Magnetic moments estimation and bounded extremal problems

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Contents

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Check and correct results numbering in [2].

1 Introduction

The present study concerns situations where, for Lipschitz-smooth connected bounded open sets S, $Q \subset \mathbb{R}^2$ and $h > 0$:
- the unknown magnetization distribution $\boldsymbol{m}$ (with values in $\mathbb{R}^3$) is supported on $\overline{S} \times \{0\} \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, $\boldsymbol{m} \in [L^2(S)]^3$.
- values $b_3[\boldsymbol{m}]$ (with values in $\mathbb{R}$) of the normal component of the magnetic field produced by $\boldsymbol{m}$ are available on $Q \times \{h\} \subset \mathbb{R}^2 \times \{h\} \subset \mathbb{R}^3_+$, and $b_3[\boldsymbol{m}] \in L^2(Q)$,
and we want to recover the net moment $\langle \boldsymbol{m} \rangle$ of $\boldsymbol{m}$ (in $\mathbb{R}^3$) which is given by its mean value on S.

Or higher order moments as well.
The present work is a sequel to [2] and [3], where silent sources and magnetizations which are equivalent to a given one are studied for thin plates.
Approx. pb et BEP, see [2, Conclu.].

2 Notations, preliminaries, framework

2.1 Notations

Notations and definitions are as in [2, Sec. 2].
Lipschitz-smooth connected bounded open sets $\Omega \subset \mathbb{R}^2$.
Hilbert-Sobolev spaces $W^{1,2}(\Omega)$, $W^{1,2}_0(\Omega)$. In Section 4: $W^{3/2,2}(\Omega)$, $W^{\beta,2}(\Omega)$ for $1/2 < \beta < 3/2$, [6] (or within the proof...). Spaces of Hölder continous functions $C^\alpha(\Omega)$, $0 \leq \alpha < 1$, [6].

2.2 Preliminary properties

Properties of Poisson and Riesz operators are discussed [2, Sec. 2], [3, Sec. 2] along with orthogonal Hodge decompositions of vector fields.
Preliminary properties in view of moment recovery are discussed in [2, Sec. 4].

2.3 Related operators

The operator $\boldsymbol{m} \to b_3[\boldsymbol{m}]$ and it's adjoint are studied in [2, Sec. 3]. We precise below those among their main properties that will be used in the sequel, see also [2, Sec. 4.3]. Let $\boldsymbol{m} = (m_1, m_2, m_3) \in [L^2(S)]^3$ and $\widetilde{\boldsymbol{m}} = \boldsymbol{m} \vee 0 \in [L^2(\mathbb{R}^2)]^3$. The operator $b_3 : [L^2(S)]^3 \to L^2(Q)$ is defined by, see [2, Sec. 3]:

$$b_3[\boldsymbol{m}] = -\frac{\mu_0}{2} \left[\partial_{x_3} P_{x_3} \star (R_1 \widetilde{m}_1 + R_2 \widetilde{m}_2 + \widetilde{m}_3)\right]_{|Q \times \{h\}},$$

and can also be written as:

$$b_3[\boldsymbol{m}] = -\frac{\mu_0}{2} \left(\partial_{x_1} P_h \star \widetilde{m}_1 + \partial_{x_2} P_h \star \widetilde{m}_2 + [\partial_{x_3} P_{x_3} \star \widetilde{m}_3]_{|x_3 = h}\right)_{|Q},$$

using properties of Poisson and Riesz operators, see [2, Sec. 2].

Say a bit more about Poisson/Riesz, see what properties are actually used.
These are to the effect that b_3 is continuous and can be rewritten as:

$$b_3[\boldsymbol{m}] = -\frac{\mu_0}{2}\left(\nabla_2 \cdot \begin{pmatrix} P_h \star \widetilde{m}_1 - R_1\left(P_h \star \widetilde{m}_3\right) \\ P_h \star \widetilde{m}_2 - R_2\left(P_h \star \widetilde{m}_3\right) \end{pmatrix}\right)_{|Q}. \tag{1}$$

The adjoint operator $b_3^* : L^2(Q) \to [L^2(S)]^3$ of b_3 acts on $\phi \in L^2(Q)$, with $\widetilde{\phi} = \phi \vee 0 \in L^2(\mathbb{R}^2)$, as, see [2, Sec. 4.3]:

$$b_3^*[\phi] = \left(\frac{\mu_0}{2}\begin{pmatrix} R_1 \\ R_2 \\ -I \end{pmatrix}\left[\partial_{x_3} P_{x_3} \star \widetilde{\phi}\right]_{|x_3=h}\right)_{|S} = \frac{\mu_0}{2}\begin{pmatrix} \partial_{x_1} P_h \star \widetilde{\phi} \\ \partial_{x_2} P_h \star \widetilde{\phi} \\ -[\partial_{x_3} P_{x_3} \star \widetilde{\phi}]_{|x_3=h} \end{pmatrix}_{|S}.$$

It is continuous (because so is b_3), and the following bound is available in [2, Sec. 3.3]:

$$\|b_3^*\| \le b \text{ with } b = \frac{\mu_0}{2}\,\frac{4\sqrt{2}}{3^{3/2}h}, \tag{2}$$

which implies that b_3^* is injective ([2, Lem. 1]) whence b_3 has a dense range in $L^2(Q)$.
For $\phi \in W_0^{1,2}(Q)$, with $\widetilde{\phi} = \phi \vee 0 \in W^{1,2}(\mathbb{R}^2)$, note that:

$$b_3^*[\phi] = \frac{\mu_0}{2}\begin{pmatrix} P_h \star \partial_{x_1}\widetilde{\phi} \\ P_h \star \partial_{x_2}\widetilde{\phi} \\ P_h \star \left(R_1\,\partial_{x_1}\widetilde{\phi} + R_2\,\partial_{x_2}\widetilde{\phi}\right) \end{pmatrix}_{|S}.$$

From [2, Prop. 1 & Lem. 2], the following properties hold true for the kernel of b_3 and the range of b_3^* in $[L^2(S)]^3$. If we set $\mathcal{D}_S = \operatorname{Ker} b_3$ then

$$\begin{cases} \mathcal{D}_S = \left\{(-\partial_{x_2}\psi, \partial_{x_1}\psi, 0)\ ,\ \psi \in W_0^{1,2}(S)\right\} \subset [L^2(S)]^3 \text{ and} \\ \mathcal{D}_S^{\perp} = \overline{\operatorname{Ran} b_3^*} = \nabla_2 W^{1,2}(S) \times L^2(S) \subset [L^2(S)]^3, \end{cases} \tag{3}$$

where $\mathcal{D}_S^{\perp}$ stands for the orthogonal space to $\mathcal{D}_S$ in $[L^2(S)]^3$. Also, since we have in $[L^2(Q)]^2$ (see [2, Rmk 1]):

$$\left[\nabla_2 W_0^{1,2}(Q)\right]^{\perp} = \left\{(-\partial_{x_2}\psi, \partial_{x_1}\psi)\ ,\ \psi \in W^{1,2}(Q)\right\} \subset [L^2(Q)]^2, \tag{4}$$

we see that vector fields in $\left[\nabla_2 W_0^{1,2}(Q)\right]^{\perp}$ are divergence free in $\mathbb{R}^2$.

