Macroscopic traffic flow models on networks - I

Paola Goatin

INRIA Sophia Antipolis - Méditerranée
paola.goatin@inria.fr

Rouen, November 5-9, 2018

Outline of the talk

Road traffic flow models

Three possible scales:

- Microscopic:
  - ODEs system

$$\dot{x}_i = v_i, \quad \dot{v}_i = C\,\frac{v_{i+1} - v_i}{x_{i+1} - x_i} \qquad \text{(“follow-the-leader”)}$$

  - numerical simulations (`http://www.traffic-simulation.de/`)
  - many parameters

Road traffic flow models

Three possible scales:

- Microscopic:
  - ODEs system

$$\dot{x}_i = v_i, \quad \dot{v}_i = C \, \frac{v_{i+1} - v_i}{x_{i+1} - x_i} \qquad \text{(``follow-the-leader'')}$$

  - numerical simulations (`http://www.traffic-simulation.de/`)
  - many parameters

- Kinetic:
  - distribution function of the microscopic states
  - Boltzmann-like equations

Road traffic flow models

Three possible scales:

- Microscopic:
  - ODEs system
$$\dot{x}_i = v_i, \quad \dot{v}_i = C\,\frac{v_{i+1} - v_i}{x_{i+1} - x_i} \qquad \text{(“follow-the-leader”)}$$
  - numerical simulations (`http://www.traffic-simulation.de/`)
  - many parameters
- Kinetic:
  - distribution function of the microscopic states
  - Boltzmann-like equations
- Macroscopic:
  - PDEs from fluid dynamics
  - analytical theory
  - few parameters
  - suitable to formulate control and optimization problems

Outline of the talk

Macroscopic models

$$\Big[\text{number of vehicles in } [a,b] \text{ at time } t\Big] = \int_a^b \rho(t,x)\ dx$$

must be conserved!

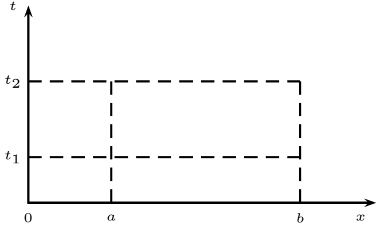
$$\begin{array}{ccc} \int_a^b \rho(t_2-,x)dx & = & \int_a^b \rho(t_1+,x)dx \\ & + & \\ \int_{t_1}^{t_2} f(t,a+)dt & - & \int_{t_1}^{t_2} f(t,b-)dt \end{array}$$

$$\Downarrow$$

divergence theorem for (ρ, f)

$$\Downarrow$$

$$\int_{t_1}^{t_2} \int_a^b \partial_t \rho + \partial_x f\ dx\ dt = 0$$



Basic requirements

$$\partial_t \rho(t,x) + \partial_x f(t,x) = 0$$

- No information propagates faster than vehicles (anisotropy)
- Flux-density relation: $f(t,x) = \rho(t,x)v(t,x)$.
- Density and mean velocity must be non-negative and bounded: $0 \leq \rho(t,x),\ v(t,x) < +\infty,\ \forall x, t > 0$.
- Different from fluid dynamics:
  - preferred direction
  - no conservation of momentum / energy
  - no viscosity
  - Avogadro number for vehicles: 106 vh/lane$\times$km $\ll 6 \cdot 10^{23}$

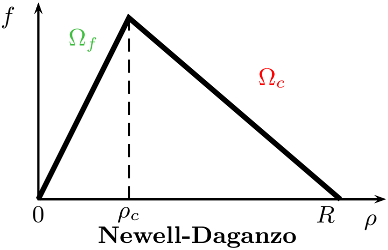
First order models

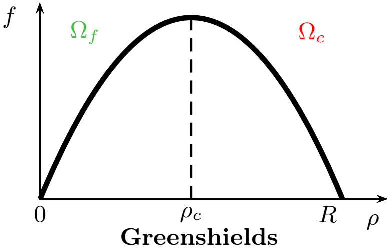
Lighthill-Whitham '55, Richards '56, Greenshields '35:

- Non-linear transport equation: scalar conservation law

$$\partial_t \rho + \partial_x f(\rho) = 0, \qquad f(\rho) = \rho v(\rho)$$

- Empirical flux function: fundamental diagram



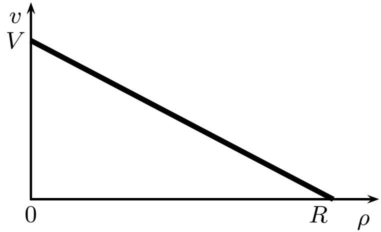


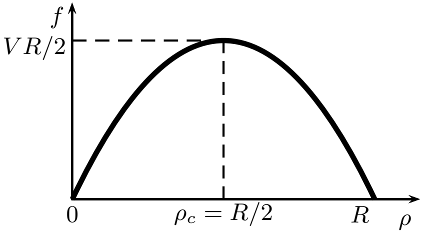
with R the maximal or *jam* density and ρ_c the critical density:

- flux is increasing for $\rho \leq \rho_c$: free-flow phase
- flux is decreasing for $\rho \geq \rho_c$: congestion phase

First order models: fundamental diagrams

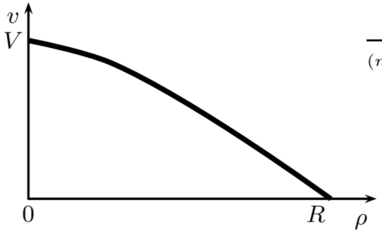
Greenshields (1935): $v(\rho) = V\left(1 - \frac{\rho}{R}\right)$

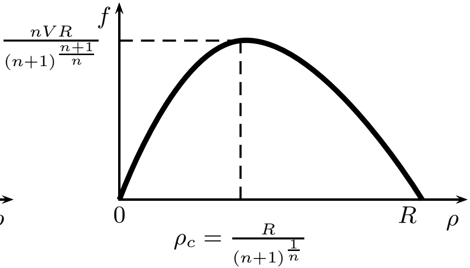




First order models: fundamental diagrams

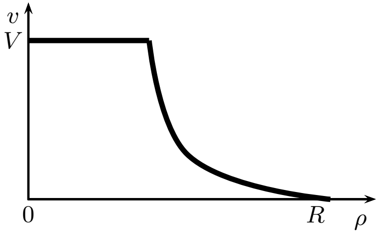
Greenshields (gen.): $v(\rho) = V\left(1 - \left(\frac{\rho}{R}\right)^n\right), \;\; n \in \mathbb{N}$

