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A well posed conservation law with a variable unilateral constraint

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Abstract

This paper considers the Cauchy problem for a conservation law with a variable unilateral constraint, its motivation being, for instance, the modeling of a toll gate along a highway. This problem is solved by means of nonclassical shocks and its well posedness is proved. Then, the solutions so obtained are shown to coincide with the limits of the classical solutions to suitable conservation laws with discontinuous flux function that approximate the constrained problem.

MSC: 35L65; 90B20

Keywords: Hyperbolic conservation laws; Continuum traffic models; Riemann problem

1. Introduction

This paper is devoted to constrained Cauchy problems of the form

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ f(\rho(t,0)) \leqslant q(t), & t \in \mathbb{R}^+, \end{cases} \tag{1.1}$$

ρ being the scalar conserved quantity. (Throughout, we let $\mathbb{R}^+ = [0, +\infty[$.) This problem is motivated by the modeling of a toll gate along a road. In this case, ρ is the (mean) traffic density,

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$f(\rho) = \rho v(\rho)$ is the (mean) traffic flow and $v(\rho)$ is the (mean) traffic speed at density ρ. In other words, we deal with the classical Lighthill–Whitham [13] and Richards [15] (LWR) model which states the conservation of the total number of vehicles and postulates that traffic speed is a function of traffic density. In this framework, $q(t)$ is the maximal flow of traffic that can go through the gate at time t. Our main result is the global well posedness of (1.1) in $\mathbf{L}^1$.

In the literature, the LWR model is typically considered on the whole real line or on the half line $x > 0$. In the latter case, the model is supplemented with both an initial datum at $t = 0$ and a boundary datum along $x = 0$ describing the inflow into the road. Similarly, but less considered in the literature, it is realistic to assume that also the outflow is subject to a constraint. The present result, as a byproduct, ensures also the well posedness of the initial-boundary value problem with an unilateral constraint on the outflow at the boundary. Similarly, these results are easily extended to the initial-boundary value problem

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0, & (t,x) \in \mathbb{R}^+ \times [0,L], \\ \rho(0,x) = \rho_0(x), & x \in [0,L], \\ f(\rho(t,0)) = f_0(t), & t \in \mathbb{R}^+, \\ f(\rho(t,L)) \leqslant q(t), & t \in \mathbb{R}^+, \end{cases} \tag{1.2}$$

that refers to a road segment of length L with an inflow at time t and $x = 0$ prescribed by $f_0(t)$ and an outflow at $x = L$ constrained by $q(t)$.

In any of the problems considered above, assume two threshold fluxes q_1 and q_2 are given with $q_1 \leqslant q_2$. Then, apparently, the solution ρ_1 corresponding to q_1 also "*solves*" the problem with constraint q_2. On the contrary, the entropy condition for (1.1) introduced below, see Definition 3.2, automatically selects only *maximal* solutions, i.e. those solutions that allow the maximal flow through the gate which is also compatible with the constraint.

The paper is organized as follows. First, in Section 2, we consider constrained Riemann problems. Section 3 is devoted to the constrained Cauchy problem proving existence and continuous dependence. Then, we approximate (1.1) with a scalar conservation law with flow discontinuous in x

$$\begin{cases} \partial_t \rho + \partial_x (k_\varepsilon(t,x) f(\rho)) = 0, \\ \rho(x,0) = \rho_0(x), \end{cases} \qquad k_\varepsilon(t,x) = \begin{cases} 1, & x \notin [-\varepsilon, \varepsilon], \\ \frac{q(t)}{\max f}, & x \in [-\varepsilon, \varepsilon], \end{cases} \tag{1.3}$$

and we show that the usual weak entropy solution to (1.3) converges to the nonclassical solution of (1.1) as $\varepsilon \to 0$.

All the proofs are collected in Section 4 and in Appendix A.

Finally, we mention that conservation laws with unilateral constraints are considered also in [1], but with entirely different tools, motivations and results.

2. The constrained Riemann problem

This section is devoted to the Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0, \\ \rho(0,x) = \begin{cases} \rho^l & \text{if } x < 0, \\ \rho^r & \text{if } x > 0 \end{cases} \end{cases} \tag{2.1}$$

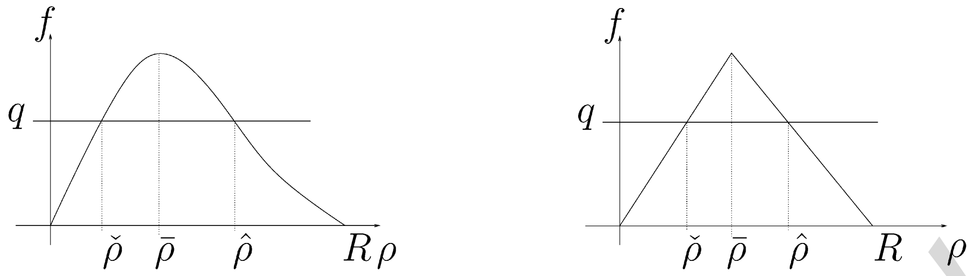

Fig. 1. Examples of fundamental diagrams considered here.

under the constraint

$$f\big(\rho(t,0)\big) \leqslant q, \tag{2.2}$$

q being constant, with the assumptions

(R1) $f:[0,R] \mapsto \mathbb{R}$ is Lipschitz, $f(0) = 0 = f(R)$, $f'(\rho)(\bar\rho - \rho) > 0$ for a.e. ρ,
(R2) $q \in [0, f(\bar\rho)]$,

for a suitable $\bar\rho \in]0, R[$. While the former is a regularity assumption, the latter is an obvious consistency requirement, see Fig. 1. We denote below by $\mathcal{R}$ the standard (i.e. without the constraint (2.2)) Lax [12] or Liu [14] Riemann solver for (2.1), i.e. the map $(t,x) \mapsto \mathcal{R}(\rho^l, \rho^r)(x/t)$ is the standard weak entropy solution to (2.1), see [3, Chapter 5] for its construction.

Definition 2.1. A Riemann solver $\mathcal{R}^q$ for (2.1)–(2.2) is defined as follows.
If $f(\mathcal{R}(\rho^l, \rho^r))(0)) \leqslant q$, then $\mathcal{R}^q(\rho^l, \rho^r) = \mathcal{R}(\rho^l, \rho^r)$.
Otherwise, $\mathcal{R}^q(\rho^l, \rho^r)(x) = \begin{cases} \mathcal{R}(\rho^l, \hat\rho)(x) & \text{if } x < 0, \\ \mathcal{R}(\check\rho, \rho^r)(x) & \text{if } x > 0. \end{cases}$

Above, $\check\rho$ and $\hat\rho$, with $\check\rho \leqslant \hat\rho$, are the solutions to $f(\rho) = q$, see Fig. 1. Note that when the constraint is enforced, at $x = 0$ a nonclassical shock arises. The solution so obtained is a weak solution to (2.1) but it violates the entropy condition as soon as $q < f(\bar\rho)$.

The Riemann solver $\mathcal{R}^q$ generates a semigroup S^q whose orbits are solutions to Cauchy problems. A necessary condition for the $\mathbf{L}^1$ continuity of S^q is the consistency of $\mathcal{R}^q$, see [5,6].

Definition 2.2. The Riemann Solver $\mathcal{R}$ is *consistent* if the following two conditions hold:

$$\text{(C1)}\quad \left.\begin{array}{l} \mathcal{R}(u^l, u^m)(\bar x) = u^m \\ \mathcal{R}(u^m, u^r)(\bar x) = u^m \end{array}\right\} \Rightarrow \mathcal{R}\big(u^l, u^r\big) = \begin{cases} \mathcal{R}(u^l, u^m), & \text{if } x < \bar x, \\ \mathcal{R}(u^m, u^r), & \text{if } x \geqslant \bar x, \end{cases}$$

$$\text{(C2)}\quad \mathcal{R}\big(u^l, u^r\big)(\bar x) = u^m \Rightarrow \begin{cases} \mathcal{R}(u^l, u^m) = \begin{cases} \mathcal{R}(u^l, u^r), & \text{if } x \leqslant \bar x, \\ u^m, & \text{if } x > \bar x, \end{cases} \\ \mathcal{R}(u^m, u^r) = \begin{cases} u^m, & \text{if } x < \bar x, \\ \mathcal{R}(u^l, u^r), & \text{if } x \geqslant \bar x. \end{cases} \end{cases}$$

Both (C1) and (C2) are enjoyed by the standard Lax [12] and Liu [14] solvers. Essentially, (C1) states that whenever two solutions to two Riemann problems can be placed side by side,

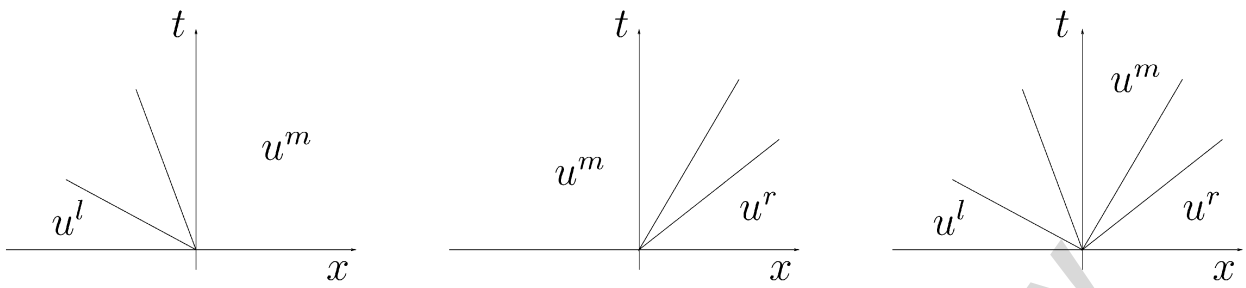

Fig. 2. Consistency of a Riemann solver.

then their juxtaposition is again a solution to a Riemann problem, see Fig. 2. (C2) is the vice versa.

