Asymptotic moments recovery from B_z

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1 Notations

Let $\mathcal{A} = [-s, s]^2 \subset \mathbb{R}^2$. Let (m_1, m_2, m_3), such that for $i = 1, 2, 3$, $m_i \in L^1(\mathcal{A})$ has planar support included in $\mathcal{A} \times \{0\}$. For any m defined on $\mathcal{A}$, we denote by $\langle m \rangle$ the moment of m:

$$\langle m \rangle = \iint_{\mathcal{A}} m(t_1, t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

In the following, we will denote by Q_R the square $[-R, R]^2$. The height h of the measurement plane is set once and for all. We will denote by $d_h(x, y)$ the quantity $\sqrt{x^2 + y^2 + h^2}$. Most of the time, since h is fixed, we will simply write $d(x, y)$.
We recall that $\mathbf{B} = -\mu_0 \nabla \phi$ where $\mu_0 = 4\pi \times 10^{-7}$ and ϕ is defined at any point (x, y, h) with $h \neq 0$ by

$$\phi(x, y, h) = \frac{1}{4\pi} \iint_{\mathcal{A}} \frac{m_1(t_1, t_2)(x - t_1)}{d_h(x - t_1, y - t_2)^3} + \frac{m_2(t_1, t_2)(y - t_2)}{d_h(x - t_1, y - t_2)^3} + \frac{m_3(t_1, t_2)h}{d_h(x - t_1, y - t_2)^3}\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

We can also write ϕ by means of Poisson kernel and Riesz transforms, namely,

$$\phi = \frac{1}{2}\left(P_h \star R_1(m_1) + P_h \star R_2(m_2) + P_h \star m_3\right),$$

where $P_h(x, y) = \frac{1}{2\pi} \cdot \frac{h}{d_h(x,y)^3}$.

2 Results

The following properties hold:

$$\iint_{Q_R} x\,B_z(x, y, h)\,\mathrm{d}x\,\mathrm{d}y = \frac{\mu_0}{2}\langle m_1 \rangle + \frac{3\mu_0}{\pi R\sqrt{2}}\left(\langle t_1\,m_3 \rangle - h\langle m_1 \rangle\right) + \mathcal{O}(1/R^3), \quad (1)$$

$$\iint_{Q_R} y\,B_z(x, y, h)\,\mathrm{d}x\,\mathrm{d}y = \frac{\mu_0}{2}\langle m_2 \rangle + \frac{3\mu_0}{\pi R\sqrt{2}}\left(\langle t_2\,m_3 \rangle - h\langle m_2 \rangle\right) + \mathcal{O}(1/R^3), \quad (2)$$

$$\iint_{Q_R} R\,B_z(x, y, h)\,\mathrm{d}x\,\mathrm{d}y = \frac{2\mu_0}{\pi\sqrt{2}}\langle m_3 \rangle + \mathcal{O}(1/R^2). \quad (3)$$

For the proof, we need the following Lemma.

Lemma 1. *For any $i = 1, 2, 3$ and $j = 1, 2$ the following equations hold:*

$$\iint_{Q_R} x\,\partial_x\,(P_h \star m_i)\,(x,y)\,\mathrm{d}x\,\mathrm{d}y = -\langle m_i\rangle + \frac{6h}{\pi R\sqrt{2}}\langle m_i\rangle + \mathcal{O}(1/R^3), \tag{4}$$

$$\iint_{Q_R} R\,\partial_x\,(P_h \star m_i)\,(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{5h}{\pi R^2\sqrt{2}}\langle t_1\, m_i\rangle + \mathcal{O}(1/R^4), \tag{5}$$

$$\iint_{Q_R} x\,\partial_y\,(P_h \star m_i)\,(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{7h}{2\pi R^3\sqrt{2}}\,\langle t_1\, t_2\, m_i\rangle + \mathcal{O}(1/R^4), \tag{6}$$

$$\begin{aligned}\iint_{Q_R} x\,\partial_x\,(P_h \star (R_j\, m_i))\,(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \frac{\langle t_2\, m_i\rangle}{\pi R\sqrt{2}} + \mathcal{O}(1/R^3) \textit{ for } j=2 \\ &= \frac{5\langle t_1\, m_i\rangle}{\pi R\sqrt{2}} + \mathcal{O}(1/R^3) \textit{ for } j=1\end{aligned} \tag{7}$$

$$\iint_{Q_R} R\,\partial_x\,(P_h \star (R_1\, m_i))\,(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{2\langle m_i\rangle}{\pi\sqrt{2}} + \mathcal{O}(1/R^2), \tag{8}$$

$$\begin{aligned}\iint_{Q_R} x\,\partial_y\,(P_h \star (R_j\, m_i))\,(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \frac{\langle t_2\, m_i\rangle}{\pi R\sqrt{2}} + \mathcal{O}(1/R^3) \textit{ for } j=1 \\ &= \frac{\langle t_1\, m_i\rangle}{\pi R\sqrt{2}} + \mathcal{O}(1/R^3) \textit{ for } j=2\end{aligned} \tag{9}$$

Proof.
Equation (4)**:** we begin with an integration by parts:

$$\begin{aligned}\iint_{Q_R} x\,\partial_x\,(P_h \star m_i)\,(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-R}^{R} \left[x\,(P_h \star m_i)\,(x,y)\right]_{x=-R}^{R}\,\mathrm{d}y \\ &\quad - \iint_{Q_R} (P_h \star m_i)(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Using Fubini we have

$$\begin{aligned}\iint_{Q_R} (P_h \star m_i)(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{\mathbb{R}^2} m_i(t_1,t_2)\left(\iint_{Q_R} \frac{h}{2\pi}\cdot\frac{1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}t_1\,\mathrm{d}t_2 \\ &= \iint_{\mathbb{R}^2} m_i(t_1,t_2)\left(1 - \frac{2h\sqrt{2}}{R\pi} + \frac{h}{2\pi}\,\widetilde{\delta_4}(R,t_1,t_2)\right)\mathrm{d}t_1\,\mathrm{d}t_2 \\ &= \left(1 - \frac{2h\sqrt{2}}{R\pi}\right)\langle m_i\rangle + \frac{h}{2\pi}\iint_{\mathbb{R}^2} m_i(t_1,t_2)\,\widetilde{\delta_4}(R,t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2.\end{aligned}$$

