Chapter 1

Invariant Sets for Controlled Degenerate Diffusions: a Viscosity Solutions Approach

Martino Bardi[1]
Paola Goatin[2]

Dedicated to Wendell Fleming for his 70th birthday.

[1]Dipartimento di Matematica P. e A., Università di Padova, via Belzoni 7, 35131 Padova, Italy; e-mail: bardi@math.unipd.it. Partially supported by M.U.R.S.T. project "Problemi nonlineari nell'analisi e nelle applicazioni fisiche, chimiche, biologiche".

[2]SISSA-ISAS,via Beirut 2-4, 34014 Trieste, Italy; e-mail: goatin@sissa.it.

Abstract

We study invariance and viability properties of a closed set for the trajectories of either a controlled diffusion process or a controlled deterministic system with disturbances. We use the value functions associated to suitable optimal control problems or differential games and analyze the related Dynamic Programming equation within the theory of viscosity solutions.

degenerate diffusion; invariance; viability; stochastic control; differential games; viscosity solutions; Hamilton-Jacobi-Bellman equations; nonsmooth analysis.

1.1 Introduction

Consider the controlled Ito stochastic differential equation in $I\!R^N$

$$(SDE)\begin{cases} dX_t = \sigma^{\alpha_t}(X_t)dB_t + f^{\alpha_t}(X_t)dt, & t > 0, \\ X_0 = x. \end{cases} \tag{1.1}$$

where B_t is an M-dimensional Brownian motion and α_t is the control taking values in a given set A. A set K is *invariant* for (SDE) if for all initial points $x \in K$ and all admissible controls α, the trajectory X_t of (SDE) remains in K for all $t > 0$ almost surely. One of the main results of this paper is a characterization of the closed invariant sets for (SDE) under general assumptions on the data. Of course this problem is interesting when the diffusion process described by (SDE) is degenerate, namely, the matrix $\sigma\sigma^T$ is merely positive semidefinite (in the nondegenerate case the only invariant set is the whole $I\!R^N$). From now on we will assume $K \subseteq I\!R^N$ is a closed set.

Let us recall the classical theorem of Nagumo (1942) that solves this problem in the deterministic case (i.e., $\sigma \equiv 0$) and without controls, that is, for the ODE

$$\begin{cases} X'_t = f(X_t), & t > 0, \\ X_0 = x. \end{cases}$$

In this case K *is invariant if and only if* $f(x) \in T_K(x)$ *for all* $x \in \partial K$, where $T_K(x)$ is the Bouligand-Severi contingent cone to K at x (see Sect. 4 for the definition; $T_K(x)$ coincides with the tangent space if K is a smooth manifold near x). An equivalent characterization can be given in terms of the cone of generalized interior normal vectors, or *1st order normal cone*,

$$\mathcal{N}^1_K(x) := \left\{p \in I\!R^N : \ p \cdot (y - x) \geq o(|y - x|) \ \text{ as } \ K \ni y \to x\right\}.$$

This is the positive polar cone of $T_K(x)$, and it reduces to the half line generated by the interior normal to K at x if K is the closure of an open set with smooth boundary and $x \in \partial K$. We can reformulate Nagumo's theorem as follows:

K is invariant if and only if $f(x) \cdot p \geq 0$ for all $p \in \mathcal{N}^1_K(x)$ and all $x \in \partial K$.

For deterministic systems with control, that is,

$$\begin{cases} X'_t = f(X_t, \alpha_t), & t > 0, \\ X_0 = x. \end{cases} \tag{1.2}$$

the property characterizing the invariance of a closed set K becomes

$$f(x, a) \cdot p \geq 0 \quad \forall\, a \in A,\ p \in \mathcal{N}^1_K(x),\ x \in \partial K. \tag{1.3}$$

The corresponding result for the controlled diffusion process (SDE) requires the following *2nd order normal cone*

$$\mathcal{N}_K^2(x) \;:=\; \{(p,Y)\in I\!R^N\times S(N):\ \text{for }\ K\ni y\to x$$
$$p\cdot(y-x)+\frac{1}{2}(y-x)\cdot Y(y-x)\geq o(|y-x|^2)\},$$

where $S(N)$ is the set of symmetric $N\times N$ matrices. The relation between the 1st and the 2nd order normal cones is transparent, in particular $p\in\mathcal{N}_K^1(x)$ if $(p,Y)\in\mathcal{N}_K^2(x)$. Here is the Invariance Theorem for (SDE):

all the trajectories of (SDE) *starting in* K *remain forever in* K *a.s. if and only if*

$$f^\alpha(x)\cdot p+\frac{1}{2}trace(\sigma^\alpha(x)\sigma^\alpha(x)^TY)\geq 0\quad \forall\alpha\in A,\ (p,Y)\in\mathcal{N}_K^2(x),\ x\in\partial K. \tag{1.4}$$

This result is better understood if K is the closure of an open set whose boundary is twice differentiable at the point $x\in\partial K$. Let us consider for simplicity an uncontrolled equation of the form

$$dX_t=f(X_t)dt+\sigma(X_t)dB_t$$

where B_t is a one-dimensional Brownian motion and σ is a vector field. In this case the condition (1.4) becomes

$$\sigma(x)\cdot\vec{n}(x)=0\ \text{ and }\ \frac{1}{2}S(P\sigma(x))\geq f(x)\cdot\vec{n}(x), \tag{1.5}$$

where $\vec{n}(x)$ is the exterior normal to K, P is the orthogonal projection on the tangent space to K at x, S is the second fundamental form of ∂K at x (oriented with $\vec{n}(x)$). This says that the diffusion vector σ is tangential to ∂K and the component of the drift pointing outward K is smaller than a quantity depending on the curvature matrix of ∂K and the diffusion vector. This is the stochastic generalization of the condition $f(x)\cdot\vec{n}(x)\leq 0$ in the deterministic case.

Previous sufficient conditions of invariance of closed sets for uncontrolled diffusions can be found in the books of Friedman (1976) (for K closure of a smooth open set) and of Ikeda and Watanabe (1981) (for K smooth manifold and stochastic differential equations in the Stratonovich sense). Aubin and Da Prato (1990) (1995) characterized the invariance of random closed sets by means of a notion of stochastic contingent set, see also Gautier and Thibault (1993) and Milian (1993) (1995) (1997).