Proposition 2.3. *The Riemann Solver defined by Definition* 2.1 *enjoys the following properties, for all* $\rho^l, \rho^r \in [0, R]$:

(RS1) $(t, x) \mapsto (\mathcal{R}^q(\rho^l, \rho^r))(x/t)$ *is a self similar weak solution to* (2.1) *in the sense of Definition* 2.1;
(RS2) $\mathcal{R}^q(\rho^l, \rho^r) \in \mathbf{BV}(\mathbb{R}; [0, R])$;
(RS3) $\mathcal{R}^q(\rho^l, \rho^r)$ *satisfies the constraint* (2.2) *in the sense that*

$$\lim_{x\to 0-} f\big(\mathcal{R}^q\big(\rho^l, \rho^r\big)(x)\big) \leqslant q \quad \textit{and} \quad \lim_{x\to 0+} f\big(\mathcal{R}^q\big(\rho^l, \rho^r\big)(x)\big) \leqslant q;$$

(RS4) $\mathcal{R}^q$ *is consistent in the sense of Definition* 2.2.

Moreover, the map $\mathcal{R}^q : [0, R]^2 \mapsto \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$ *is uniformly continuous.*

The proof is deferred to Section 4.1.

Aiming at the initial-boundary value problem (1.2), the whole construction above should be repeated with several natural modifications. Alternatively, we achieve the same goal defining as solution to the constrained Riemann problem at the boundary

$$\begin{cases} \partial_t \rho + \partial_x(f(\rho)) = 0, & (t, x) \in \mathbb{R}^+ \times\,]-\infty, 0], \\ \rho(0, x) = \rho_0, & x \in\,]-\infty, 0], \\ f(\rho(t, 0)) \leqslant q, & t \in \mathbb{R}^+, \end{cases} \tag{2.3}$$

with $\rho_0 \in [0, R]$ and $q \in [0, f(\bar\rho)]$, the restriction to $]-\infty, 0]$ of the solution to (2.1)–(2.2) with $\rho^l = \rho_0$ and any $\rho^r \in [0, \bar\rho]$. In fact, any such choice of the right state yields the same solution for $x \leqslant 0$. We only remark here that the "*maximality*" intrinsic in the entropy condition implies, for instance, that if $f(\rho_0) < q$ and $\rho_0 > \bar\rho$, then the constant function does *not* solve (2.3).

3. The constrained Cauchy problem

Consider now the Cauchy problem (1.1) under assumptions (R1) and

(R3) $q \in \mathbf{BV}(\mathbb{R}^+; [0, f(\bar\rho)])$.

The constraint (2.2) and the consequent Definition 2.1 may well cause sharp increases in $\mathrm{TV}(\rho(t,\cdot))$. The simplest example is provided by a constant initial datum $\rho_0(x)=\bar\rho$ and a constraint

$$q(t)=\begin{cases} f(\bar\rho) & \text{if } t<1,\\ \frac{1}{2}f(\bar\rho) & \text{if } t>1. \end{cases}$$

At time $t=1$, two shocks arise from $x=0$ and the total variation jumps from 0 to $2(\hat\rho-\check\rho)$, where $\check\rho<\hat\rho$ and $f(\hat\rho)=f(\check\rho)=\frac{1}{2}f(\bar\rho)$.

To overcome this difficulty, following [4,16],we use the nonlinear mapping

$$\Psi(\rho)=\operatorname{sgn}(\rho-\bar\rho)\big(f(\bar\rho)-f(\rho)\big) \tag{3.1}$$

and bound the total variation of $\Psi\circ\rho$. In fact, Ψ is one-to-one, but possibly singular at $\rho=\bar\rho$. Indeed, it is immediate to see that if $\rho\in\mathbf{BV}(\mathbb{R};\mathbb{R})$, then $\mathrm{TV}(\Psi\circ\rho)\leqslant\|f'\|_{\mathbf{C}^0}\cdot\mathrm{TV}(\rho)$, while $\mathrm{TV}(\rho)$ may well be infinite with $\Psi(\rho)$ finite, as in the case of $f(\rho)=\rho(1-\rho)$, $\bar\rho=1/2$ and $\rho=\sum_{n=3}^{+\infty}\frac{1}{n}\chi_{[\frac{1}{2}+\frac{1}{n},\frac{1}{2}+\frac{2n+1}{2n(n+1)}[}$.

Definition 3.1. A weak solution to (1.1), is a map

(1) $\rho\in\mathbf{C}^0(\mathbb{R}^+;\mathbf{L}^1(\mathbb{R},[0,R]))$;
(2) for all $t\in\mathbb{R}^+$, $\Psi(\rho(t))\in\mathbf{BV}(\mathbb{R};[0,R])$;
(3) $\int_{\mathbb{R}}\varphi(0,x)\rho_0(x)\,dx+\int_0^{+\infty}\int_{\mathbb{R}}(\rho\,\partial_t\varphi+f(\rho)\partial_x\varphi)\,dx\,dt=0$, for $\varphi\in\mathbf{C}^1_c(\mathbb{R}^2;\mathbb{R})$;
(4) for a.e. $t\in\mathbb{R}^+$, $\lim_{x\to0-}f(\rho(t,x))\leqslant q(t)$, $\lim_{x\to0+}f(\rho(t,x))\leqslant q(t)$.

The above limits exist and are finite because of (2). The present nonclassical setting allows the introduction of the following entropy condition.

Definition 3.2. A weak solution $\rho\in\mathbf{C}^0(\mathbb{R}^+;\mathbf{L}^1(\mathbb{R},[0,R]))$ is an entropy solution to (1.1) if for every $k\in\mathbb{R}$ and for every $\varphi\in\mathbf{C}^1_c(\mathbb{R}^2;\mathbb{R}^+)$

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}}\big(|\rho-k|\partial_t\varphi+\operatorname{sgn}(\rho-k)\big(f(\rho)-f(k)\big)\partial_x\varphi\big)\,dx\,dt\\&\quad+\int_{\mathbb{R}}|\rho_0-k|\varphi(0,x)\,dx+2\int_0^{+\infty}\left(1-\frac{q(t)}{f(\bar\rho)}\right)f(k)\varphi(t,0)\,dt\geqslant0.\end{aligned} \tag{3.2}$$

Remark 3.3. Definition 3.2 selects the *maximal* solution, for a nonclassical stationary shock at $x=0$ separating states $\hat\rho$ and $\check\rho$ with $f(\hat\rho)=f(\check\rho)<q(t)$ turns out to be nonentropic.

To state the first well posedness result, it is useful to introduce the translation $\mathcal{T}_t$ by $(\mathcal{T}_tq)(\tau)=q(\tau+t)$. Below we introduce a map $S^q:\mathbb{R}^+\times\mathcal{D}\mapsto\mathcal{D}$, $\mathcal{D}$ being a suitable subset of $\mathbf{L}^1$ containing the initial data of (1.1). We then denote by $\bar S^q$ the map $\bar S^q:\mathbb{R}^+\times\bar{\mathcal{D}}\mapsto\bar{\mathcal{D}}$ defined by $\bar S^q_t(\rho,q)=(S^q_t\rho,\mathcal{T}_tq)$ with $\bar{\mathcal{D}}=\mathcal{D}\times\mathbf{BV}$.

Theorem 3.4. *Let* (R1) *and* (R3) *hold. Then, for every constraint* $q \in \mathbf{BV}(\mathbb{R}^+; [0, f(\bar\rho)])$ *there exists a map* $S^q : \mathbb{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$ *such that*

(CRS1) $\mathcal{D} \supseteq \{\rho \in \mathbf{L}^1(\mathbb{R}; [0, R]):\ \Psi(\rho) \in \mathbf{BV}(\mathbb{R}; \mathbb{R})\}$;
(CRS2) $\bar S^q$ *is a semigroup, i.e.* $\bar S^q_0 = \mathrm{Id}$ *and* $\bar S^q_{t_1} \circ \bar S^q_{t_2} = \bar S^q_{t_1+t_2}$;
(CRS3) S^q *is nonexpansive in* ρ, *i.e. for all* $\rho_1, \rho_2 \in \mathcal{D}$

$$\left\| S^q_t \rho_1 - S^q_t \rho_2 \right\|_{\mathbf{L}^1} \leqslant \|\rho_1 - \rho_2\|_{\mathbf{L}^1};$$

(CRS4) *if* ρ_0 *and* q *are piecewise constant, then for* t *sufficiently small,* $S^q_t \rho_0$ *coincides with the gluing of the solutions to standard Riemann problems centered at the points of jump of* ρ_0 *and to* (2.1)–(2.2) *at* $x = 0$;
(CRS5) *for all* $\rho_0 \in \mathcal{D}$, *the orbit* $t \mapsto S^q_t \rho_0$ *yields a weak entropy solution to* (1.1), *according to Definitions* 3.1 *and* 3.2.