2.4 A density result

Moment recovery issues (define $\boldsymbol{e}_i$, see [2]):
Given $b_3[\boldsymbol{m}]$... recover $\langle \boldsymbol{m} \rangle = (\langle m_1 \rangle, \langle m_2 \rangle, \langle m_3 \rangle)$, through the scalar product of $b_3[\boldsymbol{m}]$ by ϕ:

$$\langle m_i \rangle = \langle \boldsymbol{m}\,,\ \boldsymbol{e}_i \rangle_{[L^2(S)]^3}\,,\ \langle b_3[\boldsymbol{m}]\,,\ \phi \rangle_{L^2(Q)} = \langle \boldsymbol{m}\,,\ b_3^*[\phi] \rangle_{[L^2(S)]^3} \ldots$$

such that $b_3^*[\phi] \simeq \boldsymbol{e}_i$..., see Rmk 1.

Because Q is bounded, Poincaré inequality [5, Cor. IX.19] is to the effect that there exists a constant $C > 0$ (depending on Q) such that

$$\|\phi\|_{L^2(Q)} \leq C \, \|\nabla_2 \, \phi\|_{[L^2(Q)]^2} \, , \; \forall \phi \in W_0^{1,2}(Q) \, . \tag{5}$$

It implies that $\|\cdot\|_{W^{1,2}(Q)}$ and $\|\nabla_2 \, [\cdot]\|_{[L^2(Q)]^2}$ are equivalent norms on $W_0^{1,2}(Q)$.
From this property and [2, Lem. 4], we get the following density and unstability properties. For $\boldsymbol{e} \in \overline{\operatorname{Ran} b_3^*} \subset [L^2(S)]^3$,

$$\inf_{\phi \in W_0^{1,2}(Q)} \|b_3^* \, [\phi] - \boldsymbol{e}\|_{[L^2(S)]^3} = 0 \, .$$

Whenever $\phi_n \in W_0^{1,2}(Q)$ is such that $\|b_3^* \, [\phi_n] - \boldsymbol{e}\|_{[L^2(S)]^3} \to 0$ as $n \to \infty$, then either $\boldsymbol{e} \in b_3^* \left[W_0^{1,2}(Q)\right]$ or $\|\nabla \phi_n\|_{[L^2(Q)]^2} \to \infty$. Note that $\boldsymbol{e} \in b_3^* \left[W_0^{1,2}(Q)\right]$ is the only case where the above inf is reached.

Comment about constraint on $\phi \in W_0^{1,2}(Q)$ and $\|\nabla \phi\|_{[L^2(Q)]^2}$ rather than constraint on $\|\phi\|_{L^\infty(Q)}$ and $\phi \in C_0(Q)$ which we indeed need (for constructive reasons, a vanishing boundary condition being used for solving Dirichlet problems, see Section 4, and the continuity property of ϕ will be ensured from further results, see Proposition 3).
Comment about situations with $\boldsymbol{e} \in [L^2(S)]^3$, $\boldsymbol{e} \notin \overline{\operatorname{Ran} b_3^*}$: the best we can do is to approximate the orthogonal projection $P_{\mathcal{D}_S^\perp} \boldsymbol{e} \in \overline{\operatorname{Ran} b_3^*}$.
Comment moments recovery, $\boldsymbol{e}_i \in \overline{\operatorname{Ran} b_3^*}$; discuss (state?) [2, Lem. 7] for $\boldsymbol{e}_i$ (and other interesting functions towards higher order moments estimation).

Remark 1 *From the above density result (see also [2, Sec. 4.3]), the quantity:*

$$\left| \langle b_3 \, [\boldsymbol{m}] \, , \, \phi \rangle_{L^2(Q)} - \langle \boldsymbol{m} \, , \, \boldsymbol{e}_i \rangle_{[L^2(S)]^3} \right| \leq \|b_3^* \, [\phi] - \boldsymbol{e}_i\|_{[L^2(S)]^3} \, \|\boldsymbol{m}\|_{[L^2(S)]^3} \, .$$

can be made arbitrarily small, at the expense of an unbounded $\|\nabla_2 \, \phi\|_{[L^2(Q)]^2}$.
Note that the left hand side of the above inequality vanishes if and only if $\boldsymbol{m} \in \mathcal{D}_S$ *(*$\boldsymbol{m}$ *is a silent sources). Indeed, we have*

$$\langle b_3 \, [\boldsymbol{m}] \, , \, \phi \rangle_{L^2(Q)} - \langle \boldsymbol{m} \, , \, \boldsymbol{e}_i \rangle_{[L^2(S)]^3} = \langle b_3^* \, [\phi] - \boldsymbol{e}_i \, , \, \boldsymbol{m} \rangle_{[L^2(S)]^3} \, .$$

Moreover, from [2, Lem. 7], $\boldsymbol{e}_i \in \mathcal{D}_S^\perp$*, hence* $b_3^* \, [\phi] - \boldsymbol{e}_i \in \mathcal{D}_S^\perp$*, using* (3)*. Therefore, if* $\boldsymbol{m} \in \mathcal{D}_S$*, then the above quantity vanishes. Conversely, assume that* $\langle b_3^* \, [\phi] - \boldsymbol{e}_i \, , \, \boldsymbol{m} \rangle_{[L^2(S)]^3} = 0$*. From [2, Lem. 7] again,* $\boldsymbol{e}_i \notin \operatorname{Ran} b_3^*$*, whence* $b_3^* \, [\phi] - \boldsymbol{e}_i$ *cannot identically vanish and must be orthogonal to* $\boldsymbol{m}$ *in* $[L^2(S)]^3$*. This implies that* $\boldsymbol{m} \in \mathcal{D}_S$*.*
For $\boldsymbol{m} \in [L^2(S)]^3$*, the solution* $\phi = \phi_o$ *to (BEP) below will furnish a trade-off between the error* $\|b_3^* \, [\phi] - \boldsymbol{e}_i\|_{[L^2(S)]^3}$ *and of the constraint* M *on* $\|\nabla_2 \, \phi\|_{[L^2(Q)]^2}$*.*

3 Bounded extremal problems (BEP)

Consider the following bounded extremal problem (BEP, or norm constrained best approximation issue), for $\boldsymbol{e} \in \overline{\operatorname{Ran} b_3^*} \subset [L^2(S)]^3$ (see the comment above for $\boldsymbol{e} \in [L^2(S)]^3$)

and $M > 0$:

(BEP) Find $\phi_o \in W_0^{1,2}(Q)$, $\|\nabla_2 \phi_o\|_{[L^2(Q)]^2} \leq M$ such that

$$\min_{\phi \in W_0^{1,2}(Q),\, \|\nabla_2 \phi\|_{[L^2(Q)]^2} \leq M} \|b_3^* [\phi] - \boldsymbol{e}\|_{[L^2(S)]^3} = \|b_3^* [\phi_o] - \boldsymbol{e}\|_{[L^2(S)]^3} \,.$$

3.1 Well posedness

Proposition 1 *There exists a unique solution ϕ_o to (BEP); whenever $\boldsymbol{e} \notin b_3^* \left[W_0^{1,2}(Q)\right]$, the constraint is saturated: $\|\nabla_2 \phi_o\|_{[L^2(Q)]^2} = M$, for any $M > 0$.*

Note that some constraints $M > 0$ would be saturated as well if $\boldsymbol{e} \in b_3^* \left[W_0^{1,2}(Q)\right]$ with $\boldsymbol{e} = b_3^* [\phi]$ for some $\phi \in W_0^{1,2}(Q)$ with $\|\nabla_2 \phi\|_{[L^2(Q)]^2} \geq M$.

Proof: First, because of the equivalence of the norms already mentionned in Section 2.4, the convex set

$$\{\phi \in W_0^{1,2}(Q)\,,\ \|\nabla_2 \phi_o\|_{[L^2(Q)]^2} \leq M\}$$

is closed in the Hilbert space $W_0^{1,2}(Q)$ thus in $W^{1,2}(Q)$ (for $W_0^{1,2}(Q)$ is closed in $W^{1,2}(Q)$). Then, since b_3^* is linear and continuous, the set of approximants

$$\mathcal{A} = b_3^* \left[\{\phi \in W_0^{1,2}(Q)\,,\ \|\nabla_2 \phi_o\|_{[L^2(Q)]^2} \leq M\}\right]$$

is convex and closed in $[L^2(S)]^3$. This implies that there exists a best approximation projection from $[L^2(S)]^3$ onto $\mathcal{A}$ and ensures both existence and uniqueness of the solution $\phi_o \in \mathcal{A}$.
Next, assume that $\|\nabla_2 \phi_o\|_{[L^2(Q)]^2} < M$. In this case, the minimum value of the criterion is achieved by ϕ_o interior to the approximation set. We then get by differentiating the square $\|b_3^* [\phi_o] - \boldsymbol{e}\|^2_{[L^2(S)]^3}$ of the criterion with respect to ϕ_o that for every $\delta_\phi \in W_0^{1,2}(Q)$,

$$\langle b_3^* \, [\phi_o] - \boldsymbol{e} \,,\, b_3^*[\delta_\phi]\rangle_{[L^2(S)]^3} = \langle b_3 \, b_3^* \, [\phi_o] - b_3 \, [\boldsymbol{e}] \,,\, \delta_\phi \rangle_{L^2(Q)} = 0 \,.$$

Hence, $b_3 \, b_3^* \, [\phi_o] - b_3 \, [\boldsymbol{e}]$ is orthogonal to $W_0^{1,2}(Q)$ in $L^2(Q)$ and, by density of $W_0^{1,2}(Q)$ in $L^2(Q)$, we must have $b_3 \, b_3^* \, [\phi_o] - b_3 \, [\boldsymbol{e}] = 0$. Thus, $b_3^* \, [\phi_o] - \boldsymbol{e}$ belongs to $\mathcal{D}_S = \operatorname{Ker} b_3$. However, both $b_3^* \, [\phi_o]$ and $\boldsymbol{e}$ belong to $\mathcal{D}_S^\perp$, so does their difference. Hence $b_3^* \, [\phi_o] - \boldsymbol{e} = 0$, which implies that $\boldsymbol{e} = b_3^* \, [\phi_o] \in b_3^* \left[W_0^{1,2}(Q)\right]$. □

3.2 Critical point equation (CPE)

Proposition 2 *Let $\boldsymbol{e} \in \overline{Ran\, b_3^*} \setminus b_3^* \left[W_0^{1,2}(Q)\right] \subset \left[L^2(S)\right]^3$ and $M > 0$. The solution ϕ_o to (BEP) satisfies the following critical point equation (CPE) on Q. More precisely there exists a unique $\lambda > 0$ such that $\|\nabla_2 \phi_o\|_{[L^2(Q)]^2} = M$ and*

$$b_3 \, b_3^* \, [\phi_o] - \lambda \, \Delta_2 \, \phi_o = b_3 \, [\boldsymbol{e}] \,. \tag{6}$$

Proof: By differentiating with respect to ϕ_o the square of the criterion as above and also that of the constraint $\|\nabla_2 \, \phi_o\|^2_{[L^2(Q)]^2} = M^2$ achieved in (BEP), we obtain that there exists a unique value of the Lagrange parameter $\lambda \in \mathbb{R}$ such that for every $\delta_\phi \in W_0^{1,2}(Q)$,

$$\langle b_3^* \, [\phi_o] - \boldsymbol{e} \, , \, b_3^* \, [\delta_\phi] \rangle_{[L^2(S)]^3} + \lambda \langle \nabla_2 \, \phi_o \, , \, \nabla_2 \, \delta_\phi \rangle_{[L^2(Q)]^2} = 0 \, . \tag{7}$$

Thus, for every $\delta_\phi \in W_0^{1,2}(Q)$, because δ_ϕ vanishes on the boundary of Q,

$$\langle b_3 \, b_3^* \, [\phi_o] - b_3 \, [\boldsymbol{e}] - \lambda \, \nabla_2 \cdot \nabla_2 \, \phi_o \, , \, \delta_\phi \rangle_{L^2(Q)} = 0 \, .$$

Therefore, $b_3 \, b_3^* \, [\phi_o] - b_3 \, [\boldsymbol{e}] - \lambda \, \Delta_2 \phi_o$ is orthogonal to $W_0^{1,2}(Q)$ in $L^2(Q)$ whence to $L^2(Q)$ itself, since $W_0^{1,2}(Q)$ is dense in $L^2(Q)$. This establishes (6) with $\lambda \in \mathbb{R}$.
Finally, that $\lambda \geq 0$ can be seen as follows. We get from (7) that $\forall \phi \in W_0^{1,2}(Q)$:

$$\langle b_3^* \, [\phi_o] - \boldsymbol{e} \, , \, b_3^* \, [\phi_o] \rangle_{[L^2(S)]^3} = -\lambda \, \|\nabla_2 \, \phi_o\|^2_{[L^2(Q)]^2} = -\lambda \, M^2 \, . \tag{8}$$

Because ϕ_o achieves a minimum, the above quantity is negative as detailed in the proof of [5, Thm V.2 (3)]. Thus $\lambda \geq 0$. That $\lambda \neq 0$ is finally ensured by assumption on $\boldsymbol{e}$ (namely, $\boldsymbol{e} \in \overline{\mathrm{Ran} \, b_3^*} \setminus b_3^* \left[W_0^{1,2}(Q) \right]$). $\square$

Alternative proofs of Proposition 2 are available. One could directly obtain (CPE) from the result established in [4, Thm 2.1] and recalled in [1, Prop. 4] which furnishes critical point equations associated to solutions of quite general extremal problems in Hilbert spaces.
Observe that (8) links together the Lagrange parameter λ, the constraint M and the error (criterion) in (BEP) and implies that $\lambda \to 0$ as $M \to +\infty$. Argument: use density result of Section 2.4.