Our approach is completely different from the preceding. We start from the observation that K is invariant if and only if the first exit time $\hat{t}_x(\alpha_.)$ of the trajectory of (SDE) from K is $+\infty$ for all $x\in K$ and all admissible controls $\alpha_.$ almost surely. Then we consider the value function

$$v(x):=\inf_{\alpha_.}E(1-e^{-\hat{t}_x(\alpha_.)}), \tag{1.6}$$

and note that

$$K \text{ is invariant} \Longleftrightarrow v(x) = 1 \ \ \forall x \in K.$$

A Hamilton-Jacobi-Bellman equation is associated to the stochastic control problem of (1.6) via Dynamic Programming, and even if the value function is not continuous in general, it satisfies such nonlinear PDE in the viscosity sense in the interior of K, and a generalized Dirichlet boundary condition on ∂K, see P.L. Lions (1983a) (1983b), Ishii (1989), Ishii and Lions (1990). We proved in Bardi and Goatin (1997) that v is indeed the maximal subsolution of a suitable boundary value problem (BVP) for the HJB equation, see Sect. 2. Thus we can establish if K is invariant by checking whether the constant 1 is the maximal subsolution of (BVP), and this leads naturally to the condition (1.4) involving the 2nd order normal cone.

The approach is rather flexible and we believe it applies to other related problems, such as the *viability* of the set K. This weaker property says that for all initial points $x \in K$ there exists at least one trajectory of the system that remains in K for all $t > 0$ almost surely. Viability has a large literature in the case of deterministic differential inclusions, see Aubin and Cellina (1981), Aubin (1991), Ledyaev (1994) and the references therein, and was studied recently by Aubin and Da Prato (1997) in the stochastic case.

Here we test our method to prove a viability theorem for controlled systems affected by a bounded disturbance whose statistics are not known. More precisely we consider

$$\begin{cases} X_t' = f(X_t, a_t, b_t), & t > 0 \\ X_0 = x, \end{cases}$$

where a_t is an unknown measurable disturbance taking values in a compact set A, b_t is the control taking values in a compact set B, and the controller is allowed to use relaxed controls. We say that K is viable if for all $x \in K$ there exists a nonanticipating strategy for the controller that keeps the trajectory X_t forever in K for all disturbances (see Guseinov, Subbotin and Ushakov (1985) for a class of related problems formulated within the Krasovskii-Subbotin theory of positional differential games). In this problem we consider the value function of a suitable differential game, see Sect. 2 for all the definitions. By means of the Hamilton-Jacobi-Isaacs equation of this game we prove that K is viable if and only if

$$\forall\, x \in \partial K,\ a \in A,\ p \in \mathcal{N}_K^1(x),\ \exists\, b_x \in B\ :\ f(x, a, b_x) \cdot p \geq 0.$$

Note that if B is a singleton we obtain the characterization of invariance (1.3) for system (1.2), while in the case that A is a singleton we obtain a viability theorem for a (parametrized) differential inclusion. In Sect. 4 we also reformulate these results in terms of the contingent cone $T_K(x)$, which is the usual tool in viability theory, and of other objects of nonsmooth analysis (such as proximal normals and Clarke tangent cone).

We refer the reader to Crandall, Ishii, Lions (1992) and Bardi, Crandall, Evans, Soner, Souganidis (1997) for surveys of the theory of viscosity solutions, to the books by Fleming and Soner (1993) and Bardi and Capuzzo-Dolcetta (1997) for the applications, respectively, to stochastic control and to deterministic control and games, to Fleming and Souganidis (1989) for stochastic differential games.

The paper is organized as follows. Section 2 recalls some results on discontinuous viscosity solutions and their relation with the value functions of stochastic control problems and differential games involving the exit time from a closed set. Section 3 is devoted to the invariance theorem for (SDE) and to several examples. Section 4 deals with viability and invariance for deterministic systems with and without disturbances, and gives several equivalent characterizing properties based on various tools of nonsmooth analysis.

1.2 Value functions and Dynamic Programming Equations

We consider the boundary value problem

$$(BVP)\begin{cases} F(x,u,Du,D^2u)=0 & \text{in } \Omega := \overset{\circ}{K}, \\ u=0 \text{ or } F(x,u,Du,D^2u)=0 & \text{on } \partial K, \end{cases} \tag{2.1}$$

where F is a nonlinear degenerate elliptic operator and K is an arbitrary closed set and $\overset{\circ}{K}$ is its interior. The boundary condition in (BVP) is to be understood in the pointwise viscosity sense, see the precise definition (2.6) below. In particular, we are interested in the Isaacs-Bellman operator

$$F(x,r,p,X)=\sup_{\alpha}\inf_{\beta}[-trace(A^{\alpha,\beta}(x)X)-f^{\alpha,\beta}(x)\cdot p+c^{\alpha,\beta}(x)r-l^{\alpha,\beta}(x)], \tag{2.2}$$

and we assume that for all $x\in I\!R^N$, $A^{\alpha,\beta}(x)=\frac{1}{2}\sigma^{\alpha,\beta}(x)\sigma^{\alpha,\beta}(x)^T$, where $\sigma^{\alpha,\beta}(x)$ is a $N\times M$ matrix, ^{T} denotes the transpose matrix, $\sigma^{\alpha,\beta}$, $f^{\alpha,\beta}$, $c^{\alpha,\beta}$, $l^{\alpha,\beta}$ are bounded and uniformly continuous, uniformly with respect to α,β, so that the operator is continuous. Moreover we require that $A^{\alpha,\beta}(x)\geq 0$ for all α,β and there exist $C>0$ and $c_0>0$ such that

$$||\sigma^{\alpha,\beta}(x)-\sigma^{\alpha,\beta}(y)||\leq C|x-y|, \text{ for all } x,y\in\overline{\Omega} \text{ and all } \alpha,\beta \tag{2.3}$$

$$|f^{\alpha,\beta}(x)-f^{\alpha,\beta}(y)|\leq C|x-y|, \text{ for all } x,y\in\overline{\Omega} \text{ and all } \alpha,\beta, \tag{2.4}$$

$$c^{\alpha,\beta}(x)\geq c_0, \text{ for all } x\in\overline{\Omega}, \text{ and all } \alpha,\beta. \tag{2.5}$$