The proof uses the standard technique of wave front tracking, see [3], and is deferred to Section 4.2. The above statements (CRS1)–(CRS4) are clearly modeled on the definition of *Standard Riemann Semigroup*, see [3, Definition 9.1] and provide an analogue to it in the present constrained (and nonautonomous) setting. The Lipschitz estimate (CRS3) is proved with suitable modifications of the techniques in [11] or [3, Section 6.3].

It is now easy to tackle the initial boundary value problem

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0, & (t,x) \in \mathbb{R}^+ \times \left]-\infty, 0\right], \\ \rho(0,x) = \rho_0(x), & x \in \left]-\infty, 0\right], \\ f(\rho(t,0)) \leqslant q(t), & t \in \mathbb{R}^+. \end{cases} \tag{3.3}$$

Indeed, as in the case of the Riemann problem, a solution to (3.3) is obtained restricting to $x < 0$ a solution to (1.1) with initial data, say, $\rho_0(x) = 0$ for $x > 0$. The extension to (1.2) is immediate, see [7] as a general reference on initial boundary value problems for scalar conservation laws.

The above nonclassical construction can be seen as a singular limit of the classical theory. Indeed, recall the conservation law (1.3) where q satisfies (R3). As $\varepsilon \to 0$, the flow $k_\varepsilon(t,x) f(\rho)$ converges in $\mathbf{L}^1$ to the flow in (1.1). As noted in a similar example in [2] the solution ρ_ε to (1.3) fails to converge to the (classical) solution of

$$\partial_t \rho + \partial_x f(\rho) = 0. \tag{3.4}$$

Actually, we show below that the solutions to (1.3) converge in $\mathbf{L}^1$ to the weak entropy solution to (1.1) in the sense of Definitions 3.1 and 3.2.

More precisely, (1.3) essentially fits in the framework provided by [9, Theorems 4.5, 5.5 and 6.5], see also [4,8]. Nevertheless, we state the following well posedness result for (1.3) in a slightly different setting than that in [4,8,9]. In the present form, the theorem below can be proved with the same techniques used in Theorem 3.4, see Section 4.3 and Appendix A for the proof.

Theorem 3.5. *Let* (R1) *and* (R3) *hold. Then, for every positive* ε *and every constraint* $q \in \mathbf{BV}(\mathbb{R}^+; [0, f(\bar\rho)])$ *there exists a map* $S^\varepsilon : \mathbb{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$ *such that*

(εRS1) $\mathcal{D} \supseteq \{\rho \in \mathbf{L}^1(\mathbb{R}; [0, R]): \Psi(\rho) \in \mathbf{BV}(\mathbb{R}; \mathbb{R})\}$;
(εRS2) $\bar{S}^\varepsilon$ *is a semigroup, i.e.* $\bar{S}^\varepsilon_0 = \mathrm{Id}$ *and* $\bar{S}^\varepsilon_{t_1} \circ \bar{S}^\varepsilon_{t_2} = \bar{S}^\varepsilon_{t_1+t_2}$;
(εRS3) S^ε *is nonexpansive in* ρ, *i.e. for all* $\rho_1, \rho_2 \in \mathcal{D}$

$$\left\| S^\varepsilon_t \rho_1 - S^\varepsilon_t \rho_2 \right\|_{\mathbf{L}^1} \leqslant \|\rho_1 - \rho_2\|_{\mathbf{L}^1};$$

(εRS4) *if* ρ_0 *and* q *are piecewise constant, then for* t *sufficiently small,* $S^\varepsilon_t \rho_0$ *coincides with the gluing of the solutions to standard Riemann problems for* (1.3) *centered at the points of jump of* ρ_0;
(εRS5) *for all* $\rho_0 \in \mathcal{D}$, *the orbit* $t \mapsto S^\varepsilon_t \rho_0$ *yields a weak entropy solution to* (1.3), *according to* [9, Formula (4.19) and Definition 5.1].

Finally, the following result provides the connection between the nonclassical construction in Theorem 3.4 and the classical one in Theorem 3.5.

Theorem 3.6. *Fix* $\rho_0 \in \mathcal{D}$. *For* $\varepsilon > 0$, *call* ρ_ε *the solution to* (1.3). *Then, as* $\varepsilon \to 0$, ρ_ε *converges in* $\mathbf{L}^1_{\mathrm{loc}}$ *to a function* ρ *which is the* (*unique*) *weak entropy solution to* (1.1) *in the sense of Definitions* 3.1 *and* 3.2.

4. Technical proofs

4.1. Proof of Proposition 2.3

Preliminarily, we introduce the following notations: $\lambda(\rho) = f'(\rho)$ is the characteristic speed at ρ while $\Lambda(\rho^l, \rho^r) = \frac{f(\rho^l) - f(\rho^r)}{\rho^l - \rho^r}$ is the speed of a (possibly nonentropic) shock between ρ^l and ρ^r.

(RS1) Is immediate, since the standard solution to Riemann problems is in **BV** and Definition 2.1 amounts to juxtapose standard solutions.
(RS2) Self similarity is obvious. Off from $x = 0$, $\mathcal{R}^q$ yields weak solution because so does $\mathcal{R}$. Along $x = 0$, the Rankine–Hugoniot conditions are satisfied, since the jump between ρ^l and ρ^r is a (possibly nonentropic) stationary shock.
(RS3) Note first that both limits exists and are finite since $\mathcal{R}^q(\rho^l, \rho^r) \in \mathbf{BV}(\mathbb{R})$. For simplicity, let us consider the case $f'' < 0$. We look at the left limit $\lim_{x\to 0-} f(\mathcal{R}^q(\rho^l, \rho^r)(x))$ (the study of the right limit being essentially analogous):
  (a) If $\mathcal{R}^q(\rho^l, \rho^r) = \mathcal{R}(\rho^l, \rho^r)$, by Definition 2.1, (2.2) holds.
  (b) If $\mathcal{R}(\rho^l, \hat{\rho})$ consists of a classical Lax shock with negative speed, then $\lim_{x\to 0-} \mathcal{R}(\rho^l, \hat{\rho})(x) = \hat{\rho}$, and (2.2) holds.
  (c) If $\mathcal{R}(\rho^l, \hat{\rho})$ consists of a shock with positive speed, then $f(\rho^l) < q$ and $\rho^l < \check{\rho}$, so that $\mathcal{R}^q(\rho^l, \rho^r) = \mathcal{R}(\rho^l, \rho^r)$ and (2.2) holds.
  (d) If $\mathcal{R}(\rho^l, \hat{\rho})$ is a rarefaction, then $\rho^l > \hat{\rho}$, so that the rarefaction fan has strictly negative speed and $\lim_{x\to 0-} \mathcal{R}(\rho^l, \hat{\rho})(x) = \hat{\rho}$, which implies (2.2).
(RS4) Consistency directly follows from the analogous property of the standard Liu solver.

Finally, note that the region $\mathcal{S}$ on the left and above the thick line on Fig. 3 corresponds to the states (ρ^l, ρ^r) where $\mathcal{R}^q(\rho^l, \rho^r) = \mathcal{R}(\rho^l, \rho^r)$, i.e. when Definition 2.1 yields the standard Lax solution. In the region $\mathcal{N}$, on the right and below the thick line, a nonclassical solution is selected

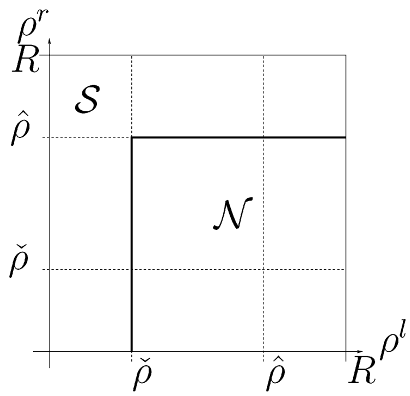

Fig. 3. Standard (above left) and nonstandard (below right) solution to (2.1)–(2.2) as provided by $\mathcal{R}^q$.

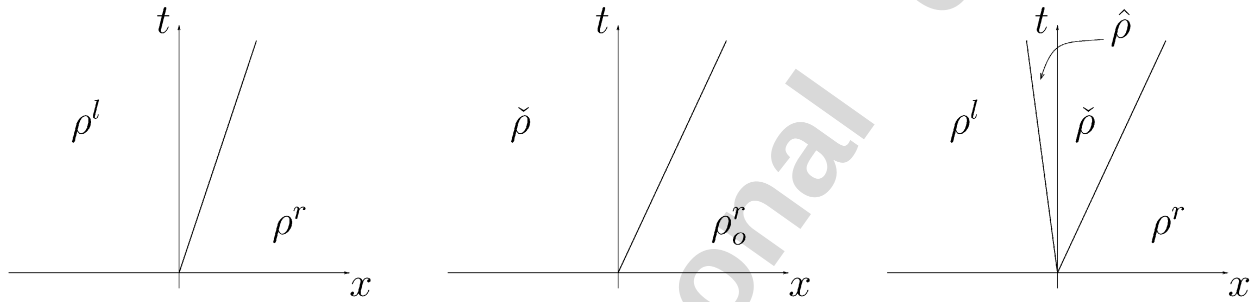

Fig. 4. Representation of $\mathcal{R}^q(\rho^l, \rho^r)$ for case $\rho^l_0 = \check{\rho}$, $\rho^r_0 \in [0, \hat{\rho}[$: left, for $(\rho^l, \rho^r) \in \mathcal{S}$, middle for $(\rho^l, \rho^r) = (\rho^l_0, \rho^r_0)$ and right for $(\rho^l, \rho^r) \in \mathcal{N}$.

and $\mathcal{R}^q(\rho^l, \rho^r) \neq \mathcal{R}(\rho^l, \rho^r)$. Therefore, due to the properties of the standard Riemann solver, it is sufficient to prove the continuity of $\mathcal{R}^q$ in each point (ρ^l_0, ρ^r_0) along the thick line.