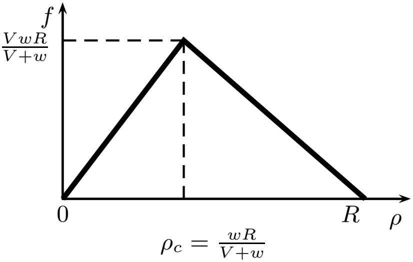




First order models: fundamental diagrams

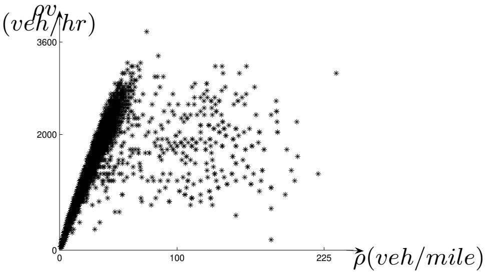
Newell-Daganzo (triangular): $v(\rho) = \min\left\{V, w\left(\frac{R}{\rho} - 1\right)\right\}$





Motivation for higher order models

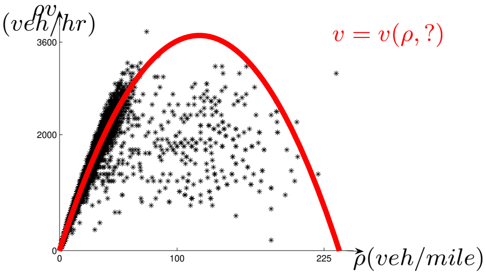
- Traffic satisfies "mass" conservation. What about other fundamental conservation principles from fluid dynamics: conservation of momentum, conservation of energy?
- Experimental observations of fundamental diagrams are more complex than postulated by first order traffic models



Viale Muro Torto, Roma

Motivation for higher order models

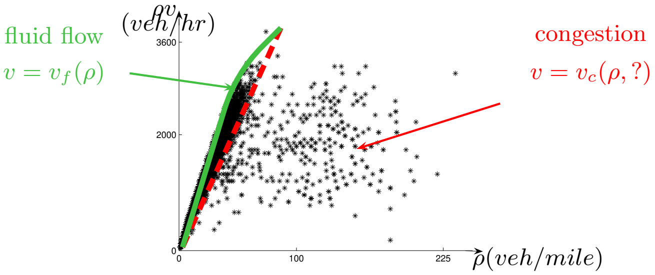
- Traffic satisfies "mass" conservation. What about other fundamental conservation principles from fluid dynamics: conservation of momentum, conservation of energy?
- Experimental observations of fundamental diagrams are more complex than postulated by first order traffic models



Viale Muro Torto, Roma

Motivation for higher order models

- Traffic satisfies "mass" conservation. What about other fundamental conservation principles from fluid dynamics: conservation of momentum, conservation of energy?
- Experimental observations of fundamental diagrams are more complex than postulated by first order traffic models



Viale Muro Torto, Roma

Second order models

- **Payne '71:**

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t v + v\partial_x v = -\dfrac{c_0^2}{\rho}\partial_x \rho + \dfrac{v_*(\rho) - v}{\tau} \end{cases}$$

Critics (Del Castillo et al. '94, Daganzo '95):

- drivers should have only positive speeds;
- anisotropy: drivers should react only to stimuli from the front.

Second order models

- **Payne '71:**

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t v + v \partial_x v = -\dfrac{c_0^2}{\rho} \partial_x \rho + \dfrac{v_*(\rho) - v}{\tau} \end{cases}$$

- **Aw-Rascle '00:**

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho w) + \partial_x(\rho v w) = 0 \end{cases} \qquad v = v(\rho, w)$$

$w = v + p(\rho)$ Lagrangian marker, $p = p(\rho)$ "pressure"

Second order models

- **Payne '71:**

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t v + v \partial_x v = -\dfrac{c_0^2}{\rho} \partial_x \rho + \dfrac{v_*(\rho) - v}{\tau} \end{cases}$$

- **Aw-Rascle '00:**

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho w) + \partial_x(\rho v w) = 0 \end{cases} \qquad v = v(\rho, w)$$

$w = v + p(\rho)$ Lagrangian marker, $p = p(\rho)$ “pressure"

- **Colombo '02:**

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t q + \partial_x((q - Q)v) = 0 \end{cases} \qquad v = v(\rho, q)$$

q “momentum”, Q road parameter

Other macroscopic models (not a complete list)

- **Non-local LWR:**
  S. Blandin, P. Goatin. Well-posedness of a conservation law with non-local flux arising in traffic flow modeling. Numer. Math. 2016.
- **Multi-class LWR:**
  S. Benzoni-Gavage, R. M. Colombo. An n-populations model for traffic flow. European J. Appl. Math. 2003.
- **Phase-transition:**
  R. M. Colombo. Hyperbolic phase transitions in traffic flow. SIAM J. Appl. Math. 2002.
- **Multilane:**
  J. M. Greenberg, A. Klar, M. Rascle. Congestion on multilane highways. SIAM J. Appl. Math. 2003.
- **Third order:**
  D. Helbing. Improved fluid-dynamic model for vehicular traffic. Phys. Rev. E 1995.

Outline of the talk

1. Traffic flow models
2. Introduction to macroscopic models
3. Conservation laws

Hyperbolic systems of conservation laws

$$\partial_t \mathbf{u} + \mathrm{div}_x f(t, x, \mathbf{u}) = 0 \qquad \begin{array}{l} t \in \mathbb{R}^+ \\ x \in \mathbb{R}^N \\ \mathbf{u} \in \mathbb{R}^n \end{array}$$

$N = 1$, $n \geq 1$

Existence: Glimm (1965)

Well posedness:
Bressan, Colombo (1995)
Bressan, Liu, Yang (1999)
Bressan, Crasta, Piccoli (2000)
Bianchini, Bressan (2005)

$N \geq 1$, $n = 1$

Existence: Kružkov (1970)

Well posed.: Kružkov (1970)

Hyperbolic systems of conservation laws

We deal with a system of PDEs of the form

$$\begin{aligned} &\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0, \\ &\mathbf{u}(0,x) = \mathbf{u}_0(x), \end{aligned}$$

where $t \in [0, +\infty[$, $x \in \mathbb{R}^1$,
$\mathbf{u} = \mathbf{u}(t,x) \in \mathbb{R}^n$ conserved quantities,
$f : \mathbb{R}^n \to \mathbb{R}^n$ flux.

Hyperbolic systems of conservation laws

We deal with a system of PDEs of the form

$$\begin{aligned} &\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0, \\ &\mathbf{u}(0,x) = \mathbf{u}_0(x), \end{aligned}$$

where $t \in [0,+\infty[$, $x \in \mathbb{R}^{1}$,
$\mathbf{u} = \mathbf{u}(t,x) \in \mathbb{R}^n$ conserved quantities,
$f : \mathbb{R}^n \to \mathbb{R}^n$ flux.