4 Critical point equation (CPE): iterative resolution scheme

For $\varrho > 0$ and $n \geq 1$, write: Precise what $\lambda > 0$.

$$b_3 \, b_3^* \, [\phi_{n-1}] - \lambda \, \Delta_2 \, \phi_n = b_3 \, [\boldsymbol{e}] - \frac{1}{\varrho} \, (\phi_n - \phi_{n-1}) \, ,$$

or equivalently:

$$\varrho \, (b_3 \, b_3^* \, [\phi_{n-1}] - \lambda \, \Delta_2 \, \phi_n) = \varrho \, b_3 \, [\boldsymbol{e}] - (\phi_n - \phi_{n-1}) \, . \tag{9}$$

Proposition 3 *Let $\phi_0 \in W_0^{1,2}(Q)$. Then, for ϱ small enough, (9) defines a sequence (ϕ_n) of functions in $W_0^{1,2}(Q)$ that converges in $L^2(Q)$ to the unique solution $\phi_o \in W_0^{1,2}(Q)$ of the critical point equation (6).*
Actually, ϕ_n (for $n \geq 1$) and $\phi_o \in C^\alpha(Q)$ are Hölder continuous functions for $0 \leq \alpha < 1/2$ and (ϕ_n) converges to ϕ_o in $C^\alpha(Q)$.

Proof: For $n \geq 1$ and $\phi_{n-1} \in W_0^{1,2}(Q)$, we first show that there exists a unique solution $\phi_n \in W_0^{1,2}(Q)$ to (9). Indeed, for $\phi\,,\ \psi \in W_0^{1,2}(Q)$:

$$a(\phi\,,\ \psi) = \langle \phi\,,\ \psi \rangle_{L^2(Q)} + \varrho\,\lambda\,\langle \nabla_2 \phi\,,\ \nabla_2 \psi \rangle_{[L^2(Q)]^2}\,.$$

defines a continuous positive definite (coercive) bilinear form a on $[W_0^{1,2}(Q)]^2$. Then, the scalar product of (9) with any $\psi \in W_0^{1,2}(Q)$ can be written as:

$$a(\phi_n\,,\ \psi) = \langle (1 - \varrho\, b_3\, b_3^*)\ \phi_{n-1} + \varrho\, b_3\,[\boldsymbol{e}]\,,\ \psi \rangle_{L^2(Q)}\,,$$

which admits a unique solution $\phi_n \in W_0^{1,2}(Q)$ from Lax-Milgram theorem [5, Cor. V.8]. Next, substract (9) from (6) to obtain:

$$-\varrho\,\lambda\,\Delta_2\,(\phi_n - \phi_o) + (\phi_n - \phi_o) = -\varrho\, b_3\, b_3^*\,[\phi_{n-1} - \phi_o] + (\phi_{n-1} - \phi_o)\ , \tag{10}$$

and take the scalar product with $\phi_n - \phi_o$ in $L^2(Q)$:

$$\begin{aligned} \varrho\,\lambda\,\|\nabla_2\,(\phi_n - \phi_o)\|^2_{[L^2(Q)]^2} + \|\phi_n - \phi_o\|^2_{L^2(Q)} &= \langle (1 - \varrho\, b_3\, b_3^*)\ [\phi_{n-1} - \phi_o]\ ,\ \phi_n - \phi_o \rangle_{L^2(Q)} \\ &\leq \|I - \varrho\, b_3\, b_3^*\|\ \|\phi_{n-1} - \phi_o\|_{L^2(Q)}\ \|\phi_n - \phi_o\|_{L^2(Q)}\ . \end{aligned} \tag{11}$$

The Poincaré inequality (5) in $W_0^{1,2}(Q)$ implies that there exists a constant $C > 0$ (depending only on Q) such that:

$$\frac{\varrho\,\lambda}{C^2}\ \|\phi_n - \phi_o\|^2_{L^2(Q)} \leq \varrho\,\lambda\ \|\nabla_2\,(\phi_n - \phi_o)\|^2_{[L^2(Q)]^2}\ ,$$

whence, back to (11) and dividing both sides by $\|\phi_n - \phi_o\|_{L^2(Q)}$, we obtain:

$$\|\phi_n - \phi_o\|_{L^2(Q)} \leq \frac{\|I - \varrho\, b_3\, b_3^*\|}{1 + \frac{\varrho\,\lambda}{C^2}}\ \|\phi_{n-1} - \phi_o\|_{L^2(Q)}\ .$$

Next, the operator $b_3\, b_3^*\ :\ L^2(Q) \to L^2(Q)$ is positive definite since b_3^* is injective, whence Cauchy-Shwarz inequality implies that

$$\|b_3\, b_3^*\| = \sup_{\substack{\phi \in L^2(Q) \\ \|\phi\|_{L^2(Q)} \leq 1}} \langle b_3\, b_3^*\,\phi, \phi \rangle_{L^2(Q)} = \sup_{\substack{\phi \in L^2(Q) \\ \|\phi\|_{L^2(Q)} \leq 1}} \|b_3^*\,\phi\|^2_{L^2(Q)} = \|b_3^*\|^2\ .$$

Ici, preuve pédestre de $\|b_3 b_3^*\| = \|b_3^*\|^2$ et ci-dessous pour $\|I - \varrho\, b_3\, b_3^*\| =$... ; références bouquins opérateurs [Kato, Chap. I, Section 6.4, (6.25)][1].

Together with (2) this ensures that $0 < \|b_3\, b_3^*\| = \|b_3^*\|^2 \leq b^2$ for $b > 0$. In particular, if $1 - \rho\, b^2 > 0$ (if $0 < \rho < 1/b^2$), the operator $I - \varrho\, b_3\, b_3^*$ is also positive definite on $L^2(Q)$ and again

$$0 < \|I - \varrho\, b_3\, b_3^*\| = \sup_{\substack{\phi \in L^2(Q) \\ \|\phi\|_{L^2(Q)} \leq 1}} \langle (I - \varrho\, b_3\, b_3^*)\phi, \phi \rangle_{L^2(Q)} \leq 1\,.$$

[1]Ou comme corollaire d'Hahn-Banach car b_3^* continu, voir e.g. cours M2 d'Emmanuel Fricain, Analyse fonctionelle et théorie des opérateurs, `math.univ-lille1.fr/$\sim$fricain/cours-M2-2009-2010.pdf`.