Let us consider the case $\rho^l_0 = \check{\rho}$, $\rho^r_0 \in [0, \hat{\rho}[$. Then, $\mathcal{R}^q(\rho^l_0, \rho^r_0) = \mathcal{R}(\rho^l_0, \rho^r_0)$. The solution $\mathcal{R}^q(\rho^l, \rho^r)$ for (ρ^l, ρ^r) in a neighborhood of (ρ^l_0, ρ^r_0) is represented in Figure 4. Let $[a, b]$ be any bounded real interval and let (ρ^l, ρ^r) vary in a neighborhood of $(\check{\rho}, \rho^r_0)$. If $(\rho^l, \rho^r) \in \mathcal{S}$, then $\mathcal{R}^q(\rho^l, \rho^r) = \mathcal{R}(\rho^l, \rho^r)$ and continuity follows from the standard properties of the Liu solver [14]. If $(\rho^l, \rho^r) \in \mathcal{N}$, then

$$
\begin{aligned}
&\int_a^b \left|\mathcal{R}^q\big(\rho^l, \rho^r\big)(x) - \mathcal{R}^q\big(\rho^l_0, \rho^r_0\big)(x)\right| dx \\
&\quad = \int_{[a,b]\cap\mathbb{R}^-} \left|\mathcal{R}^q\big(\rho^l, \rho^r\big)(x) - \check{\rho}\right| dx + \int_{[a,b]\cap\mathbb{R}^+} \left|\mathcal{R}\big(\check{\rho}, \rho^r\big)(x) - \mathcal{R}\big(\check{\rho}, \rho^r_0\big)(x)\right| dx \\
&\quad \leqslant \big(\rho^l - \check{\rho}\big)(b-a) + (\hat{\rho} - \check{\rho})\Lambda\big(\rho^l, \hat{\rho}\big) \\
&\qquad + \int_{[a,b]\cap\mathbb{R}^+} \left|\mathcal{R}\big(\check{\rho}, \rho^r\big)(x) - \mathcal{R}\big(\check{\rho}, \rho^r_0\big)(x)\right| dx,
\end{aligned}
$$

which is arbitrarily small as (ρ^l, ρ^r) approaches (ρ^l_0, ρ^r_0).

The cases $\rho^l_0 \in]\check{\rho}, R]$, $\rho^r_0 = \hat{\rho}$ and $\rho^l_0 = \check{\rho}$, $\rho^r_0 = \hat{\rho}$ are similar.

4.2. Proofs of Theorem 3.4

Fix a positive $n \in \mathbb{N}$, $n > 0$ and introduce in $[0, R]$ the mesh $\mathcal{M}_n$ by

$$\mathcal{M}_n = f^{-1}\big(2^{-n}\mathbb{N}\big).$$

Let $\mathbf{PLC}_n$ be the set of piecewise linear and continuous functions defined on $[0, R]$ whose derivative exists in $[0, R] \setminus \mathcal{M}_n$. Let $f^n \in \mathbf{PLC}_n$ coincide with f on $\mathcal{M}_n$. Clearly, if f satisfies (R1) and (R3), then so does f^n.

Similarly, introduce $\mathbf{PC}_n$, respectively $\mathbf{PC}_n^+$, as the set of piecewise constant functions defined on $\mathbb{R}$, respectively $\mathbb{R}^+$, with values in $\mathcal{M}_n$, respectively in $f(\mathcal{M}_n)$. Let $q^n \in \mathbf{PC}_n^+$ coincide with q on $f(\mathcal{M}_n)$. Note that if q satisfies (R2), then so does q^n. We write

$$\begin{aligned} \rho^n &= \sum_\alpha \rho_\alpha^n \chi_{]x_{\alpha-1}, x_\alpha]} && \text{with } \rho_\alpha^n \in \mathcal{M}_n, \\ q^n &= q_0^n \chi_{[0,t_1]}(t) + \sum_{\beta \geqslant 1} q_\beta^n \chi_{]t_\beta, t_{\beta+1}]} && \text{with } q_\beta^n \in 2^{-n}\mathbb{N} \end{aligned} \tag{4.1}$$

and we agree that for $\alpha = 0$, $x_\alpha = 0$. Both the approximations above are meant in the strong $\mathbf{L}^1$ topology, that is $\lim_{n\to+\infty} \|q^n - q\|_{\mathbf{L}^1(\mathbb{R})} = 0$ and $\lim_{n\to+\infty} \|\rho^n - \rho\|_{\mathbf{L}^1(\mathbb{R})} = 0$.

Let $\mathcal{D}_n = \{\rho \in \mathbf{PC}_n\colon \Psi(\rho) \in \mathbf{BV}(\mathbb{R};\mathbb{R})\}$ and $\bar{\mathcal{D}}_n = \mathcal{D}_n \times \mathbf{PC}_n^+$. On any $(\rho^n, q^n) \in \bar{\mathcal{D}}_n$, written as in (4.1), define the Glimm type functional

$$\Upsilon\big(\rho^n, q^n\big) = \sum_\alpha \big|\Psi\big(\rho_{\alpha+1}^n\big) - \Psi\big(\rho_\alpha^n\big)\big| + 5 \sum_{t_\beta \geqslant 0} \big|q_{\beta+1}^n - q_\beta^n\big| + \gamma, \tag{4.2}$$

where γ is defined by

$$\gamma = \begin{cases} 0 & \text{if } \rho^n(0-) > \bar\rho > \rho^n(0+) \text{ and} \\ & f(\rho^n(0+)) = f(\rho^n(0-)) = q^n(0), \\ 4(f(\bar\rho) - q^n(0)) & \text{otherwise.} \end{cases}$$

For small times, an approximate solution $\rho^n = \rho^n(t,x)$ to (1.1) is constructed piecing together the solutions to the Riemann problems

$$\begin{cases} \partial_t \rho + \partial_x(f^n(\rho)) = 0, \\ \rho(x,0) = \begin{cases} \rho_0 & \text{if } x < 0, \\ \rho_1 & \text{if } x > 0, \end{cases} \\ f(\rho(t,0)) \leqslant q_0^n, \end{cases} \qquad \begin{cases} \partial_t \rho + \partial_x(f^n(\rho)) = 0, \\ \rho(x,0) = \begin{cases} \rho_\alpha & \text{if } x < x_\alpha, \\ \rho_{\alpha+1} & \text{if } x > x_\alpha, \end{cases} \\ \alpha \neq 0. \end{cases} \tag{4.3}$$

Note that the solutions to the former Riemann problem in (4.3) is constructed by means of $\mathcal{R}^q$, the solutions to the latter by means of $\mathcal{R}$. In both cases, for fixed t the solutions are piecewise constant in x. Their juxtaposition yields a well-defined (exact) weak entropy solution ρ^n to

$$\begin{cases} \partial_t \rho^n + \partial_x(f^n(\rho^n)) = 0, \\ \rho^n(x,0) = \rho_0^n(x), \\ \rho^n(t,0) \leqslant q^n(t) \end{cases} \tag{4.4}$$

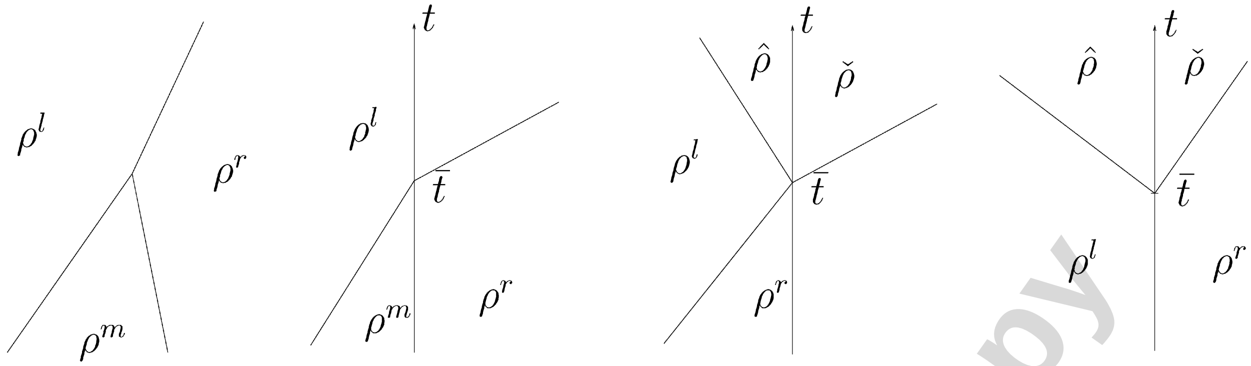

Fig. 5. Notations for the proof of Lemma 4.1.

as long as either two discontinuities collide, or the value of the constraint changes. In both cases, a new Riemann problem arises and its solution, obtained in the former case with $\mathcal{R}$ and in the latter with $\mathcal{R}^q$, allows to extend ρ^n further in time. We define $\bar S_t^n(\rho_0^n, q^n) = (\rho^n(t,\cdot), \mathcal{T}_t q^n)$ the approximate Riemann Semigroup.