We need to answer the following questions:

- Does a solution *always* exist?
- Is it unique?
- How to *find* it?

Linear case: $f(\mathbf{u}) = A\mathbf{u}$, with $A \in \mathbb{R}^{n\times n}$ matrix

The system is (strictly) hyperbolic if A admits n real distinct eigenvalues $\lambda_1 < \ldots < \lambda_n$

$$\begin{cases} \partial_t u_i + \lambda_i \partial_x u_i = 0 \\ \mathbf{u}(0,x) = \bar{\mathbf{u}}(x) \end{cases} \qquad \text{où } u_i = \mathbf{l}_i \cdot \mathbf{u},\ \ \mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{r}_i$$

Linear case: $f(\mathbf{u}) = A\mathbf{u}$, with $A \in \mathbb{R}^{n\times n}$ matrix

The system is (strictly) hyperbolic if A admits n real distinct eigenvalues $\lambda_1 < \ldots < \lambda_n$

$$\begin{cases} \partial_t u_i + \lambda_i \partial_x u_i = 0 \\ \mathbf{u}(0,x) = \bar{\mathbf{u}}(x) \end{cases} \qquad \text{où } u_i = \mathbf{l}_i \cdot \mathbf{u},\ \ \mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{r}_i$$

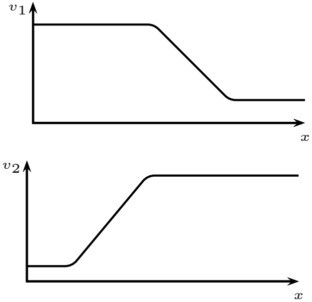
Characteristics method:

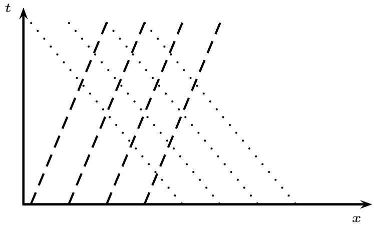
$$\begin{aligned} \dot{y}_i(t) = \lambda_i \quad &\Rightarrow \quad \frac{d}{dt} u_i(t, y_i(t)) = \partial_t u_i + \lambda_i \partial_x u_i = 0 \\ &\Rightarrow \quad u_i(t, y_i(t)) = \bar{u}_i(y_0) \\ &\Rightarrow \quad u_i(t,x) = \bar{u}_i(x - \lambda_i t) \end{aligned}$$

Linear case: $f(\mathbf{u}) = A\mathbf{u}$, with $A \in \mathbb{R}^{n\times n}$ matrix

Waves superposition:

$$\mathbf{u}(t,x) = \sum_{i=1}^{n} \bar{u}_i(x - \lambda_i t)\mathbf{r}_i$$



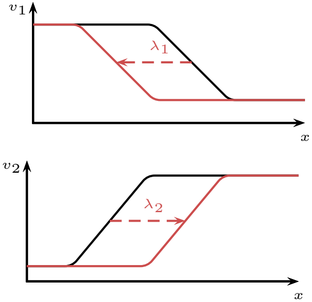


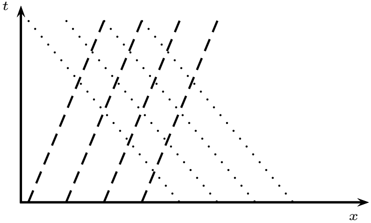
$\Rightarrow$ **existence and uniqueness**

Linear case: $f(\mathbf{u}) = A\mathbf{u}$, with $A \in \mathbb{R}^{n\times n}$ matrix

Waves superposition:

$$\mathbf{u}(t,x) = \sum_{i=1}^{n} \bar{u}_i(x - \lambda_i t)\mathbf{r}_i$$





$\Rightarrow$ **existence and uniqueness**

NON-linear case

Strictly hyperbolic: the Jacobian matrix $Df(\mathbf{u})$ has n distinct real eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \ldots < \lambda_n(\mathbf{u})$$

$$\text{eigenvectors} \quad \mathbf{r}_1(\mathbf{u}), \ldots, \mathbf{r}_n(\mathbf{u})$$

Genuinely non-linear: $\nabla\lambda_i \cdot \mathbf{r}_i > 0$ ($\sim$ convex flux)

Linearly degenerate: $\nabla\lambda_i \cdot \mathbf{r}_i \equiv 0$ ($\sim$ linear flux)

Conservation laws

Basic tools:

- Discontinuous weak solutions: Rankine-Hugoniot relation

$$\sigma(\mathbf{u}_+ - \mathbf{u}_+) = f(\mathbf{u}_+) - f(\mathbf{u}_-)$$

- Uniqueness: Lax entropy condition

$$\lambda_i(\mathbf{u}_-) \geq \sigma \geq \lambda_i(\mathbf{u}_+)$$

- Riemann problem:

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$$
$$\mathbf{u}(0,x) = \begin{cases} \mathbf{u}_l & \text{si } x < 0 \\ \mathbf{u}_r & \text{si } x > 0 \end{cases}$$

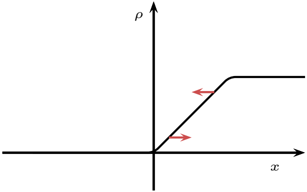
Non-linear flux ⇒ shock formation!

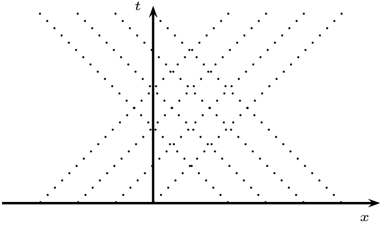
Example:

$$\partial_t \rho + \partial_x[\rho(1-\rho)] = 0$$
$$\rho_0 \text{ s.t. } \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x > 1 \end{cases}$$

Characteristics: $\rho(t, y(t)) = \rho_0(y_0)$ for $y(t)$ solution of

$$\dot{y}(t) = f'(\rho(t, y(t))) = 1 - 2\rho(t, y(t)) = 1 - 2\rho_0(y_0)$$
$$\Rightarrow \quad y(t) = (1 - 2\rho_0(y_0))\, t = \begin{cases} 1 & \text{if } y_0 < 0 \\ -1 & \text{if } y_0 > 1 \end{cases}$$





Non-linear flux $\Rightarrow$ shock formation!