When $R > C$, this last term is bounded in absolute value by $100(h^2+s^2)h^2\langle|m_i|\rangle/(\pi R^3)$. Regarding the other term, we observe that, using Fubini,

$$\int_{-R}^{R} \left[x\,(P_h \star m_i)\,(x,y)\right]_{x=-R}^{R}\,\mathrm{d}y = \frac{h}{2\pi}\iint_{\mathcal{A}} m_i(t_1,t_2)\left[x\int_{-R}^{R}\frac{\mathrm{d}y}{d(x-t_1,y-t_2)^3}\right]_{x=-R}^{R}.$$

$$= \frac{2h}{R\pi\sqrt{2}}\langle m_i\rangle + \frac{33h\langle t_1^2\, m_i\rangle}{4\pi R^3\sqrt{2}} - \frac{3h\langle t_2^2\, m_i\rangle}{4\pi R^3\sqrt{2}} - \frac{10h^3\langle m_i\rangle}{4\pi R^3\sqrt{2}} + \varepsilon$$

where $|\varepsilon| \le 100(h^2+10s^2)hs\langle|m_i|\rangle/(\pi R^4)$.
Equation (5)**:** we simply observe that

$$\iint_{Q_R} R\,\partial_x\left(P_h \star m_i\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y = R\int_{-R}^{R} \left[\left(P_h \star m_i\right)(x,y)\right]_{x=-R}^{R}\mathrm{d}y$$

and we apply Fubini as in the previous case. Therefore

$$\iint_{Q_R} R\,\partial_x\left(P_h \star m_i\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{5h}{\pi R^2\sqrt{2}}\langle t_1\,m_i\rangle + \varepsilon$$

where $|\varepsilon| \le 100(h^2+10s^2)hs\langle|m_i|\rangle/(\pi R^4)$.
Equation (6)**:** we integrate first over y and then use Fubini:

$$\begin{aligned}\iint_{Q_R} x\,\partial_y\left(P_h \star m_i\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-R}^{R} x\left[\left(P_h \star m_i\right)(x,y)\right]_{y=-R}^{R}\mathrm{d}x\\ &= \frac{h}{2\pi}\iint_{\mathcal{A}} m_i(t_1,t_2)\left(\int_{-R}^{R} x\left[\frac{1}{d(x-t_1,y-t_2)^3}\right]_{y=-R}^{R}\mathrm{d}x\right)\mathrm{d}t_1\,\mathrm{d}t_2.\end{aligned}$$

We now focus on the inner integral:

$$\begin{aligned}\int_{-R}^{R} x\left[\frac{1}{d(x-t_1,y-t_2)^3}\right]_{y=-R}^{R}\mathrm{d}x &= \left[\int_{-R}^{R}\frac{x-t_1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}x\right]_{y=-R}^{R}\\ &\quad + \left[\int_{-R}^{R}\frac{t_1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}x\right]_{y=-R}^{R}.\end{aligned}$$

We use the asymptotic expansions that we already computed, simply swapping the roles of x and y. So we have

$$\int_{-R}^{R}\frac{x}{d(x-t_1,R-t_2)^3}\,\mathrm{d}x = \frac{t_1}{\sqrt{2}}\cdot\frac{1}{R^2} + \frac{7t_1t_2}{2\sqrt{2}}\cdot\frac{1}{R^3} + \mathcal{O}(1/R^4)$$

and

$$\int_{-R}^{R}\frac{x}{d(x-t_1,R+t_2)^3}\,\mathrm{d}x = \frac{t_1}{\sqrt{2}}\cdot\frac{1}{R^2} - \frac{7t_1t_2}{2\sqrt{2}}\cdot\frac{1}{R^3} + \mathcal{O}(1/R^4)$$

Finally we get

$$\iint_{Q_R} x\,\partial_y\left(P_h \star m_i\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{7h}{2\pi R^3\sqrt{2}}\langle t_1\,t_2\,m_i\rangle + \mathcal{O}(1/R^4)$$

Equation (7)**:** we begin with the case $j=2$. Integrating by parts and using Fubini, we get

$$\begin{aligned}&\iint_{Q_R} x\,\partial_x\left(P_h \star (R_2\,m_i)\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad= \frac{1}{2\pi}\iint_{\mathcal{A}} m_i(t_1,t_2)\left[x\int_{-R}^{R}\frac{y-t_2}{d(x-t_1,y-t_2)^3}\,\mathrm{d}y\right]_{x=-R}^{R}\mathrm{d}t_1\,\mathrm{d}t_2\\ &\quad -\frac{1}{2\pi}\iint_{\mathcal{A}} m_i(t_1,t_2)\left(\int_{-R}^{R}\left(\int_{-R}^{R}\frac{y-t_2}{d(x-t_1,y-t_2)^3}\,\mathrm{d}y\right)\mathrm{d}x\right)\mathrm{d}t_1\,\mathrm{d}t_2.\end{aligned}$$

Therefore,

$$\iint_{Q_R} x\,\partial_x\left(P_h\star(R_2\,m_i)\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{\langle t_2\,m_i\rangle}{\pi R\sqrt{2}} + \varepsilon$$

where $|\varepsilon| \le 200(h^2+50s^2)s\langle|m_i|\rangle/(\pi R^3)$.
Regarding the case $j=1$, we begin the same way:

$$\begin{aligned}
&\iint_{Q_R} x\,\partial_x\left(P_h\star(R_1\,m_i)\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{2\pi}\iint_{\mathcal{A}} m_i(t_1,t_2)\left[x(x-t_1)\int_{-R}^{R}\frac{1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}y\right]_{x=-R}^{R}\mathrm{d}t_1\,\mathrm{d}t_2\\
&\qquad -\frac{1}{2\pi}\iint_{\mathcal{A}} m_i(t_1,t_2)\left(\int_{-R}^{R}\left(\int_{-R}^{R}\frac{x-t_1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}t_1\,\mathrm{d}t_2.
\end{aligned}$$