Therefore, we obtain

$$\|\phi_n - \phi_o\|_{L^2(Q)} \leq \kappa\ \|\phi_{n-1} - \phi_o\|_{L^2(Q)}\ , \text{ with } \kappa = \frac{1}{1 + \frac{\varrho\,\lambda}{C^2}} < 1\,,$$

which establishes that $\|\phi_n - \phi_o\|_{L^2(Q)}$ decreases to 0 as $n \to \infty$.
Next, since $b_3\, b_3^* : L^2(Q) \to L^2(Q)$ is continuous, it then holds that $\|b_3\, b_3^*[\phi_n - \phi_o]\|_{L^2(Q)} \to 0$. Further, we see from (6) and (9) that $\Delta_2\, \phi_o$ and $\Delta_2\, \phi_n$ belong to $L^2(Q)$, for $n \geq 1$. Because Q is bounded and Lipschitz-smooth, we use [7, Thm B, 2.] which implies that ϕ_o and ϕ_n belong to $W^{3/2,2}(Q)$, whence in particular to $W^{\beta,2}(Q)$ for $0 \leq \beta < 3/2$. Now, (10) implies that $\|\Delta_2\ (\phi_n - \phi_o)\|_{L^2(Q)} \to 0$. As a consequence of [7, Thm 0.5, (b)] it then holds that $\phi_n - \phi_o \to 0$ in $W^{\beta,2}(Q)$ for $1/2 < \beta < 3/2$. Finally, if $1 < \beta$, the continuous embedding of Sobolev spaces $W^{\beta,2}(Q)$ into spaces of Hölder continous functions $C^{\beta-1}(Q)$, see [6, Thm 4.53], ensure that $\phi_o - \phi_n \to 0$ in $C^{\beta-1}(Q)$. □

Remark that in Proposition 3, it atually holds that ϕ_n (for $n \geq 1$) and $\phi_o \in C^\alpha(\bar{Q})$, since Q is Lipschitz-smooth [6]?.

5 Conclusion

- Related spectral issues, Dmitry: about eigenfunctions of Poisson 2D and conjugate, and of $b_3\, b_3^*$. Their use in order to compute solutions to moments recovery issue and to (BEP)?
- Consider other (non zero) extensions of $b_3[\boldsymbol{m}]$ outside Q (like by dipolar field, see notes [Dmitry]) to be used as constraints? Or / and other extensions of $\boldsymbol{m}$ outside S?
- Comment about Hardy spaces of gradients of harmonic functions, express b_3^* and solutions to (BEP) in terms of projections on Hardy space, see [1, 3].

6 To be considered

6.1 More about b_3 and b_3^*

Remark 2 *Utile ?*
From Section 2.3, for $\phi \in W_0^{1,2}(Q)$: *attention, up to $\times \pm \frac{\mu_0}{2}$.*

$$-b_3\, b_3^*\ [\phi] = \left(\nabla_2 \cdot \left(P_h \star \chi_S \left(P_h \star \nabla_2\, \widetilde{\phi}\right) - \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} P_h \star \chi_S \left(P_h \star (R_1 \partial_{x_1} + R_2 \partial_{x_2})\ \widetilde{\phi}\right)\right)\right)_{|Q}$$

$$= \left(\nabla_2 \cdot \left(P_h \star \chi_S \left(P_h \star \nabla_2\, \widetilde{\phi}\right)\right) - \left[\partial_{x_3}\, P_{x_3} \star \chi_S \left(\partial_{x_3}\, P_{x_3} \star \widetilde{\phi}\right)\right]_{|x_3 = h}\right)_{|Q}$$

$$= \left(2\,\nabla_2 \cdot \left(P_h \star \chi_S \left(P_h \star \nabla_2\, \widetilde{\phi}\right)\right) - \left[\nabla_3 \cdot P_{x_3} \star \chi_S \left[\nabla_3\ \left(P_{x_3} \star \widetilde{\phi}\right)\right]\right]_{|x_3 = h}\right)_{|Q}\,.$$

Similarly, note that for $\Phi \in W^{1,2}(\mathbb{R}^2)$, using harmonicity at $x_3 = 2h$ of $P_{x_3} \star \Phi$:

$$\left[\nabla_3 \cdot P_{x_3} \star \nabla_3\ (P_{x_3} \star \Phi)\right]_{|x_3 = h} = \left[\Delta_3\ (P_{x_3} \star \Phi)\right]_{|x_3 = 2h} = 0\,.$$

Thus, if $S = \mathbb{R}^2$, the above expression for $b_3\, b_3^*[\phi]$ would coincide with $-2\, P_{2h} \star \Delta_2 \widetilde{\phi}$.

- Characterize $b_3^* \left[W_0^{1,2}(Q)\right]$ and $b_3^* \left[L^2(Q)\right]$.
- Continue analysis of $b_3\, b_3^*$. Operator $b_3\, b_3^* \ : \ L^2(Q) \to L^2(Q)$ compact, since so is $b_3^* : L^2 \to L^2$ (add proof) whence also b_3.
- Is it true that (for c, $c' > 0$) we have:

$$c \, \|\phi\|_{L^2(Q)} \leq \|b_3^* \, \phi\|_{[L^2(S)]^3} \ , \ \phi \in L^2(Q) \, ,$$

$$c' \, \|\nabla_2 \, \phi\|_{[L^2(Q)]^2} \leq \|b_3^* \, \phi\|_{[L^2(S)]^3} \ , \ \phi \in W_0^{1,2}(Q) \, .$$

Probably not uniformly in general, as discussed with Sylvain, but under additional assumptions? And probably yes from injectivity property if c, c' could depend on ϕ?
Discussed with Aline: use thm inversion locale or fonctions implicites?
- Because $\operatorname{Ker} b_3^* = \{0\}$, orthogonality property see [2] implies that $\operatorname{Ker} b_3\, b_3^* = \{0\}$! in this case, Ran $b_3\, b_3^* = b_3\, b_3^* \left[L^2(Q)\right]$ dense in ($W_0^{1,2}(Q)$? and) $L^2(Q)$?! ($b_3\, b_3^* \left[W_0^{1,2}(Q)\right]$ and continuity prop.?)

6.2 For (BEP)

- Discuss error bounds from our estimates in [9].
- Continue analysis of (BEP), study (CPE).

6.3 Operators $\boldsymbol{a}$, $\boldsymbol{a}^*$ and (CPE)

Virer ?! Garder ce qu'il faut en termes de b_3 et b_3^*.