We denote by $\mathcal{S}_v$ the set of all subsolutions of (BVP), that is

$$\mathcal{S}_v = \{w : K\to I\!R \text{ bounded, upper semicontinuous, and subsolution of } (BVP)\},$$

and by $\mathcal{Z}_e$ the following set

$$\begin{aligned}\mathcal{Z}_e \quad = \quad & \{W : \overline{\mathcal{O}} \to I\!R \text{ is bounded and lower semicontinuous,} \\ & \mathcal{O} \text{ is open, } K \subset \mathcal{O}, \\ & W \text{ is supersolution of } F = 0 \text{ in } \mathcal{O},\ W \geq 0 \text{ on } \partial\mathcal{O}\}.\end{aligned}$$

For the definitions of viscosity sub- and supersolution we refer to Ishii, Lions (1990) or Crandall, Ishii, Lions (1992). By a subsolution of (BVP) we mean a subsolution in viscosity sense not only of the PDE in Ω but also of the boundary condition; this means that, for all $x \in \partial K$,

$$\min\{u(x), F(x, u(x), p, Y)\} \leq 0 \quad \forall (p, Y) \in \mathcal{J}_K^{2,+}u(x), \tag{2.6}$$

where

$$\begin{aligned}\mathcal{J}_K^{2,+}u(x) := \{&(p, Y) \in I\!R^N \times S(N) : \text{ for } K \ni y \to x \\ & u(y) \leq u(x) + p \cdot (y - x) + \frac{1}{2}(y - x) \cdot Y(y - x) + o(|y - x|^2)\}\end{aligned}$$

In Bardi and Goatin (1997) we proved that for general nonlinear degenerate elliptic equations, there exists the maximal subsolution of (BVP) and

$$u(x) := \max_{w \in \mathcal{S}_v} w(x) = \inf_{W \in \mathcal{Z}_e} W(x). \tag{2.7}$$

This is the natural generalized solution of (BVP), and we call it *envelope solution* (briefly *e-solution*), or Perron-Wiener solution, of (BVP). Here is a precise statement for the Isaacs-Bellman operator (2.2).

Theorem 2.2.1 *Assume* $A^{\alpha,\beta} = \frac{1}{2}\sigma^{\alpha,\beta}(\sigma^{\alpha,\beta})^T$ *for some* $N \times M$ *matrix valued functions* $\sigma^{\alpha,\beta}$*, and* $\sigma^{\alpha,\beta}, f^{\alpha,\beta}, c^{\alpha,\beta}, l^{\alpha,\beta}$ *be uniformly bounded with respect to* α, β *and satisfy (2.3), (2.4), (2.5) with* $c_0 > 0$*. Suppose also* $l^{\alpha,\beta}$ *are uniformly continuous, uniformly in* α, β*. Then for any closed set* K *there exists the maximal subsolution* u *of* (BVP) *and it satisfies (2.7).*

For the proof see Bardi and Goatin (1997); see also Bardi, Goatin and Ishii (1998) for further properties of the e-solution.

In the sequel we will study the time-optimal stochastic control problem and a deterministic differential game. The two following theorems state that the value function of the control problem and the differential game are the e-solutions of the corresponding (BVP).

For the time-optimal stochastic control problem we consider a probability space $(\Omega', \mathcal{F}, P)$ with a right-continuous increasing filtration of complete sub-σ fields $\{\mathcal{F}_t\}$, a Brownian motion B_t in $I\!R^M$ $\mathcal{F}_t$-adapted, a compact set A, and call $\mathcal{A}$ the set of progressively measurable processes α_t taking values in A. We are given bounded maps σ from $I\!R^N \times A$ into the set of $N \times M$ matrices and

$f : I\!R^N \times A \to I\!R^N$ satisfying (2.3), (2.4) and consider the controlled stochastic differential equation

$$(SDE) \begin{cases} dX_t = \sigma^{\alpha_t}(X_t)dB_t + f^{\alpha_t}(X_t)dt, & t > 0, \\ X_0 = x. \end{cases}$$

This has a pathwise unique solution X_t which is $\mathcal{F}_t$-progressively measurable and has continuous sample paths. We are given also two bounded and uniformly continuous maps $l, c : I\!R^N \times A \to I\!R$, $c^a(x) \geq c_0 > 0$ for all x, a, and consider the cost functional

$$J(x, \alpha_.) := E\left(\int_0^{\hat{t}_x(\alpha_.)} l^{\alpha_t}(X_t) e^{-\int_0^t c^{\alpha_s}(X_s)ds} dt \right),$$

where E denotes the expectation and

$$\hat{t}_x(\alpha_.) := \inf\{t \geq 0 : X_t \notin K\}$$

for a given compact set K (of course $\hat{t}_x(\alpha_.) = +\infty$ if $X_t \in K$ for all $t \geq 0$). We define the value function

$$v(x) := \inf_{\alpha_. \in \mathcal{A}} J(x, \alpha_.),$$

and consider the Bellman equation

$$\begin{aligned} F(x, u, Du, D^2u) := \quad & \max_{\alpha \in A}\{-trace(A^\alpha(x)D^2u - f^\alpha(x) \cdot Du + \\ & +c^\alpha(x)u - l^\alpha(x)\} = 0, \end{aligned}$$

where the matrix $A^\alpha(x) := \frac{1}{2}\sigma^\alpha(x)\sigma^\alpha(x)^T$. We consider the boundary value problem (BVP) under the additional assumption

$$l^\alpha(x) \geq 0 \ \text{ for all } x \in I\!R^N, \ \alpha \in A. \tag{2.8}$$

Theorem 2.2.2 *Assume (2.8). Then the value function*

$$v(x) = \inf_{\alpha_. \in \mathcal{A}} E\left(\int_0^{\hat{t}_x(\alpha_.)} l^{\alpha_t}(X_t) e^{-\int_0^t c^{\alpha_s}(X_s)ds} dt \right)$$

is the unique e-solution of (BVP), i.e. $v = u$ with the property (2.7).

For the proof see Bardi and Goatin (1997).