Example:

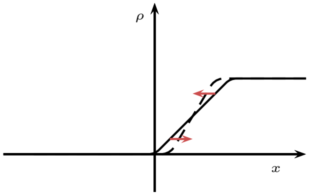
$$\partial_t \rho + \partial_x[\rho(1-\rho)] = 0$$

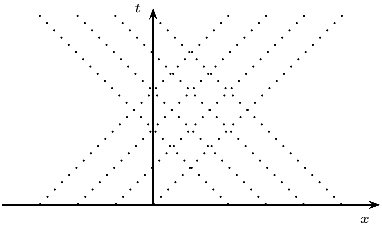
$$\rho_0 \text{ s.t. } \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x > 1 \end{cases}$$

Characteristics: $\rho(t, y(t)) = \rho_0(y_0)$ for $y(t)$ solution of

$$\dot{y}(t) = f'(\rho(t, y(t))) = 1 - 2\rho(t, y(t)) = 1 - 2\rho_0(y_0)$$

$$\Rightarrow \quad y(t) = (1 - 2\rho_0(y_0))\, t = \begin{cases} 1 & \text{if } y_0 < 0 \\ -1 & \text{if } y_0 > 1 \end{cases}$$





Non-linear flux $\Rightarrow$ shock formation!

Example:

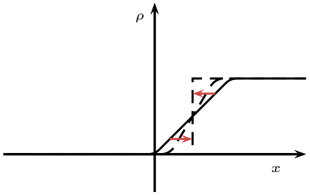
$$\partial_t \rho + \partial_x[\rho(1-\rho)] = 0$$

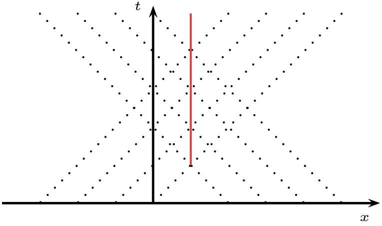
$$\rho_0 \text{ s.t. } \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x > 1 \end{cases}$$

Characteristics: $\rho(t, y(t)) = \rho_0(y_0)$ for $y(t)$ solution of

$$\dot{y}(t) = f'(\rho(t, y(t))) = 1 - 2\rho(t, y(t)) = 1 - 2\rho_0(y_0)$$

$$\Rightarrow \quad y(t) = (1 - 2\rho_0(y_0))\, t = \begin{cases} 1 & \text{if } y_0 < 0 \\ -1 & \text{if } y_0 > 1 \end{cases}$$





Weak solutions

Distributions: if $\mathbf{u}$ smooth

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \left(\partial_t \mathbf{u} + \partial_x f(\mathbf{u})\right) \phi \, dx \, dt =$$

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \mathbf{u} \, \partial_t \phi + f(\mathbf{u}) \, \partial_x \phi \, dx \, dt = 0 \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^+ \times \mathbb{R})$$

Definition

$\mathbf{u} \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^n)$ is a **weak solution** if

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \left(\mathbf{u} \, \partial_t \phi + f(\mathbf{u}) \, \partial_x \phi\right) dx \, dt = 0 \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^n)$$

Weak solutions

Distributions: if $\mathbf{u}$ smooth

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} (\partial_t \mathbf{u} + \partial_x f(\mathbf{u}))\, \phi \; dx \; dt =$$

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} \mathbf{u} \; \partial_t \phi + f(\mathbf{u}) \; \partial_x \phi \; dx \; dt = 0 \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^+ \times \mathbb{R})$$

Definition

$\mathbf{u} \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^n)$ is a **weak solution** if

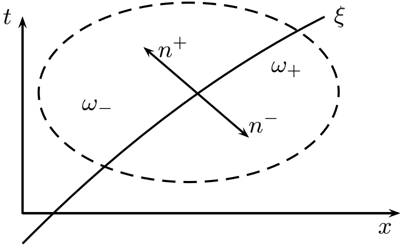
$$\int_0^{+\infty}\int_{-\infty}^{+\infty} (\mathbf{u} \; \partial_t \phi + f(\mathbf{u}) \; \partial_x \phi)\, dx \; dt = 0 \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^n)$$

Adding an initial condition $\mathbf{u}(0,x) = \mathbf{u}_0(x)$, this becomes

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} (\mathbf{u} \; \partial_t \phi + f(\mathbf{u}) \; \partial_x \phi)\, dx \; dt + \int_{-\infty}^{+\infty} \mathbf{u}_0(x)\phi(0,x)\; dx = 0 \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}\times\mathbb{R}; \mathbb{R}^n)$$

Weak solutions

Along a discontinuity (shock) $x = \xi(t)$:



$$n^+ = (-\dot{\xi}, 1), \qquad n^- = (\dot{\xi}, -1)$$

Rankine-Hugoniot conditions

Using Green's formula:

$$
\begin{aligned}
0 &= \iint_{\omega} \mathbf{u}\ \partial_t \phi + f(\mathbf{u})\ \partial_x \phi\ dx\ dt \\
&= \iint_{\omega_-} + \iint_{\omega_+} \mathbf{u}\ \partial_t \phi + f(\mathbf{u})\ \partial_x \phi\ dx\ dt \\
&= \int_{\partial\omega_-} (\mathbf{u}_- n_t^- + f(\mathbf{u}_-) n_x^-)\phi\ ds - \iint_{\omega_-} (\partial_t \mathbf{u} + \partial_x f(\mathbf{u})) dt\ dx \\
&\quad + \int_{\partial\omega_+} (\mathbf{u}_+ n_t^+ + f(\mathbf{u}_+) n_x^+)\phi\ ds - \iint_{\omega_+} (\partial_t \mathbf{u} + \partial_x f(\mathbf{u})) dt\ dx \\
&= \int_{x=\xi(t)} (\mathbf{u}_- n_t^- + f(\mathbf{u}_-) n_x^-)\phi\ ds + \int_{x=\xi(t)} (\mathbf{u}_+ n_t^+ + f(\mathbf{u}_+) n_x^+)\phi\ ds \\
&= \int_{x=\xi(t)} \big((\mathbf{u}_+ - \mathbf{u}_-) n_t + (f(\mathbf{u}_+) - f(\mathbf{u}_-)) n_x\big)\phi\ ds
\end{aligned}
$$

$$
\Rightarrow \qquad \dot{\xi}(\mathbf{u}_+ - \mathbf{u}_-) = f(\mathbf{u}_+) - f(\mathbf{u}_-)
$$

Rankine-Hugoniot condition

In the previous example:

$$\partial_t \rho + \partial_x[\rho(1-\rho)] = 0$$
$$\rho_0 \text{ s.t. } \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x > 1 \end{cases}$$

therefore

$$\rho_- = 0, \ \rho_+ = 1 \quad \Rightarrow \quad \dot{\xi} = \frac{f(\rho_+) - f(\rho_-)}{\rho_+ - \rho_-} = 1 - \rho_+ - \rho_- = 0$$

it is a *stationary* shock!