Proceeding as before we get

$$\iint_{Q_R} x\,\partial_x\left(P_h\star(R_1\,m_i)\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{5\langle t_1\,m_i\rangle}{\pi R\sqrt{2}} + \varepsilon$$

where $|\varepsilon| \le 200(h^2+50s^2)s\langle|m_i|\rangle/(\pi R^3)$.
Equation (8)**:** To establish this equation, we integrate first over x:

$$\iint_{Q_R} R\,\partial_x\left(P_h\star(R_1\,m_i)\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y = R\int_{-R}^{R}\left[(P_h\star(R_1\,m_i))(x,y)\right]_{x=-R}^{R}\mathrm{d}y.$$

Then, we use Fubini and integrate over y:

$$\begin{aligned}
&R\int_{-R}^{R}\left[(P_h\star(R_1\,m_i))(x,y)\right]_{x=-R}^{R}\mathrm{d}y\\
&= \frac{1}{2\pi}\iint_{\mathcal{A}} m_i(t_1,t_2)\left[R(x-t_1)\int_{-R}^{R}\frac{1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}y\right]_{x=-R}^{R}\mathrm{d}t_1\,\mathrm{d}t_2\\
&\qquad = \frac{2\langle m_i\rangle}{\pi\sqrt{2}} + \mathcal{O}(1/R^2).
\end{aligned}$$

Equation (9)**:** by remarking that $\partial_y(P_h\star(R_1 m_i)) = \partial_x(P_h\star(R_2 m_i))$, we observe that the case $j=1$ is actually the same as Equation (7) with $j=2$. Regarding the case $j=2$, we first integrate over y and then use Fubini:

$$\begin{aligned}
&\iint_{Q_R} x\,\partial_y\left(P_h\star(R_j\,m_i)\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad = \frac{1}{2\pi}\iint_{\mathcal{A}} m_i(t_1,t_2)\left(\int_{-R}^{R} x\left[\frac{y-t_2}{d(x-t_1,y-t_2)^3}\right]_{y=-R}^{R}\mathrm{d}x\right)\mathrm{d}t_1\,\mathrm{d}t_2\\
&\qquad = \frac{-\langle t_2\,m_i\rangle}{\pi R\sqrt{2}} + \mathcal{O}(1/R^3).
\end{aligned}$$

□

Proof of Equations (1), (2) *and* (3). First of all, we observe that

$$\begin{aligned}
-\frac{2}{\mu_0}\, B_z(x,y,h) &= \partial_z\left(P_z \star (R_1\, m_1) + P_z \star (R_2\, m_2) + P_z \star m_3\right)_{|z=h}(x,y) \\
&= \partial_x(P_h \star m_1 - P_h \star (R_1\, m_3))(x,y) + \partial_y(P_h \star m_2 - P_h \star (R_2\, m_3))(x,y).
\end{aligned}$$

Using Equations (4), (6), (7) and (9), we hence have

$$\iint_{Q_R} x\, B_z(x,y,h)\, \mathrm{d}x\, \mathrm{d}y = \frac{-\mu_0}{2}\left(-\langle m_1\rangle + \frac{6}{\pi R\sqrt{2}}(h\langle m_1\rangle - \langle t_1\, m_3\rangle)\right) + \mathcal{O}(1/R^3).$$

By permutation of variables x and y we get Equation (2). Finally, using (5), (8) and their symmetrical formulas when variables x and y are permuted, we have

$$\iint_{Q_R} R\, B_z(x,y,h)\, \mathrm{d}x\, \mathrm{d}y = \frac{2\mu_0}{\pi\sqrt{2}}\langle m_3\rangle + \mathcal{O}(1/R^2).$$

□

3 Technical lemmas

This technical lemma explains how to compose Taylor expansions of order 2 with rigorous bounds. It will be used several times for the proof of Lemma 4.

Lemma 2. *Consider a function g whose Taylor expansion at order 2 is given by $g(y) = b_0 + b_1 y + b_2 y^2 + \varepsilon_g(y)$. We suppose that $\forall y \in [-1/2, 1/2], |\varepsilon_g(y)| \le B_3\,|y|^3$ for some constant B_3. We consider two real numbers a_1 and a_2 and corresponding bounds A_1 and A_2 such that $|a_1| \le A_1$ and $|a_2| \le A_2$. Finally, we consider a function $\varepsilon_f : \mathbb{R} \to \mathbb{R}$.*
For any $x \in \mathbb{R}$ such that

i) $|\varepsilon_f(x)| \le A_2\,|x|^2$,

ii) $|a_2 x^2 + \varepsilon_f(x)| \le A_1\,|x|$,

iii) $2A_1|x| \le 1/2$,

we have $g(a_1 x + a_2 x^2 + \varepsilon_f(x)) = b_0 + b_1 a_1 x + (b_1 a_2 + b_2 a_1^2)x^2 + \varepsilon(x)$ where

$$|\varepsilon(x)| \le |b_1|\cdot|\varepsilon_f(x)| + (6A_1A_2|b_2| + 8B_3A_1^3)\,|x|^3.$$

Remark that condition ii) can sometimes conveniently be replaced by $2A_2\,|x|^2 \le A_1|x|$ since that condition together with condition i) imply condition ii).

Proof. Let x be a real number satisfying the three conditions. From the definition of g, we get

$$\varepsilon(x) = b_1\varepsilon_f(x) + b_2(2a_1x + a_2x^2 + \varepsilon_f(x))(a_2x^2 + \varepsilon_f(x)) + \varepsilon_g(a_1x + a_2x^2 + \varepsilon_f(x)).$$

Triangle inequality and point i), we get $|a_2x^2 + \varepsilon_f(x)| \le 2A_2|x|^2$. Moreover, triangle inequality and point ii) show that $|2a_1x + a_2x^2 + \varepsilon_f(x)| \le 2A_1|x| + A_1|x| = 3A_1|x|$.
The same argument shows that $|a_1x + a_2x^2 + \varepsilon_f(x)| \le 2A_1|x|$ which is smaller than 1/2 by point iii). Therefore, $|\varepsilon_g(a_1x + a_2x^2 + \varepsilon_f(x))| \le B_3(2A_1|x|)^3 = 8B_3A_1^3|x|^3$. □

The following lemma recalls some Taylor expansions with rigorous bounds for their reminders. They will be used in the proof of Lemma 4.