From (1), using commutation relations between convolution with the Poisson kernel and application of the Riesz transform, we can set:

$$b_3[\boldsymbol{m}] = -\frac{\mu_0}{2} \, \nabla_2 \cdot \boldsymbol{a}[\boldsymbol{m}] \, ,$$

with the operator $\boldsymbol{a} \ : \ \left[L^2(S)\right]^3 \to \left[L^2(Q)\right]^2$,

$$\boldsymbol{a}[\boldsymbol{m}] = \left(\begin{array}{c} P_h \star \widetilde{m}_1 - P_h \star (R_1 \, \widetilde{m}_3) \\ P_h \star \widetilde{m}_2 - P_h \star (R_2 \, \widetilde{m}_3) \end{array} \right)_{|Q} = \left(P_h \star \left(\begin{array}{c} \widetilde{m}_1 - R_1 \, \widetilde{m}_3 \\ \widetilde{m}_2 - R_2 \, \widetilde{m}_3 \end{array} \right) \right)_{|Q} \, .$$

The adjoint operator $\boldsymbol{a}^* \ : \ \left[L^2(Q)\right]^2 \to \left[L^2(S)\right]^3$ is defined at $\Phi \in \left[L^2(Q)\right]^2$ by

$$\boldsymbol{a}^*[\Phi] = P_h \star \left[\begin{array}{c} \Phi \vee 0 \\ \left[\begin{array}{c} R_1 \\ R_2 \end{array} \right] \cdot \widetilde{\Phi} \end{array} \right] = P_h \star \left[\begin{array}{l} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \\ R_1 \, \widetilde{\Phi}_1 + R_2 \, \widetilde{\Phi}_2 \end{array} \right] \ \text{on } S \, .$$

Indeed, one can check that $\langle \boldsymbol{a}^*[\Phi] \, , \, \boldsymbol{m} \rangle_{[L^2(S)]^3} = \langle \Phi \, , \, \boldsymbol{a}[\boldsymbol{m}] \rangle_{[L^2(Q)]^2}$.
Whenever $\phi \in W_0^{1,2}(Q) \subset L^2(Q)$, Up to $\mu_0/2$, check sign

$$\boldsymbol{a}^*[\nabla_2 \phi] = b_3^*[\phi] = P_h \star \left[\begin{array}{c} \nabla_2 \widetilde{\phi} \\ \left[\begin{array}{c} R_1 \\ R_2 \end{array} \right] \cdot \nabla_2 \widetilde{\phi} \end{array} \right] = P_h \star \left[\begin{array}{l} \partial_{x_1} \\ \partial_{x_2} \\ R_1 \, \partial_{x_1} + R_2 \, \partial_{x_2} \end{array} \right] \widetilde{\phi} \ \text{ on } S \, .$$

because for $\phi \in W_0^{1,2}(Q)$, $\psi \in W^{1,2}(S)$:

$$\langle \boldsymbol{a}^* [\nabla_2\phi] \,,\, \nabla_2\psi\rangle_{[L^2(S)]^3} = \langle \nabla_2\phi \,,\, \boldsymbol{a}\,[\nabla_2\psi]\rangle_{[L^2(Q)]^2} = -\langle \phi \,,\, \nabla_2\cdot \boldsymbol{a}\,[\nabla_2\psi]\rangle_{[L^2(Q)]^2}$$
$$= \langle \phi \,,\, b_3\,[\nabla_2\psi]\rangle_{[L^2(Q)]^2} = \langle b_3^*\,[\phi] \,,\, \nabla_2\psi\rangle_{[L^2(S)]^3}\,.$$

Working with $\boldsymbol{a}$, $\boldsymbol{a}^*$ rather than with b_3, b_3^* could simplify. Discuss kernels, ranges, and others, of $\boldsymbol{a}$, $\boldsymbol{a}^*$ from similar considerations for b_3, b_3^*.
Observe that $\boldsymbol{a}$ could be added a silent for b_3 term $\boldsymbol{d}$, $\nabla_2\cdot\boldsymbol{d} = 0$ (divergence free).

6.3.1 Formulations of (BEP), (CPE)

(BEP) can the be stated as: find $\phi_o \in W_0^{1,2}(Q)$, $\|\nabla_2\,\phi_o\|_{[L^2(Q)]^2} \leq M$ such that

$$\min_{\phi\in W_0^{1,2}(Q),\, \|\nabla_2\,\phi\|_{[L^2(Q)]^2}\leq M} \|\boldsymbol{a}^*\,[\nabla_2\,\phi] - \boldsymbol{e}\|_{[L^2(S)]^3} = \|\boldsymbol{a}^*\,[\nabla_2\,\phi_o] - \boldsymbol{e}\|_{[L^2(S)]^3}\,.$$

The above critical point equation (CPE) stated as (6) can then be derived directly under the following form (on Q), see also [1], for $\lambda > 0$:

$$\nabla_2\cdot\boldsymbol{a}\,\boldsymbol{a}^*[\nabla_2\phi_o] + \lambda\,\nabla_2\cdot\nabla_2\phi_o = \nabla_2\cdot[\boldsymbol{a}\,\boldsymbol{a}^* + \lambda\,I]\,[\nabla_2\phi_o] = -\nabla_2\cdot\boldsymbol{a}\,[\boldsymbol{e}]\,.$$

Hence, using (3), see also Remark 2:

$$\nabla_2\cdot([\boldsymbol{a}\,\boldsymbol{a}^* + \lambda\,I]\,[\nabla_2\phi_o] + \boldsymbol{a}\,[\boldsymbol{e}]) = 0 \tag{12}$$
$$\Leftrightarrow [a\,\boldsymbol{a}^* + \lambda\,I]\,[\nabla_2\phi_o] + \boldsymbol{a}\,[\boldsymbol{e}] \perp \nabla_2\,W_0^{1,2}(Q) \text{ in } \left[L^2(Q)\right]^2\,.$$

6.3.2 Computation of $a\,[\boldsymbol{e}_i]$, $b_3\,[\boldsymbol{e}_i]$

Consider now the functions $\boldsymbol{t}\mapsto\boldsymbol{e}_i\in[L^2(S)]^3$ introduced in Section 2.4 for $i = 1,2,3$.

New Installer avant, continuer...
On Q,

$$b_3\,[\boldsymbol{e}_1] = -\frac{\mu_0}{2}\,\partial_{x_1}\,P_h\star(1\vee 0)\,.$$

For $\boldsymbol{x}\in Q$,

$$\partial_{x_1}\,P_h\star(1\vee 0)(\boldsymbol{x}) = \frac{h}{2\pi}\partial_{x_1}\iint_S \frac{d\,\boldsymbol{t}}{d_h(\boldsymbol{x}-\boldsymbol{t})^3}\,d\,\boldsymbol{t}\,,$$

$$\partial_{x_1}\iint_S \frac{d\,\boldsymbol{t}}{d_h(\boldsymbol{x}-\boldsymbol{t})^3}\,d\,\boldsymbol{t} = \iint_S \partial_{x_1}\frac{d\,\boldsymbol{t}}{d_h(\boldsymbol{x}-\boldsymbol{t})^3}\,d\,\boldsymbol{t} = -\iint_S \partial_{t_1}\frac{d\,\boldsymbol{t}}{d_h(\boldsymbol{x}-\boldsymbol{t})^3}\,d\,\boldsymbol{t}$$
$$= -\int_{-s}^{s}\left[\frac{1}{d_h(\boldsymbol{x}-\boldsymbol{t})^3}\right]_{t_1=-s}^{s} d\,t_2 = -\left[\int_{-s}^{s}\frac{d\,t_2}{d_h(\boldsymbol{x}-\boldsymbol{t})^3}\right]_{t_1=-s}^{s} = \left[\left[f_{\boldsymbol{t}}(\boldsymbol{x})\right]_{t_2=-s}^{s}\right]_{t_1=-s}^{s},$$

from [9, Prop. 1].