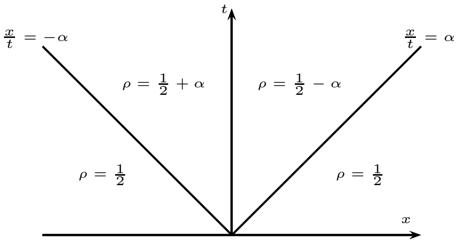
Non-uniqueness of weak solutions

Example:

$$\partial_t \rho + \partial_x[\rho(1-\rho)] = 0$$
$$\rho_0(x) \equiv 1/2$$

We can construct an infinite number of solutions satisfying RH $\forall \alpha > 0$:

$$\rho(t,x) = \begin{cases} 1/2 & x < -\alpha t \\ 1/2 + \alpha & -\alpha t < x < 0 \\ 1/2 - \alpha & 0 < x < \alpha t \\ 1/2 & x > \alpha t \end{cases}$$



Entropy conditions

$$\mathbf{u}^\varepsilon \text{ solution of } \partial_t \mathbf{u}^\varepsilon + \partial_x f(\mathbf{u}^\varepsilon) = \varepsilon \partial_{xx} \mathbf{u}^\varepsilon$$
$$\text{converges to } \mathbf{u} \text{ solution of } \partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$$

Entropy conditions

$$\mathbf{u}^\varepsilon \text{ solution of } \partial_t \mathbf{u}^\varepsilon + \partial_x f(\mathbf{u}^\varepsilon) = \varepsilon \partial_{xx} \mathbf{u}^\varepsilon$$
$$\text{converges to } \mathbf{u} \text{ solution of } \partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$$

Entropy: $E = E(\mathbf{u})$ convex entropy: $D^2 E(\mathbf{u}) > 0$

$F = F(\mathbf{u})$ s.t. $\nabla F(\mathbf{u}) = \nabla E(\mathbf{u}) Df(\mathbf{u})$ entropy flux

Entropy conditions

$$\mathbf{u}^\varepsilon \text{ solution of } \partial_t \mathbf{u}^\varepsilon + \partial_x f(\mathbf{u}^\varepsilon) = \varepsilon \partial_{xx} \mathbf{u}^\varepsilon$$
$$\text{converges to } \mathbf{u} \text{ solution of } \partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$$

Entropy: $E = E(\mathbf{u})$ convex entropy: $D^2 E(\mathbf{u}) > 0$

$F = F(\mathbf{u})$ s.t. $\nabla F(\mathbf{u}) = \nabla E(\mathbf{u}) D f(\mathbf{u})$ entropy flux

then $\mathbf{u}^\varepsilon$ satisfies

$$\begin{aligned}\partial_t E(\mathbf{u}^\varepsilon) + \partial_x F(\mathbf{u}^\varepsilon) =& \nabla E(\mathbf{u}^\varepsilon)\big(\partial_t \mathbf{u}^\varepsilon + \partial_x f(\mathbf{u}^\varepsilon)\big) = \varepsilon \nabla E(\mathbf{u}^\varepsilon) \partial_{xx} \mathbf{u}^\varepsilon \\ =& \varepsilon \partial_{xx} E(\mathbf{u}^\varepsilon) - \varepsilon D^2 E(\mathbf{u}^\varepsilon)(\partial_x \mathbf{u}^\varepsilon \otimes \partial_x \mathbf{u}^\varepsilon) \leq \varepsilon \partial_{xx} E(\mathbf{u}^\varepsilon)\end{aligned}$$

Entropy conditions

$$\mathbf{u}^\varepsilon \text{ solution of } \partial_t \mathbf{u}^\varepsilon + \partial_x f(\mathbf{u}^\varepsilon) = \varepsilon \partial_{xx} \mathbf{u}^\varepsilon$$
$$\text{converges to } \mathbf{u} \text{ solution of } \partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$$

Entropy: $E = E(\mathbf{u})$ convex entropy: $D^2 E(\mathbf{u}) > 0$

$F = F(\mathbf{u})$ s.t. $\nabla F(\mathbf{u}) = \nabla E(\mathbf{u}) Df(\mathbf{u})$ entropy flux

then $\mathbf{u}^\varepsilon$ satisfies

$$\begin{aligned} \partial_t E(\mathbf{u}^\varepsilon) + \partial_x F(\mathbf{u}^\varepsilon) =& \nabla E(\mathbf{u}^\varepsilon)\big(\partial_t \mathbf{u}^\varepsilon + \partial_x f(\mathbf{u}^\varepsilon)\big) = \varepsilon \nabla E(\mathbf{u}^\varepsilon) \partial_{xx} \mathbf{u}^\varepsilon \\ =& \varepsilon \partial_{xx} E(\mathbf{u}^\varepsilon) - \varepsilon D^2 E(\mathbf{u}^\varepsilon)(\partial_x \mathbf{u}^\varepsilon \otimes \partial_x \mathbf{u}^\varepsilon) \leq \varepsilon \partial_{xx} E(\mathbf{u}^\varepsilon) \end{aligned}$$

Passing to the limit $\varepsilon \to 0$, $\mathbf{u}$ we get

$$\partial_t E(\mathbf{u}) + \partial_x F(\mathbf{u}) \leq 0$$

or, in weak sense,

$$\iint E(\mathbf{u}) \partial_t \phi + F(\mathbf{u}) \partial_x \phi \; dx \; dt \geq 0, \quad \forall \phi \in \mathcal{C}_c^1, \phi \geq 0$$

Entropy conditions

Definition

$E \in C^1(\mathbb{R}^n;\mathbb{R})$ is an **entropy** if it is convex and there exists $F \in C^1(\mathbb{R}^n;\mathbb{R})$ such that

$$\nabla E(\mathbf{u}) \cdot Df(\mathbf{u}) = \nabla F(\mathbf{u}) \quad \forall \mathbf{u}$$

F is an **entropy flux** for E.

Entropy conditions

Definition

$E \in C^1(\mathbb{R}^n;\mathbb{R})$ is an **entropy** if it is convex and there exists $F \in C^1(\mathbb{R}^n;\mathbb{R})$ such that

$$\nabla E(\mathbf{u}) \cdot Df(\mathbf{u}) = \nabla F(\mathbf{u}) \quad \forall \mathbf{u}$$

F is an **entropy flux** for E.