Next we consider a two-person zero-sum deterministic differential game. We are given a controlled dynamical system

$$\begin{cases} y' = f(y(t), a(t), b(t)), & t > 0 \\ y(0) = x, \end{cases} \tag{2.9}$$

where $f : I\!\!R^N \times A \times B \to I\!\!R^N$ is continuous, A, B are compact metric spaces and $a = a(\cdot) \in \mathcal{A} := \{\text{measurable functions } [0, +\infty) \to A\}$ is the control function of the first player. For the second player we will use relaxed controls $b = b(\cdot) \in \mathcal{B}^r := \{\text{measurable functions } [0, +\infty) \to B^r\}$ where B^r is the set of Radon probability measures on B. For the definitions of relaxed trajectories of (2.9) we refer for instance to Warga (1972), Bardi and Capuzzo Dolcetta (1997). We will always assume that the system satisfies, for some constant L,

$$(f(x, a, b) - f(y, a, b)) \cdot (x - y) \le L|x - y|^2 \tag{2.10}$$

for all $x, y \in I\!\!R^N$, $a \in A$, $b \in B$.

The cost functional, which the first player wants to minimize and the second player wants to maximize, is

$$J(x, a, b) = \int_0^{\hat{t}_x(a,b)} e^{-t} dt = 1 - e^{-\hat{t}_x(a,b)},$$

where $\hat{t}_x(a, b)$ is the first exit time from a given closed set $K \subseteq I\!\!R^N$, i.e. $\hat{t}_x(a, b) := \inf\{t \in [0, +\infty) : y_x(t, a, b) \notin K\}$, $y_x(t, a, b)$ being the solution of (2.9) corresponding to $a \in \mathcal{A}$, $b \in \mathcal{B}^r$. Note that $\hat{t}_x$ is the time taken by the system to reach the *open target* $I\!\!R^N \setminus K$, and that J is a bounded and increasing rescaling of $\hat{t}_x$.

A relaxed strategy for the second player is a map $\beta : \mathcal{A} \to \mathcal{B}^r$; it is nonanticipating if, for any $t > 0$ and $a, \tilde{a} \in \mathcal{A}$, $a(s) = \tilde{a}(s)$ for all $s \le t$ implies $\beta[a](s) = \beta[\tilde{a}](s)$ for all $s \le t$, see Elliot and Kalton (1972), Bardi and Capuzzo-Dolcetta (1997). We will denote with Δ^r the set of nonanticipating relaxed strategies for the second player.

Now we can define the upper value of this differential game, which is

$$\hat{u}(x) = \sup_{\beta \in \Delta^r} \inf_{a \in \mathcal{A}} J(x, a, \beta[a]).$$

It is well known by the Dynamic Programming Principle that the upper value is a viscosity solution of the upper Hamilton-Jacobi-Isaacs (briefly HJI) equation

$$u(x) + \tilde{H}(x, Du(x)) := u(x) + \max_{a \in A} \min_{b \in B} \{-f(x, a, b) \cdot Du(x) - 1\} = 0 \ \text{ in } \overset{\circ}{K}.$$

Theorem 2.2.3 *Assume (2.10). Then the upper value of the minimum time problem*

$$\hat{u}(x) = \sup_{\beta \in \Delta^r} \inf_{a \in \mathcal{A}} \int_0^{\hat{t}_x(a, \beta[a])} e^{-t} dt$$

is the maximal subsolution of

$$(BVP^{\sim}) \begin{cases} u + \tilde{H}(x, Du) = 0 & \text{in } \overset{\circ}{K}, \\ u = 0 \text{ or } u + \tilde{H}(x, Du) = 0 & \text{on } \partial K, \end{cases} \tag{2.11}$$

and therefore the e-solution $(BVP^{\sim})$.

The proof can be found in Bardi and Goatin (1997). Let us recall that, since the PDE in (2.11) is of 1st order, a formulation of the boundary condition equivalent to (2.6) is the following:

$$u(x) \leq 0 \quad \text{or} \quad u(x) + \tilde{H}(x,p) \leq 0 \;\; \forall p \in \mathcal{J}_K^{1,+} u(x), \tag{2.12}$$

for all $x \in \partial K$, where

$$\mathcal{J}_K^{1,+} u(x) := \{p \in I\!R^N : u(y) \leq u(x) + p \cdot (y - x) + o(|y - x|) \text{ for } K \ni y \to x\}.$$

2.3 Invariant sets for diffusions

In this section we consider the stochastic controlled system (SDE) of equation (1.1), a closed set $K \subset I\!R^N$, and characterize the invariance of K for the trajectories of (SDE). We recall that K is invariant for (SDE) if $\hat{t}_x(\alpha_.) = +\infty$ almost surely for all $x \in K$ and admissible controls $\alpha_.$, see the definitions in Section 2. Following the approach outlined in the Introduction we consider the value function v defined by (1.6) that satisfies Theorem 2.2.2 with $l = c \equiv 1$. The corresponding HJB equation is then

$$u + \max_{\alpha \in A} \left\{ -\frac{1}{2} trace(\sigma^\alpha (\sigma^\alpha)^T D^2 u) - f^\alpha \cdot Du \right\} - 1 = 0.$$

Theorem 3.3.1 *Assume σ^α, f^α be uniformly bounded with respect to α and satisfy (2.3), (2.4). Then K is invariant if and only if for any $x \in \partial K$ and any $\alpha \in A$*

$$\frac{1}{2} trace(\sigma^\alpha(x) \sigma^\alpha(x)^T Y) + f^\alpha(x) \cdot p \geq 0 \;\; \forall \, (p, Y) \in \mathcal{N}_K^2(x). \tag{3.1}$$

Proof. By the definitions, the set K is viable if and only if the value function $v \equiv 1$. To prove this we observe that $v \leq 1$ by definition and, by Theorem 2.2.2, v is the maximal subsolution of the boundary value problem (BVP) of equation (2.1). Then $v \equiv 1$ if and only if the constant function $u(x) := 1$ is a subsolution of (BVP). The HJB equation is trivially satisfied by u in $\mathring{K}$, while the boundary condition (2.6) holds if and only if for all $x \in \partial K$ and $(p, Y) \in \mathcal{J}_K^{2,+} u(x)$

$$\max_{\alpha \in A} \left\{ -\frac{1}{2} trace(\sigma^\alpha(x) \sigma^\alpha(x)^T Y) - f^\alpha \cdot p \right\} \leq 0.$$

Now we note that $\mathcal{J}_K^{2,+} u(x) = \mathcal{N}_K^2(x)$ and get the conclusion. □

In the sequel we will show what (3.1) becomes in some particular cases.