Lemma 4.1. *For any $n \in \mathbb{N}$ and $(\rho_0^n, q^n) \in \bar{\mathcal{D}}_n$, at any interaction, the map $t \mapsto \Upsilon(t) = \Upsilon(\bar S_t^n(\rho_0^n, q^n))$ either decreases by at least 2^{-n}, or remains constant and the number of waves does not increase.*

Proof. The proof is obtained considering the different interactions separately, depending on the position of the interaction point $\bar x$ and on the flows of the interacting states. We will consider interaction points $\bar x \leqslant 0$, the case $\bar x \geqslant 0$ being symmetric. It is not restrictive to assume that at any interaction time either two waves interact or a single wave hits $x = 0$.

(I1) $\bar x \neq 0$. As in the usual scalar case, either two shocks collide (and the number of waves diminishes) or a shock and a rarefaction cancel (and Ψ diminishes), see Fig. 5, left.

(I2) A wave hits $\bar x = 0$ coming from the left and $f(\rho^l) \leqslant q^n(\bar t)$, the front crosses $\bar x = 0$ and no new wave is created. With reference to Fig. 5, second from the left, if $\rho^m = \rho^r$ then each of the three terms in (4.2) remains constant, hence $\Upsilon(\bar t+) = \Upsilon(\bar t-)$. Otherwise, if ρ^m and ρ^r result from an application of $\bar{\mathcal{R}}$, then γ increases at $\bar t$. However, Υ remains constant and no new waves are created.

$$\begin{aligned}\Delta\Upsilon(\bar t) &= \left|\Psi\left(\rho^l\right) - \Psi\left(\rho^r\right)\right| + 4\left(f(\bar\rho) - q(\bar t)\right) \\ &\quad - \left(\left|\Psi\left(\rho^l\right) - \Psi\left(\rho^m\right)\right| + \left|\Psi\left(\rho^m\right) - \Psi\left(\rho^r\right)\right|\right) \\ &= 0.\end{aligned}$$

(I3) A wave hits $\bar x = 0$ coming from the left and $f(\rho^l) > q^n(\bar t)$. Then, necessarily, $\rho^r < \rho^l \leqslant \bar\rho$. In this case, new waves are created at $(\bar t, 0)$ (see Fig. 5, second from the right). The functional changes as follows:

$$\begin{aligned}\Delta\Upsilon(\bar t) &= \Upsilon(\bar t+) - \Upsilon(\bar t-) \\ &= \left(\left|\Psi\left(\rho^l\right) - \Psi(\hat\rho)\right| + \left|\Psi(\hat\rho) - \Psi(\check\rho)\right| + \left|\Psi(\check\rho) - \Psi\left(\rho^r\right)\right|\right) \\ &\quad - \left(\left|\Psi\left(\rho^l\right) - \Psi\left(\rho^r\right)\right| + 4\left(f(\bar\rho) - q^n(\bar t)\right)\right)\end{aligned}$$

$$\begin{aligned}&= -2\big(f(\rho_l) + f(\hat\rho) - 2f(\rho_r)\big)\\&\leqslant -2^{2-n},\end{aligned}$$

where $\check\rho < \hat\rho$ are defined by $f(\check\rho) = q^n(\bar t) = f(\hat\rho)$ and we used the inequalities $f(\rho^l) > f(\rho^r)$ and $f(\hat\rho) > f(\rho^r)$.

(I4) $\rho^l = \rho^r = \rho$ and the constraint q^n jumps downward, see Fig. 5, right. Hence $q^n(\bar t+) < f(\rho) \leqslant q^n(\bar t-)$. Two waves exit the point $(\bar t, 0)$. In both cases $\rho < \bar\rho$ and $\rho > \bar\rho$, we compute:

$$\begin{aligned}\Delta\Upsilon(\bar t) &= \big(\big|\Psi(\rho) - \Psi(\hat\rho)\big| + \big|\Psi(\hat\rho) - \Psi(\check\rho)\big| + \big|\Psi(\check\rho) - \Psi(\rho)\big|\big)\\&\quad - \big(5\big|q(\bar t+) - q(\bar t-)\big| + 4\big|f(\bar\rho) - q^n(\bar t-)\big|\big)\\&= f(\hat\rho) - q(\bar t-)\\&\leqslant -2^{-n},\end{aligned}$$

where $\check\rho_+ < \hat\rho_+$ are defined by $f(\check\rho_+) = q^n(\bar t+) = f(\hat\rho_+)$.

(I5) $\rho^l < \bar\rho < \rho^r$ and the constraint q^n jumps downward. Then $q^n(\bar t+) < f(\rho^l) = f(\rho^r) \leqslant q^n(\bar t-)$.

$$\begin{aligned}\Delta\Upsilon(\bar t) &= \big|\Psi\big(\rho^l\big) - \Psi(\hat\rho)\big| + \big|\Psi(\hat\rho) - \Psi(\check\rho)\big| + \big|\Psi(\check\rho) - \Psi\big(\rho^r\big)\big|\\&\quad - \big|\Psi\big(\rho^l\big) - \Psi\big(\rho^r\big)\big| - 5\big|q^n(\bar t+) - q^n(\bar t-)\big| - 4\big|f(\bar\rho) - q(\bar t-)\big|\\&= q(\bar t+) - q(\bar t-)\\&\leqslant -2^{-n}.\end{aligned}$$

(I6) $\rho^l > \bar\rho > \rho^r$ and the constraint q^n jumps downward. We necessarily have $f(\rho^l) = f(\rho^r) = q^n(\bar t-)$ and the discontinuity at $(\bar t-, 0)$ is a nonentropic jump resulting from the constrained Riemann solver.

$$\begin{aligned}\Delta\Upsilon(\bar t) &= \big|\Psi\big(\rho^l\big) - \Psi(\hat\rho)\big| + \big|\Psi(\hat\rho) - \Psi(\check\rho)\big| + \big|\Psi(\check\rho) - \Psi\big(\rho^r\big)\big|\\&\quad - \big(\big|\Psi\big(\rho^l\big) - \Psi\big(\rho^r\big)\big| + 5\big|q(\bar t+) - q(\bar t-)\big|\big)\\&= q(\bar t+) - q(\bar t-)\\&\leqslant -2^{-n}.\end{aligned}$$

(I7) $\rho^l > \bar\rho > \rho^r$ and the constraint q^n jumps upward. Here, $\hat\rho \geqslant \bar\rho \geqslant \check\rho$. The same computations as in the latter case show that Υ is strictly decreasing:

$$\begin{aligned}\Delta\Upsilon(\bar t) &= \big|\Psi\big(\rho^l\big) - \Psi(\hat\rho)\big| + \big|\Psi(\hat\rho) - \Psi(\check\rho)\big| + \big|\Psi(\check\rho) - \Psi\big(\rho^r\big)\big|\\&\quad - \big(\big|\Psi\big(\rho^l\big) - \Psi\big(\rho^r\big)\big| + 5\big|q(\bar t+) - q(\bar t-)\big|\big)\\&= q(\bar t+) - q(\bar t-)\\&\leqslant -2^{-n}. \qquad \square\end{aligned}$$

As an immediate consequence, we have:

Corollary 4.2. *The total number of interactions is finite.*

A standard procedure based on Helly's Compactness Theorem, see [3, Theorem 2.4] allows to exhibit a weak solution to (1.1). In the present case, thanks to Definition 3.2, the limit is also an entropy solution.

Lemma 4.3. *Problem* (1.1) *admits a weak entropy solution.*

Proof. Since all other possible discontinuities are entropic in the classical sense, it is sufficient to verify the entropy inequality (3.2) against a test function with support contained in $[T_1, T_2] \times [-\delta, \delta]$, for some $\delta > 0$ sufficiently small. Let us assume the weak solution ρ presents a discontinuity along $x = 0$ which does not satisfy the usual classical entropy inequality. Hence, apart from a countable set of times, we may assume $\rho(t, 0-) = \hat{\rho}(t)$ and $\rho(t, 0+) = \check{\rho}(t)$, with $\check{\rho}(t) < \bar{\rho} < \hat{\rho}(t)$ and $f(\check{\rho}(t)) = q(t) = f(\hat{\rho}(t))$.

Integrating by parts the left-hand side of (3.2) one gets

$$\int_0^\infty \big(\operatorname{sgn}\big(\hat{\rho}(t) - k\big) - \operatorname{sgn}\big(\check{\rho}(t) - k\big)\big)\big(q(t) - f(k)\big)\varphi(t, 0)\, dt$$
$$+ 2\int_0^\infty \left(1 - \frac{q(t)}{f(\bar{\rho})}\right) f(k)\varphi(t, 0)\, dt \geqslant 0.$$

Since $\varphi(t, 0) \geqslant 0$, it is sufficient to check that

$$\big(\operatorname{sgn}\big(\hat{\rho}(t) - k\big) - \operatorname{sgn}\big(\check{\rho}(t) - k\big)\big)\big(q(t) - f(k)\big) + 2\left(1 - \frac{q(t)}{f(\bar{\rho})}\right) f(k) \geqslant 0$$

for a.e. $t \in [T_1, T_2]$, and all $k \in \mathbb{R}$. It is easy to check that if $k \leqslant \check{\rho}$ or $k \geqslant \hat{\rho}$, then one gets $2(1 - \frac{q(t)}{f(\bar{\rho})}) f(k) \geqslant 0$. On the other hand, if $\check{\rho} < k < \hat{\rho}$, easy calculations give $q(t)(1 - \frac{f(k)}{f(\bar{\rho})}) \geqslant 0$. $\square$

The rest of this section is devoted to the Lipschitz estimate (CRS3).