Lemma 3. *The following estimates hold:*

$$\forall x \in [-1/2, 5/8], \quad \frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + \varepsilon_1(x) \text{ with } |\varepsilon_1(x)| \leq |x|^3, \tag{10}$$

$$\forall x \in [-1/2, 1/2], \quad \frac{1}{1+x} = 1 - x + x^2 + \varepsilon_2(x) \text{ with } |\varepsilon_2(x)| \leq 2|x|^3. \tag{11}$$

$$\forall x \in [-1/2, 1/2], \quad \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \varepsilon_3(x) \text{ with } |\varepsilon_3(x)| \leq \frac{1}{8}|x|^3. \tag{12}$$

$$\forall x \in [-1/2, 1/2], \quad \arctan(x) = x + \varepsilon_4(x) \text{ with } |\varepsilon_4(x)| \leq \frac{1}{2}|x|^3. \tag{13}$$

$$\forall x \in [-1/2, 1/2], \quad \operatorname{argsinh}(1+x) = \operatorname{argsinh}(1) + \frac{x}{\sqrt{2}} - \frac{x^2}{4\sqrt{2}} + \varepsilon_5(x)$$
$$\text{with } |\varepsilon_5(x)| \leq \frac{5}{8\sqrt{2}}|x|^3. \tag{14}$$

Proof.
Equation (10)**:** observe that, for all $x > -1$,

$$\varepsilon_1(x) = \frac{1 - \left(1 - \frac{x}{2} + \frac{3x^2}{8}\right)\sqrt{1+x}}{\sqrt{1+x}} = \frac{1 - \left(1 - \frac{x}{2} + \frac{3x^2}{8}\right)^2(1+x)}{\sqrt{1+x}\,\left(1 + \left(1 - \frac{x}{2} + \frac{3x^2}{8}\right)\sqrt{1+x}\right)},$$

whence, after simplification,

$$\varepsilon_1(x) = \frac{-40 + 15x - 9x^2}{\sqrt{1+x} + \left(1 - \frac{x}{2} + \frac{3x^2}{8}\right)(1+x)} \cdot \frac{x^3}{64}.$$

It is easy to see that $|-40 + 15x - 9x^2|$ reaches its maximum over $[-1/2, 5/8]$ at $x = -1/2$ where it is equal to $199/4$ which we conveniently bound by 50. The polynomial $(1 - \frac{x}{2} + \frac{3x^2}{8})(1+x)$ reaches its minimum at $x = -1/2$ where it is equal to $43/64$. Finally, $\sqrt{1+x} \geq \sqrt{1/2} \geq 45/64$. Altogether we see that $|\varepsilon_1(x)| \leq \frac{50}{88}|x|^3 \leq |x|^3$.
Equation (11)**:** by definition of ε_2, we have

$$\varepsilon_2(x) = \frac{1}{1+x} - (1 - x + x^2) = \frac{-x^3}{1+x}.$$

We conclude by remarking that $|1+x| \geq 1/2$ when $x \in [-1/2, 1/2]$.
Equation (12)**:** by definition of ε_3, we have

$$\varepsilon_3(x) = \sqrt{1+x} - \left(1 + \frac{x}{2} - \frac{x^2}{8}\right) = \frac{8 - x}{\sqrt{1+x} + \left(1 + \frac{x}{2} - \frac{x^2}{8}\right)} \cdot \frac{x^3}{64}.$$

Moreover, when $x \in [-1/2, 1/2]$, $\sqrt{1+x} + 1 + \frac{x}{2} - \frac{x^2}{8} \geq \sqrt{\frac{1}{2}} + 1 - \frac{1}{4} - \frac{1}{32} \geq \frac{91}{64}$ and $|8 - x| \leq \frac{17}{2}$, hence $|\varepsilon_3(x)| \leq \frac{17}{182}|x|^3 \leq \frac{|x|^3}{8}$.

Equation (13)**:** we observe that, for any $t \in [-1/2, 1/2]$,

$$\frac{1}{1+t^2} = 1 + \widetilde{\varepsilon_4}(t)$$

where $|\widetilde{\varepsilon_4}(t)| = |\frac{-t^2}{1+t^2}| \leq |t|^2$. Now, for any $x \in [-1/2, 1/2]$ we have, by integration between 0 and x, $\arctan(x) = x + \varepsilon_4(x)$ where

$$|\varepsilon_4(x)| = \left|\int_0^x \widetilde{\varepsilon_4}(t)\,\mathrm{d}t\right| \leq \frac{1}{3}|x|^3 \leq \frac{1}{2}\,|x|^3.$$

Equation (14)**:** using Equation (10) we get, for any $x \in [-1/2, 5/8]$,

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \widetilde{\varepsilon_1}(x) \text{ with } |\widetilde{\varepsilon_1}(x)| = \left|\frac{3x^2}{8} + \varepsilon_1(x)\right| \leq |x|^2\left(\frac{3}{8} + |x|\right) \leq |x|^2.$$