Example 3.3.2 (K closure of a smooth open set) If $K = \overline{\Omega}$, where Ω is a C^2 N-submanifold of $I\!R^N$ with boundary, and $x \in \partial\Omega$, then the contingent cone $T_K(x)$ is a halfspace and Ω has an exterior normal $\vec{n}$ at x. In this event we have the following characterization of $\mathcal{N}_K^2(x)$ (see Crandall, Ishii and Lions (1992)): if we represent Ω near $x = 0$ in the form

$$\{(\tilde{x}, x_N) : x_N \leq \Psi(\tilde{x})\},$$

where $\tilde{x} = (x_1, ..., x_{N-1})$, and we assume that the boudary of Ω is twice differentiable at 0 and rotated so that the normal is $\vec{n} = e_N = (0, ..., 0, 1)$, it is not hard to see that

$$(p, Y) \in \mathcal{N}_K^2(0) \Longleftrightarrow \begin{cases} p = 0, \ Y \geq 0; \\ \text{or} \\ p = -\lambda\vec{n}, \ \lambda > 0, \text{and for } \tilde{x} \to 0 \\ \lambda\tilde{x} \cdot D^2\Psi(0)\tilde{x} - (\tilde{x}, 0) \cdot Y(\tilde{x}, 0) \leq o(|\tilde{x}|^2). \end{cases}$$

If in (SDE) we let σ^i be the i-th column of the $N \times M$ matrix σ, and we drop the dependence on α, we can write the equation as

$$dX_t = f(X_t)dt + \sum_{i=1}^{M} \sigma^i(X_t)dB_t^i, \tag{3.2}$$

where B_t^i are the components of B_t, so they are one-dimensional independent Brownian motions. Then we can write (3.1) as

$$\frac{1}{2}\sum_{i=1}^{M} \sigma^i(0) \cdot Y\sigma^i(0) + f(0) \cdot p \geq 0 \ \text{ for any } (p, Y) \in \mathcal{N}_K^2(0).$$

By the characterization of the elements of $\mathcal{N}_K^2(0)$, we have that (3.1) holds at $x = 0$ if and only if

$$\sigma_N^i(0) = 0 \ \ \forall i = 1, ..., M, \ \text{ and } \ \frac{1}{2}\sum_{i=1}^{M} \tilde{\sigma}^i(0) \cdot D^2\Psi(0)\tilde{\sigma}^i(0) \geq f(0) \cdot \vec{n}, \tag{3.3}$$

where $\tilde{\sigma}^i = (\sigma_1^i, ..., \sigma_{N-1}^i)$. Note that if K is strictly convex (near 0) and $\sigma(0) \neq 0$, the drift vector f must be directed to the interior of K, because the matrix $D^2\Psi(0)$ is negative definite. On the other hand, if $I\!R^N \setminus K$ is strictly convex and $\sigma(0) \neq 0$, the drift can have a small component directed outward K because in this case the diffusion pushes the system inside K.

Returning to the original coordinates, the condition (3.1) reads

$$\vec{n}(x)^T\sigma(x) = 0 \ \text{ and } \ \frac{1}{2}\sum_{i=1}^{M} S(P\sigma(x)) \geq f(x) \cdot \vec{n}(x), \tag{3.4}$$

where P is the orthogonal projection on the tangent space to K at x and S is the second fundamental form of ∂K at x, oriented with the exterior normal $\vec{n}(x)$.

We can also rewrite (3.4) in terms of the Hessian matrix of any function ρ of class C^2 in a neighborhood of ∂K and such that $\rho(x) = 0$ and $\nabla\rho(x) = \vec{n}(x)$ for all $x \in \partial K$ (e.g., the signed distance function from ∂K). In fact $S(Pw) = -w \cdot D^2\rho(x)w$ for any tangent vector w at $x \in \partial K$, so (3.4) is equivalent to

$$\vec{n}(x)^T\sigma^\alpha(x) = 0 \;\text{ and }\; f^\alpha(x) \cdot \vec{n}(x) + trace(A^\alpha(x)D^2\rho(x)) \leq 0, \tag{3.5}$$

where $A^\alpha = \frac{1}{2}\sigma^\alpha(\sigma^\alpha)^T$. A direct proof of the sufficiency of this condition (for all $x \in \partial K$) for the invariance of K can be also found in the book of Friedman (1976) (without control α).

Example 3.3.3 (Balls and exterior of balls) Here we consider two special cases of the previous example. The first is the ball $K = \{x : |x| \leq R\}$ (see also Aubin and Da Prato (1990)). We obtain that K is invariant if and only if, for all $\alpha \in A$ and $|x| = R$,

$$x^T\sigma^\alpha(x) = 0 \;\text{ and }\; f^\alpha(x) \cdot x + \frac{1}{2R}||\sigma^\alpha(x)||^2 \leq 0, \tag{3.6}$$

where $||\cdot||$ denotes the Euclidean matrix norm.

The second example is the complement of a ball, $K = \{x : |x| \geq R\}$. Now the condition of invariance is, for all $\alpha \in A$ and $|x| = R$,

$$x^T\sigma^\alpha(x) = 0 \;\text{ and }\; f^\alpha(x) \cdot x + \frac{1}{2R}||\sigma^\alpha(x)||^2 \geq 0.$$

Both examples show clearly the role of the curvature of ∂K for the invariance property.