Proposition 4.4. *Fix two initial data ρ_1, ρ_2. Then, the corresponding weak entropy solutions $S_t^q \rho_1$ and $S_t^q \rho_2$ to* (1.1) *with the same constraint q satisfy*

$$\big\| S_t^q \rho_1 - S_t^q \rho_2 \big\|_{\mathbf{L}^1} \leqslant \|\rho_1 - \rho_2\|_{\mathbf{L}^1}.$$

Proof. We denote $\rho_1(t, x) = S_t^q \rho_1(x)$ and $\rho_2(t, x) = S_t^q \rho_2(x)$. Take $k, k' \in \mathbb{R}$ and a smooth function $\varphi = \varphi(s, x, t, y) \geqslant 0$ with compact support contained in the set where $s, t > 0$. Since ρ_1, ρ_2 are entropy solutions to (1.1), they satisfy condition (3.2):

$$\begin{aligned}
&\iint \Big(\big|\rho_1(s,x) - k\big| \partial_s \varphi(s,x,t,y) \\
&\quad + \operatorname{sgn}\big(\rho_1(s,x) - k\big)\big(f\big(\rho_1(s,x)\big) - f(k)\big)\partial_x \varphi(s,x,t,y)\Big)\,dx\,ds \\
&\quad + 2\int \left(1 - \frac{q(s)}{f(\bar\rho)}\right) f(k)\varphi(s,0,t,y)\,ds \geqslant 0, \qquad (4.5)\\
&\iint \Big(\big|\rho_2(t,y) - k'\big| \partial_t \varphi(s,x,t,y) \\
&\quad + \operatorname{sgn}\big(\rho_2(t,y) - k'\big)\big(f\big(\rho_2(t,y)\big) - f(k')\big)\partial_y \varphi(s,x,t,y)\Big)\,dy\,dt \\
&\quad + 2\int \left(1 - \frac{q(t)}{f(\bar\rho)}\right) f(k')\varphi(s,x,t,0)\,dt \geqslant 0. \qquad (4.6)
\end{aligned}$$

Set now $k = \rho_2(t,y)$ in (4.5) and integrate w.r.t. t, y. Similarly, set $k' = \rho_1(s,x)$ in (4.6) and integrate w.r.t. s, x. Then add the two results, and get

$$\begin{aligned}
&\iiiint \Big(\big|\rho_1(s,x) - \rho_2(t,y)\big| (\partial_s\varphi + \partial_t\varphi)(s,x,t,y) \\
&\quad + \operatorname{sgn}\big(\rho_1(s,x) - \rho_2(t,y)\big)\big(f\big(\rho_1(s,x)\big) - f\big(\rho_2(t,y)\big)\big)(\partial_x\varphi + \partial_y\varphi)(s,x,t,y)\Big)\,dx\,dy\,ds\,dt \\
&\quad + 2\iiint \left(1 - \frac{q(s)}{f(\bar\rho)}\right) f\big(\rho_2(t,y)\big)\varphi(s,0,t,y)\,dy\,ds\,dt \\
&\quad + 2\iiint \left(1 - \frac{q(t)}{f(\bar\rho)}\right) f\big(\rho_1(s,x)\big)\varphi(s,x,t,0)\,dx\,ds\,dt \geqslant 0. \qquad (4.7)
\end{aligned}$$

Take

$$\varphi(s,x,t,y) = \psi\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\delta_h\left(\frac{s-t}{2}\right)\delta_h\left(\frac{x-y}{2}\right), \qquad (4.8)$$

where $\psi = \psi(T,X) \geqslant 0$ is a smooth function with compact support contained in the half plane $T > 0$, and δ_h, $h \geqslant 1$, is a smooth approximation of the Dirac mass at the origin. More precisely, given a $\mathcal{C}^\infty$ function $\delta : \mathbb{R} \to [0,1]$ such that $\int_{\mathbb{R}} \delta(z)\,dz = 1$, $\delta(z) = 0$ for all $z \notin [-1,1]$, define $\delta_h(z) = h\delta(hz)$ and $\alpha_h(z) = \int_{-\infty}^{z} \delta_h(s)\,ds$. We then have

$$(\partial_s\varphi + \partial_t\varphi)(s,x,t,y) = \partial_T\psi\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\delta_h\left(\frac{s-t}{2}\right)\delta_h\left(\frac{x-y}{2}\right), \qquad (4.9)$$

$$(\partial_x\varphi + \partial_y\varphi)(s,x,t,y) = \partial_X\psi\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\delta_h\left(\frac{s-t}{2}\right)\delta_h\left(\frac{x-y}{2}\right). \qquad (4.10)$$

For h sufficiently large, the support of φ is contained in the set where $s > 0$, $t > 0$. Replacing (4.8), (4.9) and (4.10) in (4.7), and performing the change of variables

$$T = \frac{s+t}{2}, \qquad S = \frac{s-t}{2}, \qquad X = \frac{x+y}{2}, \qquad Y = \frac{x-y}{2},$$

inequality (4.7) becomes

$$\begin{aligned}
&\iiiint \Big(\big|\rho_1(T+S,X+Y)-\rho_2(T-S,X-Y)\big|\partial_T\psi(T,X)\\
&\quad+\operatorname{sgn}\big(\rho_1(T+S,X+Y)-\rho_2(T-S,X-Y)\big)\big(f\big(\rho_1(T+S,X+Y)\big)\\
&\quad-f\big(\rho_2(T-S,X-Y)\big)\big)\partial_X\psi(T,X)\Big)\delta_h(S)\delta_h(Y)\,dX\,dY\,dS\,dT\\
&\quad+2\iiint\left(1-\frac{q(T+S)}{f(\bar\rho)}\right)f\big(\rho_2(T-S,y)\big)\delta_h(S)\delta_h\left(-\frac{y}{2}\right)\psi\left(T,\frac{y}{2}\right)dy\,dS\,dT\\
&\quad+2\iiint\left(1-\frac{q(T-S)}{f(\bar\rho)}\right)f\big(\rho_1(T+S,x)\big)\delta_h(S)\delta_h\left(\frac{x}{2}\right)\psi\left(T,\frac{x}{2}\right)dx\,dS\,dT\\
&\geqslant 0.
\end{aligned}\tag{4.11}$$

We let $h\to\infty$ in (4.11) and we rename the variables T, X, thus obtaining

$$\begin{aligned}
&\iint\Big(\big|\rho_1(t,x)-\rho_2(t,x)\big|\,\partial_t\psi(t,x)\\
&\quad+\operatorname{sgn}\big(\rho_1(t,x)-\rho_2(t,x)\big)\big(f\big(\rho_1(t,x)\big)-f\big(\rho_2(t,x)\big)\big)\partial_x\psi(t,x)\Big)\,dx\,dt\\
&\quad+2\int\left(1-\frac{q(t)}{f(\bar\rho)}\right)\big(f\big(\rho_1(t,0)\big)+f\big(\rho_2(t,0)\big)\big)\psi(t,0)\,dt\geqslant 0.
\end{aligned}\tag{4.12}$$

For $\varepsilon>0$, let us define the Lipschitz function

$$\Theta_\varepsilon(x)=\begin{cases}-\frac{1}{\varepsilon}x-1, & x\in[-2\varepsilon,-\varepsilon],\\ 0, & x\in[-\varepsilon,\varepsilon],\\ \frac{1}{\varepsilon}x-1, & x\in[\varepsilon,2\varepsilon],\\ 1, & |x|\geqslant 2\varepsilon.\end{cases}$$

Note that $\Theta_\varepsilon\to 1$ in $\mathbf{L}^1(\mathbb{R})$ as $\varepsilon\to 0$, and it vanishes in a neighborhood of $x=0$. Take any Φ smooth with compact support contained in the half-plane $t>0$, and set $\psi=\Phi\Theta_\varepsilon$ in (4.12):

$$\iint\Big(|\rho_1-\rho_2|\partial_t\Phi+\operatorname{sgn}(\rho_1-\rho_2)\big(f(\rho_1)-f(\rho_2)\big)\partial_x\Phi\Big)\Theta_\varepsilon\,dx\,dt+J(\varepsilon)\geqslant 0,$$

where we have set $J(\varepsilon)=\iint\operatorname{sgn}(\rho_1-\rho_2)(f(\rho_1)-f(\rho_2))\Phi\Theta_\varepsilon'\,dx\,dt$. We now pass to the limit as $\varepsilon\to 0$. Observe that

$$\begin{aligned}
\lim_{\varepsilon\to 0}J(\varepsilon)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\infty\Bigg\{&\int_{-2\varepsilon}^{-\varepsilon}\operatorname{sgn}(\rho_1-\rho_2)\big(f(\rho_1)-f(\rho_2)\big)\Phi\,dx\\
&-\int_{\varepsilon}^{2\varepsilon}\operatorname{sgn}(\rho_1-\rho_2)\big(f(\rho_1)-f(\rho_2)\big)\Phi\,dx\Bigg\}\,dt
\end{aligned}$$

$$= \int_0^\infty \Big\{ \operatorname{sgn}\big(\rho_1(t,0-) - \rho_2(t,0-)\big)\big(f\big(\rho_1(t,0-)\big) - f\big(\rho_2(t,0-)\big)\big)$$
$$- \operatorname{sgn}\big(\rho_1(t,0+) - \rho_2(t,0+)\big)\big(f\big(\rho_1(t,0+)\big) - f\big(\rho_2(t,0+)\big)\big)\Big\}\, dt,$$

and the integrand in the last term is non-negative for a.e. t, by the Rankine–Hugoniot conditions satisfied by the traces of ρ_1 and ρ_2 at $x = 0$, and the concavity of the function f. Hence we recover

$$\iint \big(|\rho_1 - \rho_2|\, \partial_t \Phi + \operatorname{sgn}(\rho_1 - \rho_2)\big(f(\rho_1) - f(\rho_2)\big)\partial_x \Phi\big)\, dx\, dt \geqslant 0,$$

and we can proceed with standard arguments to get the conclusion. □

4.3. Proof of Theorems 3.5 and 3.6

Proof of Theorem 3.5. The proof follows closely the construction in [4]. We explain below the main arguments, leaving the details to Appendix A.