Now, for any $t \in [-1/2, 1/2]$, $\sqrt{1+(1+t)^2} = \sqrt{2}\sqrt{1+t+\frac{t^2}{2}}$ and $t + \frac{t^2}{2} \in [-3/8, 5/8]$. Hence, using the above expansion,

$$\frac{1}{\sqrt{1+(1+t)^2}} = \frac{1}{\sqrt{2}}\left(1 - \frac{t}{2} - \frac{t^2}{4} + \widetilde{\varepsilon_1}\left(t + \frac{t^2}{2}\right)\right) = \frac{1}{\sqrt{2}}\left(1 - \frac{t}{2} + \widetilde{\varepsilon_5}(t)\right)$$

where $\widetilde{\varepsilon_5}(t) = -\frac{t^2}{4} + \widetilde{\varepsilon_1}\left(t + \frac{t^2}{2}\right)$. Since $|t + t^2/2| \leq 5|t|/4$ for $t \in [-1/2, 1/2]$ we see that $|\widetilde{\varepsilon_5}(t)| \leq 29|t|^2/16$. Integrating between 0 and x, where $x \in [-1/2, 1/2]$ we get

$$\operatorname{argsinh}(1+x) - \operatorname{argsinh}(1) = \frac{x}{\sqrt{2}} - \frac{x^2}{4\sqrt{2}} + \varepsilon_5(x)$$

where $|\varepsilon_5(x)| \leq \left|\frac{1}{\sqrt{2}}\int_0^x \frac{29}{16}t^2\,\mathrm{d}t\right| = \frac{29}{48\sqrt{2}}|x|^3 \leq \frac{5}{8\sqrt{2}}|x|^3$. This proves the third equation of the Lemma. □

Lemma 4. *There is a constant C that depends only on s such that, for any $(t_1, t_2) \in \mathcal{A}$,*

$$\frac{1}{d(R-t_1, R-t_2)} = \frac{1}{\sqrt{2}} \cdot \frac{1}{R} + \frac{t_1+t_2}{2\sqrt{2}} \cdot \frac{1}{R^2} + \frac{t_1^2 + t_2^2 + 6t_1t_2 - 2h^2}{8\sqrt{2}} \cdot \frac{1}{R^3} + \delta_1(R, t_1, t_2),$$

$$\begin{aligned}\frac{R-t_2}{(R-t_1)^2+h^2} \cdot \frac{1}{d(R-t_1, R-t_2)} = \frac{1}{\sqrt{2}} \cdot \frac{1}{R^2} + \frac{5t_1 - t_2}{2\sqrt{2}} \cdot \frac{1}{R^3}& \\ + \frac{33t_1^2 - 3t_2^2 - 6t_1t_2 - 10h^2}{8\sqrt{2}} \cdot \frac{1}{R^4} + \delta_2(R, t_1, t_2),&\end{aligned}$$

$$\begin{aligned}\operatorname{argsinh}\left(\frac{R-t_1}{\sqrt{(R-t_2)^2+h^2}}\right) = \operatorname{argsinh}(1) + \frac{t_2 - t_1}{\sqrt{2}} \cdot \frac{1}{R} + \frac{3t_2^2 - t_1^2 - 2t_1t_2 - 2h^2}{4\sqrt{2}} \cdot \frac{1}{R^2}& \\ + \delta_3(R, t_1, t_2),&\end{aligned}$$

$$\arctan\left(\frac{h\sqrt{(R-t_1)^2 + (R-t_2)^2 + h^2}}{(R-t_1)(R-t_2)}\right) = h\sqrt{2}\,\frac{1}{R} + \frac{h\sqrt{2}(t_1+t_2)}{2} \cdot \frac{1}{R^2} + \delta_4(R, t_1, t_2).$$

where, for any $R \geq C$, $|\delta_1(R,t_1,t_2)| \leq \frac{(9h^2+274s^2)s}{4\sqrt{2}} \cdot \frac{1}{R^4}$, $|\delta_2(R,t_1,t_2)| \leq \frac{(87h^2+954s^2)s}{4\sqrt{2}} \cdot \frac{1}{R^5}$, $|\delta_3| \leq \frac{(113h^2+1287s^2)s}{2\sqrt{2}} \cdot \frac{1}{R^3}$ and $|\delta_4| \leq \frac{(136h^2+563s^2)h\sqrt{2}}{16} \cdot \frac{1}{R^3}$.
The bounds for $|\delta_1|$, $|\delta_2|$, $|\delta_3|$ and $|\delta_4|$ are not sharp, so we will conveniently use the following bounds in the sequel: $|\delta_1| \leq \frac{50(h^2+10s^2)s}{R^4}$, $|\delta_2| \leq \frac{50(h^2+10s^2)s}{R^5}$, $|\delta_3| \leq \frac{50(h^2+10s^2)s}{R^3}$ and $|\delta_4| \leq \frac{50(h^2+10s^2)h}{R^3}$.

Proof. First, we remark that one may find constants C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11} and C_{12} that depend only on h and s such that, for any $(t_1,t_2) \in \mathcal{A}$:

$$\forall R \geq C_1, \quad \frac{4s}{R} \leq \frac{1}{2} \tag{15}$$
$$\forall R \geq C_2, \quad \frac{h^2+2s^2}{2R^2} \leq \frac{2s}{R} \tag{16}$$
$$\forall R \geq C_3, \quad \frac{h^2+s^2}{R^2} \leq \frac{2s}{R} \tag{17}$$
$$\forall R \geq C_4, \quad \frac{(9h^2+274s^2)s}{4R^3} \leq \frac{h^2+4s^2}{4R^2} \tag{18}$$
$$\forall R \geq C_5, \quad \frac{h^2+4s^2}{2R^2} \leq \frac{s}{R} \tag{19}$$
$$\forall R \geq C_6, \quad \frac{(12h^2+140s^2)s}{R^3} \leq \frac{h^2+3s^2}{R^2} \tag{20}$$
$$\forall R \geq C_7, \quad \frac{2h^2+6s^2}{R^2} \leq \frac{s}{R} \tag{21}$$
$$\forall R \geq C_8, \quad \frac{(44h^2+478s^2)s}{R^3} \leq \frac{h^2+4s^2}{2R^2} \tag{22}$$
$$\forall R \geq C_9, \quad \frac{h^2+4s^2}{R^2} \leq \frac{2s}{R} \tag{23}$$
$$\forall R \geq C_{10}, \quad \frac{(3h^2+38s^2)s}{4R^3} \leq \frac{h^2+2s^2}{4R^2}, \tag{24}$$
$$\forall R \geq C_{11}, \quad \frac{8h^2+563s^2}{16R^2} \leq \frac{s}{R}, \tag{25}$$
$$\forall R \geq C_{12}, \quad \frac{2h\sqrt{2}}{R} \leq \frac{1}{2}. \tag{26}$$