Example 3.3.4 (K hypersurface) If K is a (N-1)-dimensional manifold of class C^2 without boundary in $I\!R^N$, the contingent cone at a point $x \in K$ turns out to be the tangent hyperplane at x. We can change coordinates as in Example 3.3.2 and represent K near $x = 0$ as the graph of a twice differentiable function Ψ of the first $N-1$ variables, with normal vector $\vec{n}(0) = (0, \ldots, 0, 1)$. Using the same notations as in Example 3.3.2 we compute

$$(p, Y) \in \mathcal{N}_K^2(0) \Longleftrightarrow \begin{cases} p = 0,\ (\tilde{x}, 0) \cdot Y(\tilde{x}, 0) \geq 0\ \forall \tilde{x} \in I\!R^{N-1}; \\ \text{or} \\ p = \lambda\vec{n},\ \lambda \neq 0, \text{and for } \tilde{x} \to 0 \\ \lambda\tilde{x} \cdot D^2\Psi(0)\tilde{x} + (\tilde{x}, 0) \cdot Y(\tilde{x}, 0) \geq o(|\tilde{x}|^2). \end{cases}$$

So in this case (3.3) becomes

$$\sigma_N^i(0) = 0 \;\; \forall i = 1, ..., M, \;\text{ and }\; \frac{1}{2}\sum_{i=1}^{M} \tilde{\sigma}^i(0) \cdot D^2\Psi(0)\tilde{\sigma}^i(0) = f(0) \cdot \vec{n}.$$

Then the invariance condition in the original coordinates is (3.4) or (3.5) with the inequality replaced by an equality; for the sphere of radius R the condition is (3.6) with $=$ instead of $\leq$.

Example 3.3.5 (Smooth diffusion matrix and Stratonovich equations) If the matrix σ^α is twice differentiable the conditions of the previous examples can be also written as follows. We use the notation of Example 3.3.2 and observe that $\nabla\rho \cdot \sigma^i \equiv 0$ on ∂K and σ^i tangent to ∂K imply

$$\sigma^i \cdot \nabla(\nabla\rho \cdot \sigma^i) \equiv 0 \;\text{ on }\; \partial K.$$

This gives

$$\sigma^i \cdot D^2\rho\, \sigma^i = -\sigma^i \cdot D\sigma^i \nabla\rho,$$

where $D\sigma^i$ is the Jacobian matrix of the vector field σ^i. Then the inequality in (3.5) becomes

$$\vec{n}(x) \cdot \phi(x) := \vec{n}(x) \cdot \left(f(x) - \frac{1}{2} \sum_{i=1}^{M} D\sigma^i(x)\, \sigma^i(x) \right) \leq 0, \tag{3.7}$$

where we temporarily dropped α. The new vector field ϕ appearing in this inequality is the drift of the Stratonovich equation equivalent to (3.2), and we can rewrite its components as

$$\phi_j^\alpha := f_j^\alpha - \frac{1}{2} \sum_{i=1}^{M} \sum_{k=1}^{N} \frac{\partial \sigma_{ji}^\alpha}{\partial x_k} \sigma_{ki}^\alpha.$$

Now the invariance condition (3.4) or (3.5) can be written as

$$\vec{n}(x)^T \sigma^\alpha(x) = 0 \;\text{ and }\; \vec{n}(x) \cdot \phi^\alpha(x) \leq 0 \tag{3.8}$$

for all $x \in \partial K$ and $\alpha \in A$.

In the case that K is a hypersurface, as in Example 3.3.4, the inequalities in (3.7) and (3.8) become equalities, and the invariance condition now is

$$(\sigma^\alpha)^i(x) \in T_K(x) \;\; \forall i = 1, \ldots, M, \;\; \phi^\alpha(x) \in T_K(x)$$

for all $x \in \partial K$ and $\alpha \in A$, where $T_K(x)$ is the tangent vector space to K at x. This is the classical condition for the existence of a diffusion process on a manifold in Ikeda and Watanabe (1981). See also Milian (1997) for similar results for manifolds with boundary.

Example 3.3.6 (Polyhedra) Let K be a halfspace, i.e.

$$K = \{x \in I\!R^N : (x-a) \cdot \vec{n} \leq 0\} =: P(a, \vec{n})$$

for some point $a \in I\!R^N$ and some vector $\vec{n} \in I\!R^N$, or, up to rotation,

$$K = \{(\tilde{x}, x_N) : x_N \leq a + \lambda \cdot \tilde{x}\}.$$

So in this case (3.3) reads :

$$f \cdot \vec{n} \leq 0, \quad \sigma^i \cdot \vec{n} = 0 \quad \forall i = 1, ..., M.$$

In general, if we consider a polyhedron, that is any set of the form

$$K = \bigcap_{k \in I} P(a_k, \vec{n}_k),$$

where $I = \{1, ..., r\}$ is a finite set, we recover the following result of Milian (1995).

Theorem 3.3.7 *Let $K = \bigcap_{k \in I} P(a_k, \vec{n}_k)$ be a polyhedron in $I\!R^N$. Suppose that f, σ satisfy conditions (2.3), (2.4). Then K is invariant for (3.2) if and only if the following condition holds:*

$$f(x) \cdot \vec{n}_k \leq 0 \text{ and } \sigma^i(x) \cdot \vec{n}_k = 0 \quad \forall i = 1, ..., M,$$
$$\text{for all } k \in I \text{ and } x \in \partial K \text{ such that } (x - a_k, \vec{n}_k) = 0.$$

This result follows from the previous one on hyperplanes, by extending f and σ out of K in a suitable way. Of course it is not hard to extend it to the case that f and σ^i in (3.2) are controlled vector fields.

Example 3.3.8 (Union of smooth sets) Assume $K = K_1 \cup K_2$, where K_i, $i = 1, 2$, are closures of smooth open sets in $I\!R^N$ as in Example 3.3.2. If $x \in \partial K_1 \cap \partial K_2$, and $\vec{n}_1(x) \neq \vec{n}_2(x)$ ($\vec{n}_i$ denotes the exterior normal to K_i), then $\mathcal{N}_K^2(x) = \{(0, Y) : Y \geq 0\}$. In fact, since $T_K(x) = T_{K_1}(x) \cup T_{K_2}(x)$, and $T_{K_i}(x)$ are halfspaces, if $(p, Y) \in \mathcal{N}_K^2(x)$ then p is parallel to both $\vec{n}_1(x)$ and $\vec{n}_2(x)$. So at the points $x \in \partial K_1 \cap \partial K_2$ we need no conditions on f and σ for (3.1) to hold, while at the other points of ∂K the condition is the same as in Example 3.3.2.