Continue... check:

$$b_3[\boldsymbol{e}_1](\boldsymbol{x}) = \frac{\mu_0\,h\,s}{2\,\pi}\left[\frac{1}{(x_1-s)^2+h^2} + \frac{1}{(x_1+s)^2+h^2}\right],$$

comment $b_3[\boldsymbol{e}_2]$, compute:

$$b_3[\boldsymbol{e}_3](\boldsymbol{x}) \rightsquigarrow \left[\left[((x_1-t_1)+(x_2-t_2))\;f_{\boldsymbol{t}}(\boldsymbol{x})\right]_{t_2=-s}^{s}\right]_{t_1=-s}^{s}.$$

Ex Voir...

Recall from Section 6.3 that $\nabla_2 \cdot \boldsymbol{a}\,[\boldsymbol{e}_i] = -b_3\,[\boldsymbol{e}_i]$, we have on Q:

$$\boldsymbol{a}[\boldsymbol{e}_1] = P_h \star \begin{pmatrix} 1 \vee 0 \\ 0 \end{pmatrix}, \ \boldsymbol{a}[\boldsymbol{e}_2] = P_h \star \begin{pmatrix} 0 \\ 1 \vee 0 \end{pmatrix}, \ \boldsymbol{a}[\boldsymbol{e}_3] = -P_h \star \begin{pmatrix} R_1\,[1 \vee 0] \\ R_2\,[1 \vee 0] \end{pmatrix}.$$

We make use below of results in [9, Sec. 3] concerning the functions $k_{\boldsymbol{t}}$, $\ell_{\boldsymbol{t}} : \mathbb{R}^2 \to \mathbb{R}$, defined for $\boldsymbol{t} \in \mathbb{R}^2$ by:

$$\begin{aligned} k_{\boldsymbol{t}}(\boldsymbol{x}) &= \frac{1}{h}\arctan\left(\frac{(x_1-t_1)(x_2-t_2)}{h\,d_h(\boldsymbol{x}-\boldsymbol{t})}\right), \\ \ell_{\boldsymbol{t}}(\boldsymbol{x}) &= -\mathrm{argsinh}\left(\frac{x_2-t_2}{((x_1-t_1)^2+h^2)^{1/2}}\right). \end{aligned}$$

Because $k_{\boldsymbol{t}}(\boldsymbol{x}) = k_{\boldsymbol{x}}(\boldsymbol{t})$ and $\ell_{\boldsymbol{t}}(\boldsymbol{x}) = -\ell_{\boldsymbol{x}}(\boldsymbol{t})$, we have from [9, Prop. 1]:

$$\begin{aligned} \partial_{t_1 t_2}\, k_{\boldsymbol{x}}(\boldsymbol{t}) &= 1/d_h(\boldsymbol{x}-\boldsymbol{t})^3, \\ \partial_{t_1 t_2}\, \ell_{\boldsymbol{x}}(\boldsymbol{t}) &= -(x_1-t_1)/d_h(\boldsymbol{x}-\boldsymbol{t})^3. \end{aligned}$$

Consider first at $\boldsymbol{x} \in Q$,

$$P_h \star (1 \vee 0)\,(\boldsymbol{x}) = \frac{h}{2\,\pi} \iint_S \frac{d\,\boldsymbol{t}}{d_h(\boldsymbol{x}-\boldsymbol{t})^3}.$$

From [9, Prop. 1] we have:

$$P_h \star (1 \vee 0)\,(\boldsymbol{x}) = \frac{h}{2\,\pi} \iint_S \partial_{t_1 t_2} k_{\boldsymbol{x}}(\boldsymbol{t})\,d\,\boldsymbol{t} = \frac{h}{2\,\pi}\left[\left[k_{\boldsymbol{x}}(\boldsymbol{t})\right]_{t_1=-s}^{s}\right]_{t_2=-s}^{s}.$$

Therefore,

$$\boldsymbol{a}[\boldsymbol{e}_1](\boldsymbol{x}) = \frac{h}{2\,\pi}\left[\left[k_{\boldsymbol{x}}(\boldsymbol{t})\right]_{t_1=-s}^{s}\right]_{t_2=-s}^{s}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \boldsymbol{a}[\boldsymbol{e}_2](\boldsymbol{x}) = \frac{h}{2\,\pi}\left[\left[k_{\boldsymbol{x}}(\boldsymbol{t})\right]_{t_1=-s}^{s}\right]_{t_2=-s}^{s}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\left[\left[k_{\boldsymbol{x}}(\boldsymbol{t})\right]_{t_1=-s}^{s}\right]_{t_2=-s}^{s} = k_{\boldsymbol{x}}(s,s) - k_{\boldsymbol{x}}(-s,s) - k_{\boldsymbol{x}}(s,-s) + k_{\boldsymbol{x}}(-s,-s).$$

Use of [9, Prop. 2, Lem. 1]? Ou formule de sommation des arctan?

Next, in order to compute $a[\boldsymbol{e}_3]$ at $\boldsymbol{x} \in Q$,

$$P_h \star R_i\,[1 \vee 0]\,(\boldsymbol{x}) = \frac{1}{2\,\pi}\iint_S \frac{x_i - t_i}{d_h(\boldsymbol{x}-\boldsymbol{t})^3} d\,\boldsymbol{t}.$$

From [9, Prop. 1] we have:

$$P_h \star R_1\,[1 \vee 0]\,(\boldsymbol{x}) = -\frac{1}{2\,\pi}\iint_S \partial_{t_1 t_2}\ell_{\boldsymbol{x}}(\boldsymbol{t})\,d\,\boldsymbol{t} = -\frac{1}{2\,\pi}\left[\left[\ell_{\boldsymbol{x}}(\boldsymbol{t})\right]_{t_1=-s}^{s}\right]_{t_2=-s}^{s},$$

and

$$P_h \star R_2\,[1 \vee 0]\,(\boldsymbol{x}) = -\frac{1}{2\,\pi}\left[\left[\ell_{(x_2,x_1)}(\boldsymbol{t})\right]_{t_2=-s}^{s}\right]_{t_1=-s}^{s}.$$