Definition ($n \geq 1$)

A weak solution $\mathbf{u} = \mathbf{u}(t,x)$ is **entropy admissible** if for every entropy-entropy flux pairs (E,F) it holds

$$\iint E(\mathbf{u})\partial_t \phi + F(\mathbf{u})\partial_x \phi \; dx \; dt \geq 0, \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^+ \times \mathbb{R};\mathbb{R}^+)$$

Entropy conditions

Definition

$E \in C^1(\mathbb{R}^n;\mathbb{R})$ is an **entropy** if it is convex and there exists $F \in C^1(\mathbb{R}^n;\mathbb{R})$ such that

$$\nabla E(\mathbf{u}) \cdot Df(\mathbf{u}) = \nabla F(\mathbf{u}) \quad \forall \mathbf{u}$$

F is an **entropy flux** for E.

Definition ($n \geq 1$)

A weak solution $\mathbf{u} = \mathbf{u}(t,x)$ is **entropy admissible** if for every entropy-entropy flux pairs (E,F) it holds

$$\iint E(\mathbf{u})\partial_t\phi + F(\mathbf{u})\partial_x\phi \; dx \; dt \geq 0, \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^+ \times \mathbb{R};\mathbb{R}^+)$$

Definition ($n = 1$) [A.I. Volpert, Math. USSR Sbornik, 1967]

A weak solution $u = u(t,x)$ is **entropy admissible** if for every $\kappa \in \mathbb{R}$ it holds

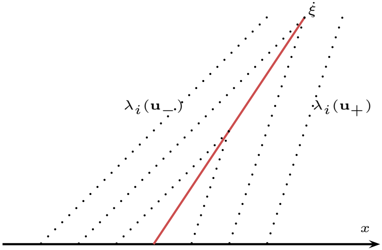
$$\iint |u-\kappa|\,\partial_t\phi + \operatorname{sgn}(u-\kappa)\,(f(u)-f(\kappa))\,\partial_x\phi \; dx \; dt \geq 0, \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^+\times\mathbb{R};\mathbb{R}^+)$$

Lax entropy conditions

If f is strictly convex ($f''(u) \geq c > 0$) or concave ($f''(u) \leq -c < 0$), the entropy condition is equivalent to

$$f'(\mathbf{u}_-) > \dot{\xi} > f'(\mathbf{u}_+) \qquad \text{(scalar case)}$$

characteristics impinge on the shock



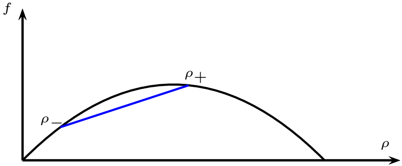
Lax condition (scalar case)

Example: concave flux

$$\partial_t \rho + \partial_x [\rho(1-\rho)] = 0$$

Lax condition writes:

$$1 - 2\rho_- > 1 - \rho_- - \rho_+ > 1 - 2\rho_+$$



$\Rightarrow$ the shock is admissible iff $\rho_- < \rho_+$

Riemann problem

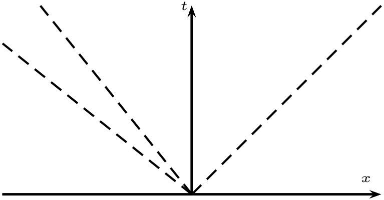
The simplest non-trivial Cauchy problem:

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$$
$$\mathbf{u}(0,x) = \begin{cases} \mathbf{u}_L & \text{si } x < 0 \\ \mathbf{u}_R & \text{si } x > 0 \end{cases}$$

Solution must be self-similar

$$\mathbf{u}(t,x) \equiv \mathbf{u}(at,ax) \quad \forall a > 0$$

$\Rightarrow$ we look for $\mathbf{u}$ of the form $\mathbf{u}(t,x) = \mathbf{v}(x/t)$



The Riemann Solver

Definition

The **Riemann Solver** corresponding to

$$(RP)\begin{cases} \partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 \\ \mathbf{u}(0,x) = \begin{cases} \mathbf{u}_L & \text{if } x < 0 \\ \mathbf{u}_R & \text{if } x > 0 \end{cases} \end{cases}$$

is the map $\mathcal{RS} : \Omega^2 \to L^1_{loc}(\mathbb{R};\Omega)$

$$(t,x) \mapsto \mathcal{RS}(\mathbf{u}_L, \mathbf{u}_R)(x/t)$$

given by the weak entropy solution of (RP)

The Riemann Solver ($n = 1$)

- if $f'(u_L) > f'(u_R) \Rightarrow$ shock of speed $\sigma = \frac{f(u_R)-f(u_L)}{u_R-u_L}$
- if $f'(u_L) < f'(u_R) \Rightarrow$ rarefaction wave:

$$
\begin{aligned}
u(t,x) = v(x/t), \ \ x/t = \lambda \quad &\Rightarrow \quad \frac{d}{d\lambda} f(v) = \lambda \frac{d}{d\lambda} v \\
&\qquad \left(f'(v) - \lambda\right) \frac{d}{d\lambda} v \\
\frac{d}{d\lambda} v \neq 0 \quad &\Rightarrow \quad f'(v) = \lambda
\end{aligned}
$$

that is: $u(t,x) = v(x/t)$ s.t. $f'(u(t,x)) = x/t$

What if f is not strictly convex/concave?

The Riemann Solver ($n > 1$)

- $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ conserved quantities
- $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ fluxes

$$\begin{cases} \partial_t u_1 + \partial_x f_1(u_1, \ldots, u_n) = 0 \\ \ldots \\ \partial_t u_n + \partial_x f_n(u_1, \ldots, u_n) = 0 \end{cases}$$

Jacobian matrix: $$Df(\mathbf{u}) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \ldots & \frac{\partial f_1}{\partial u_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial f_n}{\partial u_1} & \ldots & \frac{\partial f_n}{\partial u_n} \end{pmatrix}$$

The Riemann Solver ($n > 1$)

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$$

Definition

The system is **strictly hyperbolic** if the Jacobian matrix $Df(\mathbf{u})$ has n real and distinct eigenvalues

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \ldots < \lambda_n(\mathbf{u})$$

Right eigenvectors: $r_1(\mathbf{u}), \ldots, r_n(\mathbf{u})$
Left eigenvectors: $\ell_1(\mathbf{u}), \ldots, \ell_n(\mathbf{u})$

Definition

- The i-th field is **genuinely non-linear** if $D\lambda_i(\mathbf{u}) \cdot r_i(\mathbf{u}) > 0$ for every $\mathbf{u}$
- The i-th field is **linearly degenarate** if $D\lambda_i(\mathbf{u}) \cdot r_i(\mathbf{u}) = 0$ for every $\mathbf{u}$