3.4 Viability for deterministic systems with disturbances

In this section we consider the deterministic controlled dynamical system (2.9). We recall the definition of 1st order normal cone to the set $K \subset I\!R^N$ at the point x

$$\mathcal{N}_K^1(x) := \left\{ p \in I\!R^N : \liminf_{K \ni y \to x} \; p \cdot \frac{y - x}{|y - x|} \geq 0 \right\};$$

we note that, up to a change of sign, it coincides with the regular normal cone in the terminology of Rockafellar and Wets (1998). We recall also the definition of the Bouligand-Severi contingent cone (see Aubin and Cellina (1981) and Aubin and Frankowska (1990)), which is simply called tangent cone by Rockafellar and Wets (1998),

$$T_K(x) := \left\{ v \in I\!R^N : \liminf_{h\to 0+} \frac{d_K(x+hv)}{h} = 0 \right\},$$

where d_K is the distance function from the set K, and of the proximal normal cone generated by the Bony generalized exterior normals to K

$$P_K(x) := \left\{ q \in I\!R^N \mid \exists \varepsilon > 0 \ : \ q \cdot (y-x) \le \varepsilon |y-x|^2 \ \forall y \in K \right\}.$$

Theorem 4.4.1 *Assume (2.10). Then the following statements are equivalent:*

(i) K is viable, i.e., for all $x \in K$ there exists a relaxed strategy $\beta \in \Delta^r$ such that $\hat{t}_x(a, \beta[a]) = +\infty$ for all $a(\cdot) \in \mathcal{A}$;

(ii) for all $x \in \partial K$, $p \in \mathcal{N}^1_K(x)$, and $a \in A$, there exists $b \in B$ such that $f(x,a,b) \cdot p \ge 0$;

(iii) for all $x \in \partial K$, $q \in P_K(x)$, and $a \in A$, there exists $b \in B$ such that $f(x,a,b) \cdot q \le 0$;

(iv) for all $x \in \partial K$ and $a \in A$, there exists $b \in B$ such that $f(x,a,b) \in \overline{co} T_K(x)$.

For the proof of the equivalence of (iv) we need one more concept of nonsmooth analysis, namely, the Clarke tangent cone to K at x

$$C_K(x) := \left\{ v \in I\!R^N : \liminf_{K \ni x' \to x, h \to 0+} \frac{d_K(x'+hv)}{h} = 0 \right\}.$$

For equivalent definitions and some properties of all these cones we refer to Aubin and Frankowska (1990) and Rockafellar and Wets (1998). We only recall the following inclusions

$$C_K(x) = \liminf_{K \ni y \to x} T_K(y) = \liminf_{K \ni y \to x} \overline{co} T_K(y) \subseteq T_K(x), \tag{4.1}$$

where $\liminf_{y\to x} S(x) = \{z = \lim_n z_n : z_n \in S(y_n), y_n \to x\}$ (see, e.g., Theorem 4.1.10 in Aubin and Frankowska (1990)) . We introduce now the (positive) polar cone to a subset $S \subset I\!R^N$ defined by

$$S^\circ := \{p \in I\!R^N : p \cdot q \ge 0 \ \text{ for any } \ q \in S\}.$$

Lemma 4.4.2 $\mathcal{N}^1_K(x) = (T_K(x))^\circ$.

Proof. $\mathcal{N}^1_K(x) \subseteq (T_K(x))^\circ$ follows from the definitions. On the contrary, assume that $p \notin \mathcal{N}^1_K(x)$. Then there exists a sequence of points $K \ni y_n \to x$ such that $\lim_{n\to\infty} p \cdot \frac{y_n - x}{|y_n - x|} < 0$. We have that $\frac{y_n - x}{|y_n - x|} \to q \in T_K(x)$. Then $p \cdot q < 0$, and $p \notin (T_K(x))^\circ$. □

Lemma 4.4.3 *A vector $q \in P_K(x)$ if and only if $(-q, Y) \in \mathcal{N}_K^2(x)$ for some matrix Y.*

The proof of this Lemma is straightforward.

Proof of Theorem 4.4.1. By the definitions,the set K is viable if and only if the value function $\hat{u} \equiv 1$, and this occurs if and only if the constant $u \equiv 1$ is a subsolution of $(BVP\tilde{\ })$ in equation (2.11); in fact $\hat{u} \leq 1$ and it is the maximal subsolution of $(BVP\tilde{\ })$ by Theorem 2.2.3. Substituting $u \equiv 1$ in the equation we see that it is a solution in $\overset{\circ}{K}$, and it is a subsolution of the boundary condition (2.12) if and only if for any $x \in \partial K$ and $p \in \mathcal{J}_K^{1,+}u(x)$

$$\max_{a \in A} \min_{b \in B} \{-f(x, a, b) \cdot p\} \leq 0. \tag{4.2}$$

Since $\mathcal{J}_K^{1,+}u(x) = \mathcal{N}_K^1(x)$ we get the equivalence between (i) and (ii).

The equivalence between (ii) and (iii) is obtained in a similar way: now we use the boundary condition (2.6), we get (4.2) for all p such that $(p, Y) \in \mathcal{J}_K^{2,+}u(x) = \mathcal{N}_K^2(x)$, and then we conclude by Lemma 4.4.3.

The equivalence between (ii) and (iv) follows easily from Lemma 4.4.2, since $S^{\circ\circ} = \overline{co}S$. □

Remark 4.4.4 Guseinov, Subbotin and Ushakov (1985) analyzed a similar problem in the study of stable bridges for pursuit-evasion games in the framework of the Krasovskii-Subbotin theory of positional differential games. They found two equivalent conditions related to *(iv)*.

The next two results are consequences of Theorem 4.4.1.

Theorem 4.4.5 (Invariance) *Let (2.10) hold and B be a singleton. Then the following statements are equivalent:*

(i) K is invariant for the player a;
(ii) $f(x, a) \cdot p \geq 0$ for every $x \in \partial K$, $p \in \mathcal{N}_K^1(x)$, and $a \in A$;
(iii) $f(x, a) \cdot q \leq 0$ for every $x \in \partial K$, $q \in P_K(x)$, and $a \in A$;
(iv) $f(x, A) \subseteq \overline{co}T_K(x)$ for every $x \in \partial K$;
(v) $f(x, A) \subseteq T_K(x)$ for every $x \in \partial K$;
(vi) $f(x, A) \subseteq C_K(x)$ for every $x \in \partial K$.

Proof. By Theorem 4.4.1 it is enough to prove the equivalence of the last three statements. By (4.1) (vi) implies (v) and (iv). Now assume (iv), fix $x \in \partial K$ and take $K \ni x_n \to x$. By assumption (iv), $f(x_n, a) \in \overline{co}T_K(x_n)$ for all $a \in A$. Then $f(x, a) \in \liminf_{K \ni y \to x} \overline{co}T_K(y)$ and we get the result by (4.1). □

In the last result of this section we assume A is a singleton; then the definition of viability simplifies because we do not have to use strategies.