We denote $f_\varepsilon(\rho) = k_\varepsilon(t,x) f(\rho)$, and define the nonlinear mapping

$$\Psi_\varepsilon(\rho) = \operatorname{sgn}(\rho - \bar\rho)\big(f_\varepsilon(\bar\rho) - f_\varepsilon(\rho)\big). \tag{4.13}$$

For any $T > 0$, let ρ_0^n, q^n be piecewise constant approximations of ρ_0, q defined as in (4.1), such that

$$\lim_{n\to+\infty} \big\|\rho_0^n - \rho_0\big\|_{\mathbf{L}^1(\mathbb{R})} = 0, \qquad \Upsilon_\varepsilon\big(\rho_0^n\big) \leqslant \mathrm{TV}(\Psi_\varepsilon \circ \rho_0),$$
$$\lim_{n\to+\infty} \big\|q^n - q\big\|_{\mathbf{L}^1(\mathbb{R}^+)} = 0, \qquad \mathrm{TV}\big(q^n, [0,T]\big) \leqslant \mathrm{TV}\big(q, [0,T]\big). \tag{4.14}$$

We construct now a wave front tracking approximate solution ρ^n to (1.3) as in [4,10]. At each time $t \in\,]t_{\beta-1}, t_\beta[$, ρ^n is made of constant states separated either by ρ-waves, or by k_ε-waves sited at $x = -\varepsilon, \varepsilon$. To measure the strength of a wave w connecting two states ρ^- and ρ^+, we use the Temple functional

$$\Upsilon_\varepsilon(w) = \begin{cases} |\Delta\Psi_\varepsilon| & \text{if } w \text{ is a } \rho\text{-wave},\\ 4|\Delta f_\varepsilon(\bar\rho)| & \text{if } w \text{ is a } k_\varepsilon\text{-wave and } \rho^- < \rho^+,\\ 2|\Delta f_\varepsilon(\bar\rho)| & \text{if } w \text{ is a } k_\varepsilon\text{-wave and } \rho^- > \rho^+. \end{cases}$$

We define the functional $\Upsilon_\varepsilon(\rho^n(t))$ as the sum of the strengths of the waves. Interaction estimates similar to those in Lemma 4.1 ensure that Υ_ε verifies

$$\Upsilon_\varepsilon\big(\rho^n(t+)\big) \leqslant \Upsilon_\varepsilon\big(\rho^n(t-)\big) + 5\big|q^n(t+) - q^n(t-)\big| \tag{4.15}$$

at any interaction (see Appendix A for a detailed analysis of all possible interactions). It follows that for every $t \in [0,T]$

$$\begin{aligned}\Upsilon_\varepsilon\big(\rho^n(t)\big) &\leqslant \Upsilon_\varepsilon\big(\rho_0^n\big)+5\sum_{\beta\geqslant 1}\big|q_\beta^n-q_{\beta-1}^n\big| \leqslant \mathrm{TV}(\Psi_\varepsilon\circ\rho_0)+8f(\bar\rho)+5\,\mathrm{TV}(q)\\ &\leqslant \mathrm{TV}(\Psi\circ\rho_0)+8f(\bar\rho)+5\mathrm{TV}(q).\end{aligned}$$

Hence the total variation of $\Psi_\varepsilon(\rho^n)$ is bounded independently of n (and of ε). An application of Helly's compactness theorem [3, Chapter 2] yields the existence of solutions to (1.3). Uniqueness and the entropy inequalities are proved as in Lemma 4.3 and Proposition 4.4. □

Proof of Theorem 3.6. Fix a flow f, an initial datum $\rho_0\in\mathcal{D}$ and a constraint q. Approximate ρ_0 in $\mathbf{L}^1$ with a **PC** function ρ_0^n valued in $\mathcal{M}_n$ and such that $\Upsilon_\varepsilon(\rho_0^n)\leqslant \mathrm{TV}(\Psi_\varepsilon\circ\rho_0)$. For any positive sequence ε_n converging to 0, let ρ^n be an approximate solution to (1.3) constructed as in the proof of Theorem 3.5 with $\varepsilon=\varepsilon_n$ and ρ_{ε_n} be the exact solution to (1.3). Then, for all n,

$$\begin{aligned}\Upsilon_{\varepsilon_n}\big(\rho^n(t)\big) &\leqslant \mathrm{TV}(\Psi_{\varepsilon_n}\circ\rho_0)+8f(\bar\rho)+5\,\mathrm{TV}(q)\\ &\leqslant \mathrm{TV}(\Psi\circ\rho_0)+8f(\bar\rho)+5\,\mathrm{TV}(q).\end{aligned}$$

An application of Helly's theorem ensures the convergence of the ρ^n to a limit ρ. To show that ρ solves (1.1), it is enough to pass to the limit in n under the integrals in the definition of weak entropy solution. Finally,

$$\lim_{n\to+\infty}\big\|\rho_{\varepsilon_n}(t)-\rho(t)\big\|\leqslant \lim_{n\to+\infty}\big\|\rho_{\varepsilon_n}(t)-\rho^n(t)\big\|+\lim_{n\to+\infty}\big\|\rho^n(t)-\rho(t)\big\|=0$$

completing the proof. □

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Appendix A

We present here the study of the various interactions that leads to (4.15). We will focus on interactions involving k_ε-waves at $x=-\varepsilon$, the case $x=\varepsilon$ being completely symmetric. We consider first the interaction between a ρ-wave and a k_ε-wave, see Figs. 6–8. Then we list the various interaction types between a k_ε-wave and a q-discontinuity, see Figs. 10–13. Finally we analyze how a ρ-wave behaves across a q-discontinuity, see Fig. 15.

(Iε1): A ρ-wave hits $x=-\varepsilon$ coming from the left and $f(\rho^l)\leqslant q^n(\bar t)$. We assume $\rho^r>\bar\rho$ (the other case being entirely analogous) and recall that $f_\varepsilon(\rho_1)=q^n(\bar t)f(\rho_1)/f(\bar\rho)=f(\rho^r)$ and $f_\varepsilon(\rho_2)=q^n(\bar t)f(\rho_2)/f(\bar\rho)=f(\rho^l)$:

$$\begin{aligned}\Delta\Upsilon_\varepsilon(\bar t)&=4\big|f(\bar\rho)-q^n(\bar t)\big|+\big|\Psi_\varepsilon(\rho_2)-\Psi_\varepsilon(\rho_1)\big|\\ &\quad-\big|\Psi\big(\rho^l\big)-\Psi\big(\rho^r\big)\big|-2\big|f(\bar\rho)-q^n(\bar t)\big|=0.\end{aligned}$$

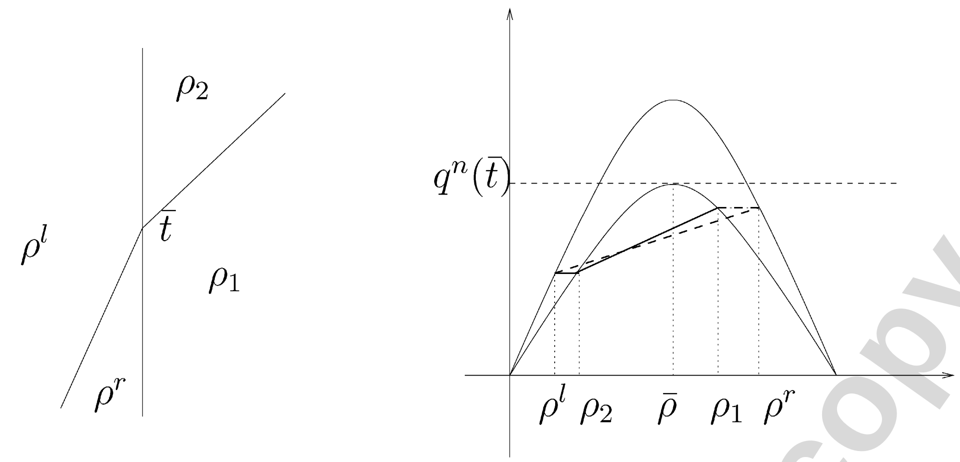

Fig. 6. Notations for case (Iε1).