The Riemann Solver $(n > 1)$

- i-**rarefaction curve** through $\mathbf{u}_0$: $\sigma \mapsto R_i(\sigma)(\mathbf{u}_0)$
  Integral curve of r_i through $\mathbf{u}_0$:

$$\begin{cases} \dfrac{d}{d\sigma} R_i(\sigma)(\mathbf{u}_0) = r_i(R_i(\sigma)(\mathbf{u}_0)) \\ R_i(0)(\mathbf{u}_0) = \mathbf{u}_0 \end{cases}$$

- i-**shock curve** through $\mathbf{u}_0$: $\sigma \mapsto S_i(\sigma)(\mathbf{u}_0)$
  Set of points $\mathbf{u}$ connected to u_0 by an i-shock:

$$f(S_i(\sigma)(\mathbf{u}_0)) - f(\mathbf{u}_0) = \lambda_i^S(\sigma)\left[S_i(\sigma)(\mathbf{u}_0) - \mathbf{u}_0\right]$$

  with $\lambda_i^S(\sigma) = \lambda_i(S_i(\sigma)(\mathbf{u}_0))$ (and $\lambda_i^S(0) = \lambda_i(\mathbf{u}_0)$)

The Riemann Solver ($n > 1$)

- i-**rarefaction curve** through $\mathbf{u}_0$: $\sigma \mapsto R_i(\sigma)(\mathbf{u}_0)$
  Integral curve of r_i through $\mathbf{u}_0$:

$$\begin{cases} \dfrac{d}{d\sigma} R_i(\sigma)(\mathbf{u}_0) = r_i(R_i(\sigma)(\mathbf{u}_0)) \\ R_i(0)(\mathbf{u}_0) = \mathbf{u}_0 \end{cases}$$

- i-**shock curve** through $\mathbf{u}_0$: $\sigma \mapsto S_i(\sigma)(\mathbf{u}_0)$
  Set of points $\mathbf{u}$ connected to u_0 by an i-shock:

$$f(S_i(\sigma)(\mathbf{u}_0)) - f(\mathbf{u}_0) = \lambda_i^S(\sigma)\left[S_i(\sigma)(\mathbf{u}_0) - \mathbf{u}_0\right]$$

with $\lambda_i^S(\sigma) = \lambda_i(S_i(\sigma)(\mathbf{u}_0))$ (and $\lambda_i^S(0) = \lambda_i(\mathbf{u}_0)$)

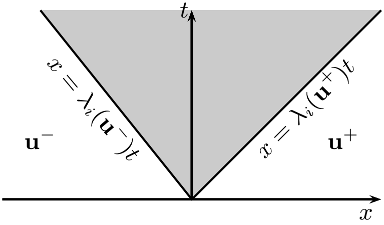
Note

If i-th field is linearly degenerate, then $R_i(\sigma)(\mathbf{u}_0) = S_i(\sigma)(\mathbf{u}_0)$ for all σ

The Riemann Solver ($n > 1$)

Centered rarefaction wave:
i-th field genuinely non-linear and $\mathbf{u}^+ = R_i(\bar\sigma)(\mathbf{u}^-)$ for some $\bar\sigma > 0$

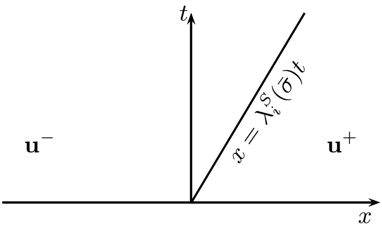
$$\mathbf{u}(t,x) = \begin{cases} \mathbf{u}^- & x < \lambda_i(\mathbf{u}^-)t \\ R_i(\sigma)(\mathbf{u}^-) & \lambda_i(\mathbf{u}^-) < x/t < \lambda_i(\mathbf{u}^+), x/t = \lambda_i(\sigma), \sigma \in [0,\bar\sigma] \\ \mathbf{u}^+ & x > \lambda_i(\mathbf{u}^+)t \end{cases}$$



The Riemann Solver ($n > 1$)

Shock or contact discontinuity:
if $\mathbf{u}^+ = S_i(\bar\sigma)(\mathbf{u}^-)$ for some $\bar\sigma > 0$

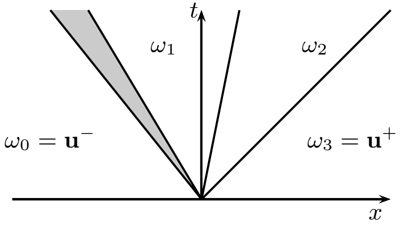
$$\mathbf{u}(t,x) = \begin{cases} \mathbf{u}^- & x < \lambda_i^S(\bar\sigma)t \\ \mathbf{u}^+ & x > \lambda_i^S(\bar\sigma)t \end{cases}$$



Solution of the general Riemann problem (P. Lax, 1957)

Find states $\omega_0, \omega_1, \ldots, \omega_n$ such that $\omega_0 = \mathbf{u}_L$ $\omega_n = \mathbf{u}_R$ and

$$\begin{array}{lll} \text{either} & \omega_i = R_i(\sigma_i)(\omega_{i-1}) & \sigma_i \geq 0 \\ \text{or} & \omega_i = S_i(\sigma_i)(\omega_{i-1}) & \sigma_i < 0 \end{array}$$



Theorem

Let the system be strictly hyperbolic and each characteristic field either genuinely non-linear or linearly degenerate. Then for every compact set $K \subset \Omega$, there exists $\delta > 0$ s.t. the Riemann problem has a unique weak solution for all $\mathbf{u}_L \in K$, $|\mathbf{u}_R - \mathbf{u}_L| \leq \delta$.

The Riemann problem for ARZ system

Aw-Rascle-Zhang:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0 \end{cases}$$

$w := v + p(\rho)$ Lagrangian marker,
$p(\rho) = \rho^\gamma$, $\gamma > 0$, pressure, $p'(\rho) > 0$

$y := \rho w = \rho(v + p(\rho)) \quad \longrightarrow \quad \mathbf{u} = \begin{pmatrix} \rho \\ y \end{pmatrix}$ conserved variables

$f(\mathbf{u}) = \begin{pmatrix} y - \rho p(\rho) \\ \frac{y^2}{\rho} - y p(\rho) \end{pmatrix}$ flux

The Riemann problem for ARZ system

In (ρ, v) variables:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t v + (v - \rho p'(\rho))\partial_x v) = 0 \end{cases}$$