Theorem 4.4.6 (Viability) *Let (2.10) hold and A be a singleton. Then the following statements are equivalent:*

(j) K is viable, i.e., for all $x \in K$ there exists a relaxed control $b(\cdot) \in \mathcal{B}^r$ such that $\hat{t}_x(b) = +\infty$;

(jj) for all $x \in \partial K$ and $p \in \mathcal{N}_K^1(x)$ there is $b \in B$ such that $f(x,b) \cdot p \geq 0$;

(jjj) for all $x \in \partial K$ and $q \in P_K(x)$ there is $b \in B$ such that $f(x,b) \cdot q \leq 0$;

(jv) $f(x,B) \cap \overline{co} T_K(x) \neq \emptyset$ for all $x \in \partial K$.

Remark 4.4.7 Under the assumptions of this theorem the solutions of (2.9) associated with relaxed controls $b(\cdot)$ correspond to the trajectories of the differential inclusion $y' \in \overline{co} f(y,B)$. Then the classical viability theorem (see Aubin and Cellina (1981)) implies that the statements *(j)* - *(jv)* are also equivalent to

(v) $\overline{co} f(x,B) \cap T_K(x) \neq \emptyset$ for all $x \in \partial K$.

Note that, for $f(x,B)$ convex, condition *(jv)* looks weaker than the standard condition *(v)*, because $T_K(x)$ is not convex in general. The equivalence between them is not trivial, Ledyaev (1994) pointed out that it follows from the work of Guseinov, Subbotin and Ushakov (1985).

We thank P. Baldi, G. Colombo, G. Da Prato and W. Runggaldier for bibliographical suggestions.

Aubin, J.P. (1991). *Viability Theory*, Birkäuser, Boston.

Aubin, J.P. and Cellina, A. (1984). *Differential Inclusions*, Springer, Berlin.

Aubin, J.P. and Da Prato, G. (1990). *Stochastic viability and invariance*, Ann. Sc. Norm. Sup. Pisa (IV) 17, 595-613.

Aubin, J.P. and Da Prato, G. (1995). *Stochastic Nagumo's viability theorem*, Stochastic Anal. Appl. 13, 1-11.

Aubin, J.P. and Da Prato, G. (1997). *The viability theorem for stochastic differential inclusions*, preprint, to appear in Stochastic Anal. Appl.

Aubin, J.P. and Frankowska, H. (1990). *Set-valued analysis*, Birkhäuser, Boston.

Bardi, M. and Capuzzo Dolcetta, I. (1997). *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston.

Bardi, M., Crandall, M.G., Evans, L.C., Soner, H.M., and Souganidis, P.E.
(1997). *Viscosity solutions and applications*, I. Capuzzo Dolcetta and P.-L. Lions eds., Springer Lecture Notes in Mathematics 1660, Berlin.

Bardi, M. and Goatin, P. (1997). *A Dirichlet type problem for nonlinear degenerate elliptic equations arising in time-optimal stochastic control*, Preprint n. 50, SISSA-ISAS, Trieste.

Bardi, M., Goatin, P. and Ishii, H. (1998). *A Dirichlet type problem for nonlinear degenerate elliptic equations arising in time-optimal stochastic control*, Preprint n. 1, Dipartimento di Matematica, Università di Padova.

Crandall, M.G., Ishii, H. and Lions, P.L. (1992). *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. 27, 1-67.

Elliott, R.J. and Kalton, N.K. (1972). *The existence of value in differential games*, Mem. Amer. Math. Soc. 126.

Fleming, W.H. and Soner, H.M. (1993). *Controlled Markov Process and Viscosity Solutions*, Springer-Verlag, New York.

Fleming, W.H. and Souganidis, P.E. (1989). *On the existence of value functions of two-players, zero-sum stochastic differential games*, Indiana Univ. Math. J. 38, 293-314.

Friedman, A. (1976). *Stochastic differential equations and applications*, vol. 2, Academic Press, New York.

Gautier, S. and Thibault, L. (1993). *Viability for constrained stochastic differential equations*, Differential Integral Equations 6, 1395-1414.

Guseinov, H.G., Subbotin, A.I. and Ushakov, V.N. (1985). *Derivatives for multivalued mappings with applications to game-theoretical problems of control*, Problems Control Inform. Theory 14, 155-167.

Ishii, H. (1989). *A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations*, Ann. Scuola Norm. Sup. Pisa (IV) 16, 105-135.

Ishii, H. and Lions, P.L. (1990). *Viscosity solutions of fully nonlinear second order elliptic partial differential equations*, J. Differential Equations 83, 26-78.

Ikeda, N. and Watanabe, S. (1981). *Stochastic differential equations and diffusion processes*, North Holland, Amsterdam.

Ledyaev, Yu.S. (1994). *Criteria for viability of trajectories of nonautonomous differential inclusions and their applications*, J. Math. Anal. Appl. 182, 165-188.

Lions, P.L. (1983a). *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Part 1: The dynamic programming principle and applications, Part 2: Viscosity solutions and uniqueness*, Comm. Partial Differential Equations 8, 1101-1174 and 1229-1276.

Lions, P.L. (1983b). *On the Hamilton-Jacobi-Bellman equations*, Acta Appl. Math. 1, 17-41.

Milian, A. (1993). *A note on the stochastic invariance for the Ito equations*, Bull. Polish Acad. Sci. Math. 41, 139-150.

Milian, A. (1995). *Stochastic viability and a comparison theorem*, Colloq. Math. 68, 297-316.

Milian, A. (1997). *Invariance for stochastic equations with regular coefficients*, Stochastic Anal. Appl. 15, 91-101.

Nagumo, N. (1942). *Uber die lage der integralkurven gewöhnlicher differentialgleichungen*, Proc. Phys. Math. Soc. Japan 24, 551-559.

Rockafellar, R.T. and Wets, R.J-B. (1998). *Variational Analysis*, Springer-Verlag, Berlin.

Warga, J. (1972) *Optimal control of differential and functional equations*, Academic Press, New York.