We define C as $C = \max\{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}\}$ and we further consider an arbitrary $R \geq C$ and an arbitrary $(t_1,t_2) \in \mathcal{A}$.
Now,

$$(R-t_1)^2 + (R-t_2)^2 + h^2 = 2R^2\left(1 - \frac{t_1+t_2}{R} + \frac{h^2+t_1^2+t_2^2}{2R^2}\right),$$

hence $\frac{1}{d(R-t_1,R-t_2)} = \frac{1}{R\sqrt{2}}\, g\left(-\frac{t_1+t_2}{R} + \frac{h^2+t_1^2+t_2^2}{2R^2}\right)$ where $g : y \mapsto 1/\sqrt{1+y}$. Therefore using Lemma 2 with Equation (10) (Equations (16) and (15) guarantee that the hypotheses of the lemma are fulfilled) we get

$$\frac{1}{d(R-t_1,R-t_2)} = \frac{1}{R\sqrt{2}}\left(1 + \frac{t_1+t_2}{2R} + \frac{t_1^2+t_2^2+6t_1t_2-2h^2}{8R^2}\right) + \varepsilon_6(R,t_1,t_2)$$

where $|\varepsilon_6(R,t_1,t_2)| \leq \frac{(9h^2+274s^2)s}{4R^3}$. This proves the first equation of the lemma.
Besides, let us remark that $(R-t_1)^2+h^2 = R^2\left(1-\frac{2t_1}{R}+\frac{h^2+t_1^2}{R^2}\right)$. Therefore, using Lemma 2 with Equation (11), we get

$$\frac{1}{(R-t_1)^2+h^2} = \frac{1}{R^2}\left(1+\frac{2t_1}{R}+\frac{3t_1^2-h^2}{R^2}+\varepsilon_7(R,t_1,t_2)\right)$$

with $|\varepsilon_7(R,t_1,t_2)| \leq \frac{(12h^2+140s^2)s}{R^3}$. The hypotheses of Lemma 2 were satisfied thanks to Equations (17) and (15).
Altogether, we get

$$\begin{aligned}
&\frac{R-t_2}{(R-t_1)^2+h^2}\cdot\frac{1}{d(R-t_1,R-t_2)}\\
&\quad=\frac{1}{R^2\sqrt{2}}\left(1-\frac{t_2}{R}\right)\left(1+\frac{2t_1}{R}+\frac{3t_1^2-h^2}{R^2}+\varepsilon_7(R,t_1,t_2)\right)\\
&\qquad\qquad\left(1+\frac{t_1+t_2}{2R}+\frac{t_1^2+t_2^2+6t_1t_2-2h^2}{8R^2}+\varepsilon_6(R,t_1,t_2)\right).
\end{aligned}$$

Let us define $A_1=\frac{2t_1}{R}$, $A_2=\frac{3t_1^2-h^2}{R^2}+\varepsilon_7(R,t_1,t_2)$, $B_1=\frac{t_1+t_2}{2R}$ and $B_2=\frac{t_1^2+t_2^2+6t_1t_2-2h^2}{8R^2}+\varepsilon_6(R,t_1,t_2)$. With these notations, we have

$$\begin{aligned}
\frac{R-t_2}{(R-t_1)^2+h^2}\cdot\frac{1}{d(R-t_1,R-t_2)} &= \frac{1}{R^2\sqrt{2}}\left(1-\frac{t_2}{R}\right)(1+A_1+A_2)(1+B_1+B_2)\\
&= \frac{1}{R^2\sqrt{2}}\left(1+\frac{5t_1-t_2}{2R}+\frac{33t_1^2-3t_2^2-6t_1t_2-10h^2}{8R^2}+\varepsilon_8(R,t_1,t_2)\right)
\end{aligned}$$

where $\varepsilon_8(R,t_1,t_2) = \varepsilon_6(R,t_1,t_2)+\varepsilon_7(R,t_1,t_2)-\frac{t_2}{R}(A_2+B_2)+A_1B_2+A_2(B_1+B_2)-\frac{t_2}{R}(A_1+A_2)(B_1+B_2)$. Now, using Equation (18), we see that $|B_2| \leq \frac{h^2+4s^2}{2R^2}$ and then, with Equation (19) we get $|B_1+B_2| \leq \frac{2s}{R}$. Similarly, Equation (20) leads to $|A_2| \leq \frac{2h^2+6s^2}{R^2}$ and then Equation (21) gives $|A_1+A_2| \leq \frac{3s}{R}$. Collecting all these results we get

$$|\varepsilon_8(R,t_1,t_2)| \leq \frac{(87h^2+954s^2)s}{4R^3}$$

which proves the second equation of the lemma.
In order to prove the third equation, observe that $\sqrt{(R-t_2)^2+h^2} = R\sqrt{1-\frac{2t_2}{R}+\frac{t_2^2+h^2}{R^2}}$, whence, using Lemma 2 and Equation (10) (which is legitimate because of Equations (17) and (15)),

$$\frac{1}{\sqrt{(R-t_2)^2+h^2}} = \frac{1}{R}\left(1+\frac{t_2}{R}+\frac{2t_2^2-h^2}{2R^2}+\varepsilon_9(R,t_1,t_2)\right)$$

where $|\varepsilon_9(R,t_1,t_2)| \leq \frac{(9h^2+137s^2)s}{2R^3}$. Therefore we obtain

$$\frac{R-t_1}{\sqrt{(R-t_2)^2+h^2}} = 1+\frac{t_2-t_1}{R}+\frac{2t_2^2-2t_1t_2-h^2}{2R^2}+\varepsilon_{10}(R,t_1,t_2)$$

where $|\varepsilon_{10}(R,t_1,t_2)| = \left|-\frac{2\,t_2^2\,t_1 - h^2 t_1}{2R^3} + \left(1-\frac{t_1}{R}\right)\varepsilon_9(R,t_1,t_2)\right| \leq \frac{(44h^2+478s^2)s}{R^3}$. To get this upper bound, we roughly bounded $\left|1-\frac{t_1}{R}\right|$ by 2 using Equation (15). Finally, using Lemma 2 and Equation (14) we get

$$\operatorname{argsinh}\left(\frac{R-t_1}{\sqrt{(R-t_2)^2+h^2}}\right) = \operatorname{argsinh}(1) + \frac{t_2-t_1}{R\sqrt{2}} + \frac{3t_2^2 - 2t_1t_2 - t_1^2 - 2h^2}{4R^2\sqrt{2}} + \varepsilon_{11}(R,t_1,t_2),$$

where

$$|\varepsilon_{11}(R,t_1,t_2)| \leq \operatorname{argsinh}(1)\,\frac{(44h^2+478s^2)s}{R^3} + \frac{(3h^2+92s^2)s}{2R^3\sqrt{2}} \leq \frac{(113h^2+1287s^2)s}{2R^3\sqrt{2}}.$$