Therefore,

$$a[\boldsymbol{e}_3](\boldsymbol{x}) = -\frac{1}{2\pi}\begin{pmatrix} \Big[\Big[\ell_{\boldsymbol{x}}(\boldsymbol{t})\Big]_{t_1=-s}^{s}\Big]_{t_2=-s}^{s} \\ \Big[\Big[\ell_{(x_2,x_1)}(\boldsymbol{t})\Big]_{t_2=-s}^{s}\Big]_{t_1=-s}^{s} \end{pmatrix} = -\frac{1}{2\pi}\Bigg[\Bigg[\begin{pmatrix} \ell_{\boldsymbol{x}}(\boldsymbol{t}) \\ \ell_{(x_2,x_1)}(\boldsymbol{t}) \end{pmatrix}\Bigg]_{t_1=-s}^{s}\Bigg]_{t_2=-s}^{s}.$$

Continue computations.
We also see from the above computations that for $\psi \in W_0^{1,2}(Q)$, $i = 1, 2$,

$$-\langle b_3\,[\boldsymbol{e}_i]\;,\;\psi\rangle_{L^2(Q)} = \langle a\,[\boldsymbol{e}_i]\;,\;\nabla_2\,\psi\rangle_{[L^2(Q)]^2}$$

$$= \frac{h}{\pi}\iint_Q \partial_{x_i}\psi(\boldsymbol{x})\iint_S \frac{d\,\boldsymbol{t}}{d_h(\boldsymbol{x}-\boldsymbol{t})^3}\,d\,\boldsymbol{x} = -\frac{h}{\pi}\iint_S\iint_Q \psi(\boldsymbol{x})\,\partial_{x_i}\frac{1}{d_h(\boldsymbol{x}-\boldsymbol{t})^3}\,d\,\boldsymbol{x}\,d\,\boldsymbol{t}$$

$$= \begin{cases} -\frac{h}{\pi}\iint_S\iint_Q \psi(\boldsymbol{x})\,f_{(t_2,t_1)}(x_2,x_1)\,d\,\boldsymbol{x}\,d\,\boldsymbol{t}\,,\; i=1\,, \\ \\ -\frac{h}{\pi}\iint_S\iint_Q \psi(\boldsymbol{x})\,f_{\boldsymbol{t}}(\boldsymbol{x})\,d\,\boldsymbol{x}\,d\,\boldsymbol{t}\,,\; i=2\,. \end{cases}$$

check signs $\cdots$, do $i = 3$
$\rightsquigarrow$ functions $f_{\boldsymbol{t}}$, $g_{\boldsymbol{t}}$, $f_{\boldsymbol{t}}(\boldsymbol{x}) = -f_{\boldsymbol{x}}(\boldsymbol{t})$, $g_{\boldsymbol{t}}(\boldsymbol{x}) = g_{\boldsymbol{x}}(\boldsymbol{t})$; links between $f_{\boldsymbol{t}}(\boldsymbol{x})$ and $\partial_{x_1}\,k_{\boldsymbol{t}}(\boldsymbol{x})$, $g_{\boldsymbol{t}}(\boldsymbol{x})$ and $\partial_{x_2}\,\ell_{\boldsymbol{t}}(\boldsymbol{x})$ from:

$$\partial_{x_2}\,f_{\boldsymbol{t}}(\boldsymbol{x}) = \partial_{x_1x_2}\,k_{\boldsymbol{t}}(\boldsymbol{x})\,,\;\; \partial_{x_1}\,g_{\boldsymbol{t}}(\boldsymbol{x}) = \partial_{x_1x_2}\,\ell_{\boldsymbol{t}}(\boldsymbol{x})\,.$$

6.3.3 Other possibility to solve (CPE)

Remettre en b_3 et b_3^*, use Fourier basis.

From (12) together with the use of a suitable (complete) family of test functions like, with $Q = [-R, R]^2$:

$$\Psi(\boldsymbol{x}) = (x_1^2 - R^2)\,(x_2^2 - R^2)\,\psi(\boldsymbol{x}) \in W_0^{1,2}(Q)\,,$$

for $\psi \in W^{1,2}(Q)$ polynomials? See [9] and related computations (use Sylvain's Maple *primitives-utiles.mw*).
Indeed, for such Ψ, we get

$$\langle [a\,\boldsymbol{a}^* + \lambda\,I]\;[\nabla_2\phi_o] + a\,[\boldsymbol{e}]\,,\,\nabla_2\Psi\rangle_{[L^2(Q)]^2} = 0$$

whence

$$(1+\lambda)\,\langle\nabla_2\phi_o\,,\,a\,\boldsymbol{a}^*[\nabla_2\Psi]\rangle_{[L^2(Q)]^2} = -\langle a\,[\boldsymbol{e}]\,,\,\nabla_2\Psi\rangle_{[L^2(Q)]^2} = -\langle\boldsymbol{e}\,,\,\boldsymbol{a}^*\,[\nabla_2\Psi]\rangle_{[L^2(S)]^3} \quad (13)$$

for the unique $\lambda > 0$ such that

$$\|\nabla_2\phi_o\|_{[L^2(Q)]^2} = M\,.$$

Section 6.3.2 for $a[\boldsymbol{e}]$.

6.4 About $\boldsymbol{m}$ and other extremal problem

Comment about dual roles of Q and S.

- Tests for unidirectionality: what is specific about unidirectional $\boldsymbol{m}_u$? Do they belong to $\overline{\operatorname{Ran} b_3^*}$?
- Tests for $\boldsymbol{m} = \boldsymbol{m}_E$ with (restricted) compact support in $\overline{E} \subset S$? Recovery of partial moments?
- What is special in Sections 2.4, 2.3, 3 whenever:

  - $\boldsymbol{m} \in \overline{\operatorname{Ran} b_3^*} = \mathcal{D}_S^{\perp}$? $\boldsymbol{m} \in \operatorname{Ran} b_3^*$? $\boldsymbol{m} \in b_3^* \left[W_0^{1,2}(Q)\right]$? Consider $\boldsymbol{m}_u$, $\boldsymbol{m}_E$, action of $b_3\, b_3^*$ (or of $\boldsymbol{a}\,\boldsymbol{a}^*$).

  - $\boldsymbol{m}$ smooth (for instance extended from $\boldsymbol{m}_E$ if E smooth): $\boldsymbol{m} \in \left[W_0^{1,2}(S)\right]^3$?
- Links with dual bounded extremal problem on magnetization (see work by Doug and Michael): minimize
$$\left\| b_3[\boldsymbol{m}] - b_3^d \right\|_{L^2(Q)}$$
among constrained $\boldsymbol{m} \in \left[L^2(S)\right]^3$, or mixed problem? Look at $b_3^*\, b_3$ or $\boldsymbol{a}^*\, \boldsymbol{a}$.

Complete bibliography below; see [2].

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