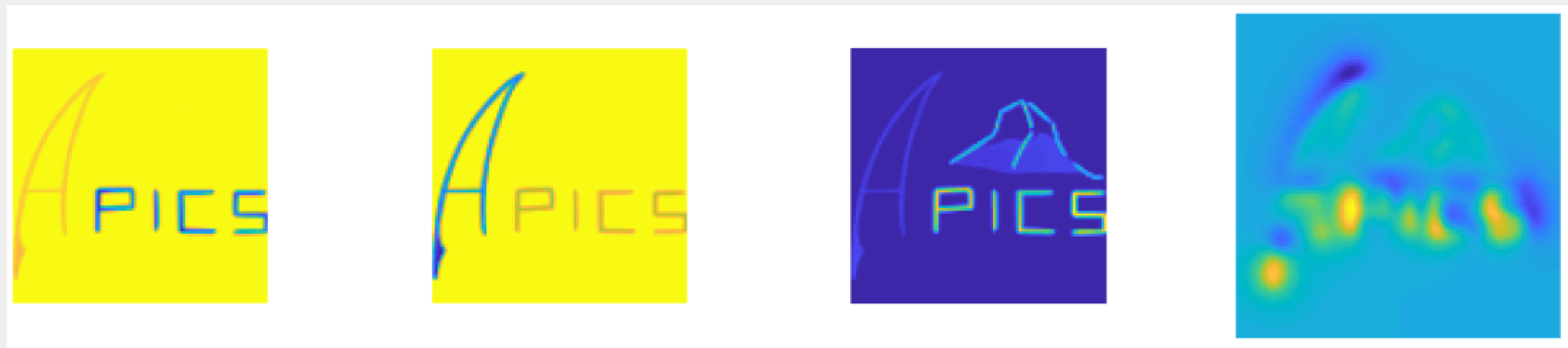

Figure: Three components of magnetization $\vec{m}$ on S, and resulting component B_3 on Q

8. Relative estimation error for various choices of λ

Total integral(2) is used to compute true moment $\langle m \rangle$.
Purely random gaussian noise is added to synthetic data B_3 (std=2%).
Estimated moments are given by formula(3) for different values of λ.

	−7.4059e−05	−1.1218e−04	4.0880e−05	
10^−6	−6.4679e−05	−9.4277e−05	3.9061e−05	14.44%
10^−7	−6.7989e−05	−1.0136e−04	3.9485e−05	8.89%
10^−8	−6.9647e−05	−1.0587e−04	3.9649e−05	5.55%
10^−9	−7.0035e−05	−1.0857e−04	3.9616e−05	3.95%
10^−10	−7.0699e−05	−1.0941e−04	3.9878e−05	3.18%
10^−11	−7.4309e−05	−1.0969e−04	4.1968e−05	1.94%
10^−12	−7.8249e−05	−1.1932e−04	4.6955e−05	7.31%

9. Conclusion

Measures in a small domain of only one component of a magnetic field are sufficient to estimate the underlying magnetization net moment. Linear estimators are obtained via direct resolution of a bounded extremal problem with careful choice of the Lagrange multiplier. Current numerical experiments on a standard PC suffer limitation of the discretization grid size. Further works will concern recursive resolution scheme, noise influence study, parallelization.

10. References

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