In order to apply Lemma 2, we bounded $|\varepsilon_{10}(R,t_1,t_2)|$ by $\frac{h^2+4s^2}{2R^2}$ using Equation (22) and we bounded $2\frac{h^2+4s^2}{2R^2}$ by $\frac{2s}{R}$ using Equation (23). Moreover, we used $\operatorname{argsinh}(1) \leq \frac{5}{4\sqrt{2}}$.
To prove the last equation, we first use Lemma 2 with Equation (12) (checking the hypotheses of the lemma with Equations (16) and (15)) to get

$$\sqrt{(R-t_1)^2+(R-t_2)^2+h^2} = R\sqrt{2}\left(1 - \frac{t_1+t_2}{2R} + \frac{t_1^2 - 2t_1t_2 + t_2^2 + 2h^2}{8R^2} + \varepsilon_{12}(R,t_1,t_2)\right)$$

where $|\varepsilon_{12}(R,t_1,t_2)| \leq \frac{(3h^2+38s^2)s}{4R^3}$. We can truncate this expansion as

$$\sqrt{(R-t_1)^2+(R-t_2)^2+h^2} = R\sqrt{2}\left(1 - \frac{t_1+t_2}{2R} + \widetilde{\varepsilon_{12}}(R,t_1,t_2)\right)$$

where $|\widetilde{\varepsilon_{12}}(R,t_1,t_2)| \leq \frac{h^2+2s^2}{2R^2}$ thanks to Equation (24).
On the other hand, using Lemma 2 with Equation (11) (checking the hypotheses of the lemma by repeated use of Equation (15)) we get

$$\frac{h}{(R-t_1)(R-t_2)} = \frac{h}{R^2}\left(1 + \frac{t_1+t_2}{R} + \frac{t_1^2+t_1t_2+t_2^2}{R^2} + \varepsilon_{13}(R,t_1,t_2)\right)$$

where $|\varepsilon_{13}(R,t_1,t_2)| \leq \frac{140s^3}{R^3}$. This expansion can be truncated as

$$\frac{h}{(R-t_1)(R-t_2)} = \frac{h}{R^2}\left(1 + \frac{t_1+t_2}{R} + \widetilde{\varepsilon_{13}}(R,t_1,t_2)\right)$$

where $|\widetilde{\varepsilon_{13}}(R,t_1,t_2)| \leq \frac{3s^2}{R^2} + \frac{140s^2}{R^2}\cdot\frac{s}{R} \leq \frac{41s^2}{2R^2}$ thanks to Equation (15).
Multiplying both expansions we see that

$$\frac{h\sqrt{(R-t_1)^2+(R-t_2)^2+h^2}}{(R-t_1)(R-t_2)} = \frac{h\sqrt{2}}{R}\left(1 + \frac{t_1+t_2}{2R} + \varepsilon_{14}(R,t_1,t_2)\right),$$

where

$$\begin{aligned}\varepsilon_{14}(R,t_1,t_2) &= \widetilde{\varepsilon_{12}}(R,t_1,t_2) + \widetilde{\varepsilon_{13}}(R,t_1,t_2)\\ &\quad + \left(-\frac{t_1+t_2}{2R} + \widetilde{\varepsilon_{12}}(R,t_1,t_2)\right)\left(\frac{t_1+t_2}{R} + \widetilde{\varepsilon_{13}}(R,t_1,t_2)\right).\end{aligned}$$

From Equation (16) we see that $|\widetilde{\varepsilon_{12}}(R,t_1,t_2)| \le \frac{2s}{R}$ and from Equation (15), $|\widetilde{\varepsilon_{13}}(R,t_1,t_2)| \le \frac{41s}{16R}$. Therefore $|\varepsilon_{14}(R,t_1,t_2)| \le \frac{8h^2+563s^2}{16R^2}$.
Now, by Equation (13),

$$\arctan\left(\frac{h\sqrt{(R-t_1)^2+(R-t_2)^2+h^2}}{(R-t_1)(R-t_2)}\right) = \frac{h\sqrt{2}}{R} + \frac{(t_1+t_2)h\sqrt{2}}{2R^2} + \varepsilon_{15}(R,t_1,t_2)$$

where

$$\varepsilon_{15}(R,t_1,t_2) = \frac{h\sqrt{2}}{R}\,\varepsilon_{14}(R,t_1,t_2) + \varepsilon_4\left(\frac{h\sqrt{2}}{R}\left(1+\frac{t_1+t_2}{2R}+\varepsilon_{14}(R,t_1,t_2)\right)\right).$$

Finally, thanks to Equation (25), $\left|\frac{t_1+t_2}{2R}+\varepsilon_{14}(R,t_1,t_2)\right| \le \frac{2s}{R}$ that we roughly bound by 1 thanks to Equation (15). Now the argument of ε_4 is less than $1/2$ in absolute value thanks to Equation (26), so we can use the bound given in Lemma 3. This concludes the proof. $\square$