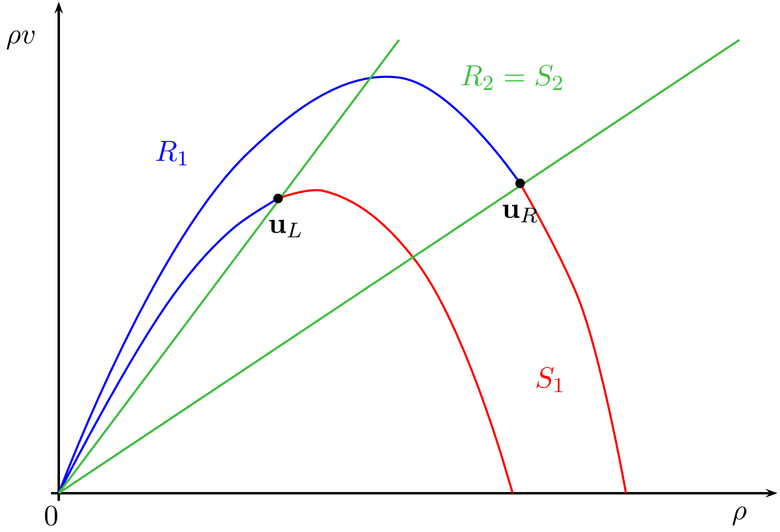
$$Df(\rho, v) = \begin{pmatrix} v & \rho \\ 0 & v - \rho p'(\rho) \end{pmatrix} \qquad \lambda_1(\rho, v) = v - \rho p'(\rho) < \lambda_2(\rho, v) = v, \;\; \rho \neq 0$$

$$r_1 = \begin{pmatrix} -1 \\ p'(\rho) \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

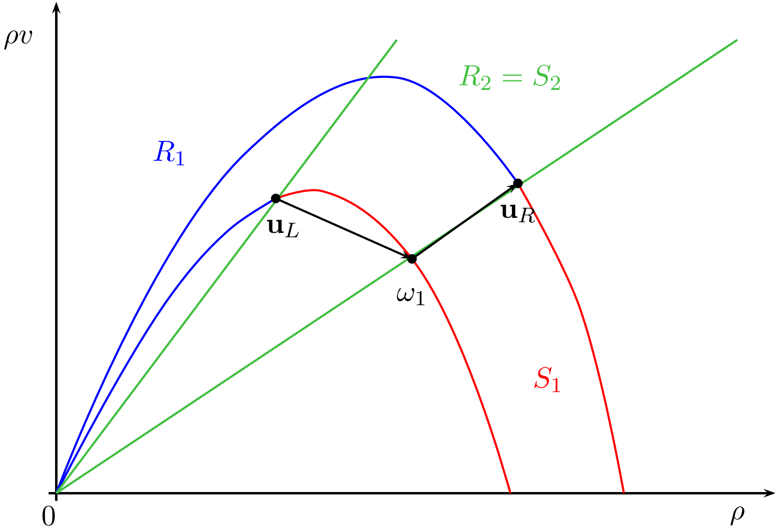
$$\begin{aligned} \nabla\lambda_1 \cdot r_1 &= \begin{pmatrix} -p'(\rho) - \rho p''(\rho) \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ p'(\rho) \end{pmatrix} = 2p'(\rho) + \rho p''(\rho) > 0 \\ \nabla\lambda_2 \cdot r_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \end{aligned}$$

The Riemann problem for ARZ system



$R_1(\mathbf{u}^-) = S_1(\mathbf{u}^-) : v + p(\rho) = v^- + p(\rho^-) \qquad R_2(\mathbf{u}^-) = S_2(\mathbf{u}^-) : v = v^-$

The Riemann problem for ARZ system



$R_1(\mathbf{u}^-) = S_1(\mathbf{u}^-) : v + p(\rho) = v^- + p(\rho^-)$ $\qquad$ $R_2(\mathbf{u}^-) = S_2(\mathbf{u}^-) : v = v^-$

Well-posedness

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0, \qquad t > 0, x \in \mathbb{R}$$
$$\mathbf{u}(0, x) = \mathbf{u}_0(x)$$

Theorem (n=1)

Let f locally Lipschitz continuous and $\mathbf{u} \in \mathrm{BV}(\mathbb{R})$. Then the Cauchy problem admits a (unique) entropy weak solution $\mathbf{u} = \mathbf{u}(t, x)$.
Moreover:

- $\mathrm{TV}(\mathbf{u}(t, \cdot)) \leq \mathrm{TV}(\mathbf{u}_0) \qquad t > 0$
- $\|\mathbf{u}(t, \cdot)\|_\infty \leq \|\mathbf{u}_0\|_\infty \qquad t > 0$
- $\|\mathbf{u}(t, \cdot) - \mathbf{u}(s, \cdot)\|_1 \leq L|t - s|\mathrm{TV}(\mathbf{u}_0) \qquad t, s > 0$
- $\|\mathbf{u}(t, \cdot) - \mathbf{v}(t, \cdot)\|_1 \leq \|\mathbf{u}_0 - \mathbf{v}_0\|_1 \qquad t > 0$
- $\mathbf{u}_0(x) \leq \mathbf{v}_0(x)\ \forall x \in \mathbb{R} \quad \Longrightarrow \quad \mathbf{u}(t, x) \leq \mathbf{v}(t, x)\ \forall x \in \mathbb{R}, t > 0$

Well-posedness

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0, \qquad t > 0, x \in \mathbb{R}$$
$$\mathbf{u}(0,x) = \mathbf{u}_0(x)$$

Theorem (n>1)

Let the system be strictly hyperbolic and each characteristic field either genuinely non-linear or linearly degenerate.
The there exists $\delta > 0$ such that if

$$\mathrm{TV}(\mathbf{u}_0) \leq \delta$$

the Cauchy problem admits a weak solution $\mathbf{u} = \mathbf{u}(t,x)$ defined for all $t > 0$.
If the system admits an entropy E, there exists a E-admissible solution.

Remark

- No TVD property
- No maximum principle
- No uniqueness
- No comparison principle

Basic bibliography

- A. Bressan. Hyperbolic systems of conservation laws. Oxford Lecture Series in Mathematics and its Applications, 20. Oxford University Press, Oxford, 2000.
- C.M. Dafermos. Hyperbolic conservation laws in continuum physics. Fourth edition. Springer-Verlag, Berlin, 2016.
- M. Garavello, B. Piccoli. Traffic flow on networks. AIMS Series on Applied Mathematics, 1. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2006.
- S.N. Kružkov. First order quasilinear equations with several independent variables. Mat. Sb. 1970.
- P.G. LeFloch. Hyperbolic systems of conservation laws. The theory of classical and nonclassical shock waves. Lectures in Mathematics ETH Zürich. Birkäuser Verlag, Basel, 2002.
- M.D. Rosini. Macroscopic Models for Vehicular Flows and Crowd Dynamics: Theory and Applications. Heidelberg: Springer, 2013.
- D. Serre. Systems of conservation laws. 1 & 2. Cambridge University Press, 2000.