As a direct corollary of the previous Lemma, we obtain asymptotic expansions of important integrals:

$$\begin{aligned}\int_{-R}^{R}\frac{\mathrm{d}y}{d(R-t_1,y-t_2)^3} &= \left[\frac{y-t_2}{(R-t_1)^2+h^2}\cdot\frac{1}{d(R-t_1,y-t_2)}\right]_{y=-R}^{R}\\ &= \frac{R-t_2}{(R-t_1)^2+h^2}\cdot\frac{1}{d(R-t_1,R-t_2)} + \frac{R+t_2}{(R-t_1)^2+h^2}\cdot\frac{1}{d(R-t_1,R+t_2)}\\ &= \frac{2}{\sqrt{2}}\cdot\frac{1}{R^2} + \frac{5t_1}{\sqrt{2}}\cdot\frac{1}{R^3} + \frac{33t_1^2-3t_2^2-10h^2}{4\sqrt{2}}\cdot\frac{1}{R^4} + \widetilde{\delta_2}(R,t_1,t_2),\end{aligned}$$

where $|\widetilde{\delta_2}(R,t_1,t_2)| \le 100(h^2+10s^2)s/R^5$ for any $R \ge C$ and any $(t_1,t_2) \in \mathcal{A}$.
By replacing t_1 with $-t_1$, we also have

$$\int_{-R}^{R}\frac{\mathrm{d}y}{d(R+t_1,y-t_2)^3} = \frac{2}{\sqrt{2}}\cdot\frac{1}{R^2} - \frac{5t_1}{\sqrt{2}}\cdot\frac{1}{R^3} + \frac{33t_1^2-3t_2^2-10h^2}{4\sqrt{2}}\cdot\frac{1}{R^4} + \widetilde{\delta_2}(R,-t_1,t_2).$$

Accordingly,

$$\begin{aligned}\int_{-R}^{R}\frac{(y-t_2)\mathrm{d}y}{d(R-t_1,y-t_2)^3} &= \left[\frac{-1}{d(R-t_1,y-t_2)}\right]_{y=-R}^{R}\\ &= \frac{1}{d(R-t_1,R+t_2)} - \frac{1}{d(R-t_1,R-t_2)}\\ &= \frac{-t_2}{\sqrt{2}}\cdot\frac{1}{R^2} - \frac{3t_1t_2}{2\sqrt{2}}\cdot\frac{1}{R^3} + \widetilde{\delta_1}(R,t_1,t_2)\end{aligned}$$

where $|\widetilde{\delta_1}(R,t_1,t_2)| \le 100(h^2+50s^2)s/R^4$ for any $R \ge C$ and any $(t_1,t_2) \in \mathcal{A}$.
And, by replacing t_1 with $-t_1$, we also have

$$\int_{-R}^{R}\frac{(y-t_2)\mathrm{d}y}{d(R-t_1,y-t_2)^3} = \frac{-t_2}{\sqrt{2}}\cdot\frac{1}{R^2} + \frac{3t_1t_2}{2\sqrt{2}}\cdot\frac{1}{R^3} + \widetilde{\delta_1}(R,-t_1,t_2).$$

Another important integral is $\int_{-R}^{R}\int_{-R}^{R}\frac{y-t_2}{d(x-t_1,y-t_2)^3}\,\mathrm{d}y\,\mathrm{d}x$ and can be explicitely obtained by

$$\begin{aligned}
\int_{-R}^{R}\int_{-R}^{R}\frac{y-t_2}{d(x-t_1,y-t_2)^3}\,\mathrm{d}y\,\mathrm{d}x &= \int_{-R}^{R}\left[\frac{-1}{d(x-t_1,y-t_2)}\right]_{y=-R}^{R}\,\mathrm{d}x \\
&= \left[\frac{-1}{((y-t_2)^2+h^2)^{1/2}}\int_{-R}^{R}\frac{1}{\sqrt{\left(\frac{x-t_1}{((y-t_2)^2+h^2)^{1/2}}\right)^2+1}}\,\mathrm{d}x\right]_{y=-R}^{R} \\
&= -\left[\left[\operatorname{argsinh}\left(\frac{x-t_1}{((y-t_2)^2+h^2)^{1/2}}\right)\right]_{x=-R}^{R}\right]_{y=-R}^{R} \\
&= \frac{-4t_2}{\sqrt{2}}\cdot\frac{1}{R}+\widetilde{\delta}_3(R,t_1,t_2)
\end{aligned}$$

where $|\widetilde{\delta}_3(R,t_1,t_2)|\leq 200(h^2+50s^2)s/R^3$ for any $R\geq C$ and any $(t_1,t_2)\in\mathcal{A}$.
Finally, the last important integral is related to Poisson's kernel: $\int_{-R}^{R}\int_{-R}^{R}\frac{1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}y\,\mathrm{d}x$.
It can be explicitely obtained by

$$\int_{-R}^{R}\int_{-R}^{R}\frac{1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}y\,\mathrm{d}x=\left[\int_{-R}^{R}\frac{(y-t_2)}{(x-t_1)^2+h^2}\cdot\frac{1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}x\right]_{y=-R}^{R}.$$

Now, succesively performing the changes of variable $\tan(t)=\frac{x-t_1}{((y-t_2)^2+h^2)^{1/2}}$, $u=\sin(t)$ and $v=\frac{u(y-t_2)}{h}$ we see that

$$\int_{-R}^{R}\frac{(y-t_2)}{(x-t_1)^2+h^2}\cdot\frac{1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}x=\frac{1}{h}\int_{f_y(-R)}^{f_y(R)}\frac{\mathrm{d}v}{1+v^2}$$

where $f_y(x)=\frac{1}{h}\cdot\frac{(x-t_1)(y-t_2)}{d(x-t_1,y-t_2)}$. Finally, using the formula $\arctan(x)=\frac{\pi}{2}-\arctan(1/x)$ that holds for any $x>0$, we get

$$\int_{-R}^{R}\int_{-R}^{R}\frac{1}{d(x-t_1,y-t_2)^3}\,\mathrm{d}y\,\mathrm{d}x=\frac{2\pi}{h}-4\sqrt{2}\,\frac{1}{R}+\widetilde{\delta}_4(R,t_1,t_2)$$

where $|\widetilde{\delta}_4(R,t_1,t_2)|\leq 200(h^2+s^2)h/R^3$.