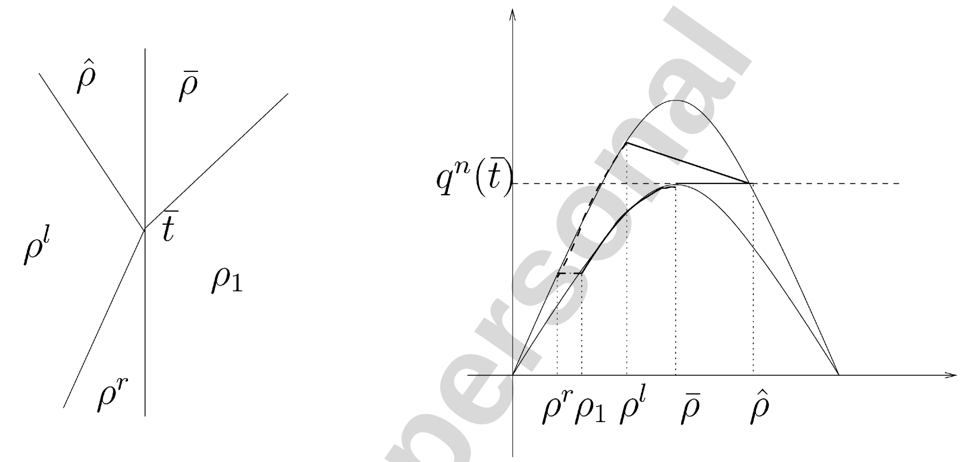

Fig. 7. Notations for case (Iε2).

(Iε2): A ρ-wave hits $x=-\varepsilon$ coming from the left and $f(\rho^l) > q^n(\bar{t})$:

$$\begin{aligned}
\Delta\Upsilon_\varepsilon(\bar{t}) &= \left|\Psi\left(\rho^l\right) - \Psi(\hat{\rho})\right| + 2\left|f(\bar{\rho}) - q^n(\bar{t})\right| + \left|\Psi_\varepsilon(\bar{\rho}) - \Psi_\varepsilon(\rho_1)\right| \\
&\quad - \left|\Psi\left(\rho^l\right) - \Psi\left(\rho^r\right)\right| - 4\left|f(\bar{\rho}) - q^n(\bar{t})\right| \\
&= -2\left(f\left(\rho^l\right) - f\left(\rho^r\right)\right) < 0.
\end{aligned}$$

(Iε3): A ρ-wave hits $x=-\varepsilon$ coming from the right and $\rho^l \geqslant \bar{\rho}$. Since $f_\varepsilon(\rho_1) = q^n(\bar{t}) f(\rho_1)/f(\bar{\rho}) = f(\rho^l)$ and $f_\varepsilon(\rho_2) = q^n(\bar{t}) f(\rho_2)/f(\bar{\rho}) = f(\rho^r)$:

$$\begin{aligned}
\Delta\Upsilon_\varepsilon(\bar{t}) &= \left|\Psi(\rho_2) - \Psi(\rho_1)\right| + 2\left|f(\bar{\rho}) - q^n(\bar{t})\right| \\
&\quad - 2\left|f(\bar{\rho}) - q^n(\bar{t})\right| - \left|\Psi_\varepsilon\left(\rho^l\right) - \Psi_\varepsilon\left(\rho^r\right)\right| = 0.
\end{aligned}$$

(Iε4): A ρ-wave hits $x=-\varepsilon$ coming from the right and $\rho^l < \bar{\rho}$:

$$\begin{aligned}
\Delta\Upsilon_\varepsilon(\bar{t}) &= \left|\Psi(\rho_2) - \Psi(\rho_1)\right| + 2\left|f(\bar{\rho}) - q^n(\bar{t})\right| \\
&\quad - 4\left|f(\bar{\rho}) - q^n(\bar{t})\right| - \left|\Psi_\varepsilon\left(\rho^l\right) - \Psi_\varepsilon\left(\rho^r\right)\right| = 0.
\end{aligned}$$

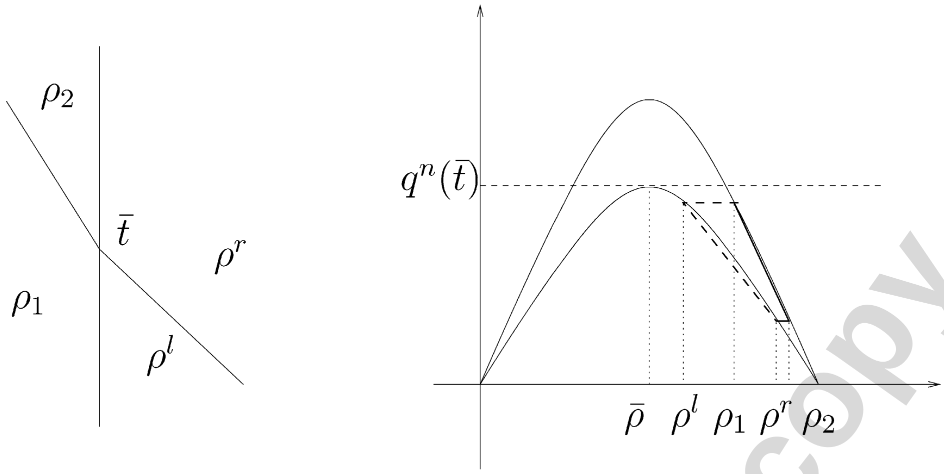

Fig. 8. Notations for case (Iε3).

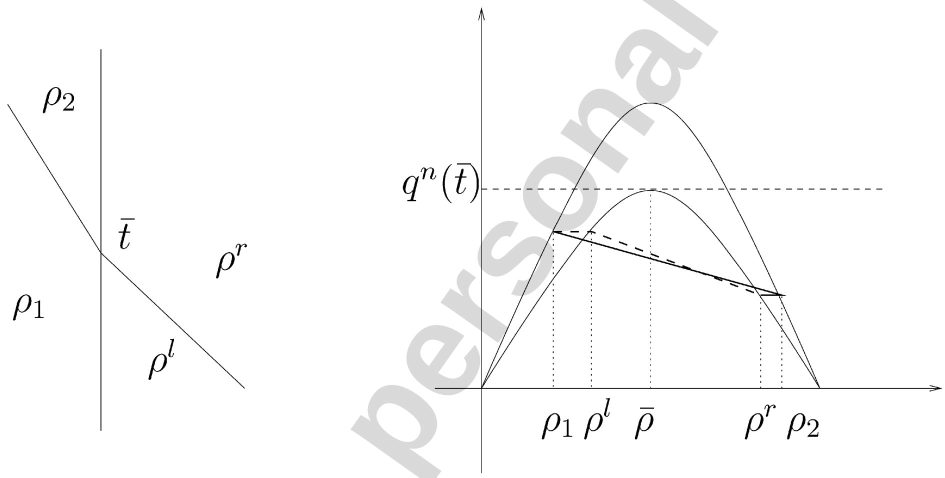

Fig. 9. Notations for case (Iε4).

(Iε5): The constraint q^n jumps downward at $\bar{t}$. Assume first that $\rho \leqslant \bar{\rho}$ and $q^n(\bar{t}-) \geqslant f(\rho) > q^n(\bar{t}+)$:

$$
\begin{aligned}
\Delta \Upsilon_\varepsilon(\bar{t}) &= \left|\Psi(\rho) - \Psi(\hat{\rho})\right| + 2\left|f(\bar{\rho}) - q^n(\bar{t}+)\right| + \left|\Psi_\varepsilon(\bar{\rho}) - \Psi_\varepsilon(\rho_1)\right| \\
&\quad - 4\left|f(\bar{\rho}) - q^n(\bar{t}-)\right| \\
&= 4q^n(\bar{t}-) - 2q^n(\bar{t}+) - f(\rho)\left(1 + \frac{q^n(\bar{t}+)}{q^n(\bar{t}-)}\right) \\
&\leqslant 4q^n(\bar{t}-) - 2q^n(\bar{t}+) - q^n(\bar{t}+)\left(1 + \frac{q^n(\bar{t}+)}{q^n(\bar{t}-)}\right) \\
&= \left(4 + \frac{q^n(\bar{t}+)}{q^n(\bar{t}-)}\right)\left(q^n(\bar{t}-) - q^n(\bar{t}+)\right) \\
&\leqslant 5\left(q^n(\bar{t}-) - q^n(\bar{t}+)\right),
\end{aligned}
$$

since $k_\varepsilon(\bar{t}+, x) f(\rho_1) = \frac{q^n(\bar{t}+)}{q^n(\bar{t}-)} f(\rho)$.

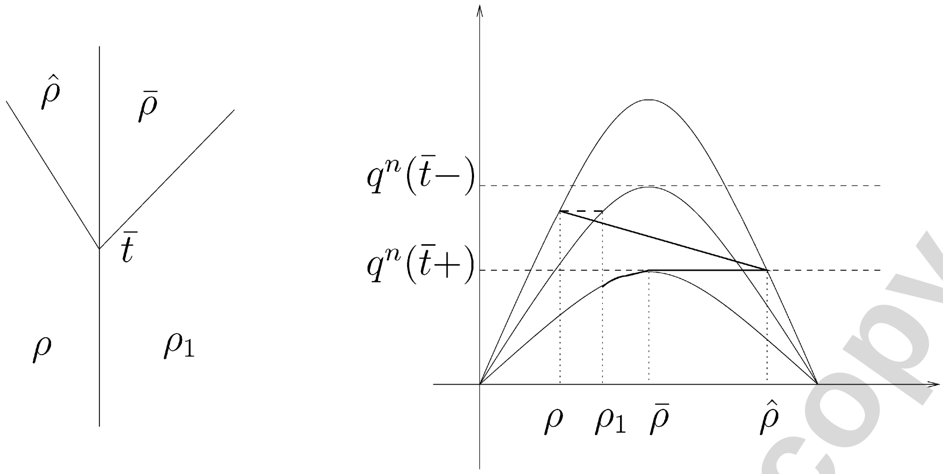

Fig. 10. Notations for case (Iε5).

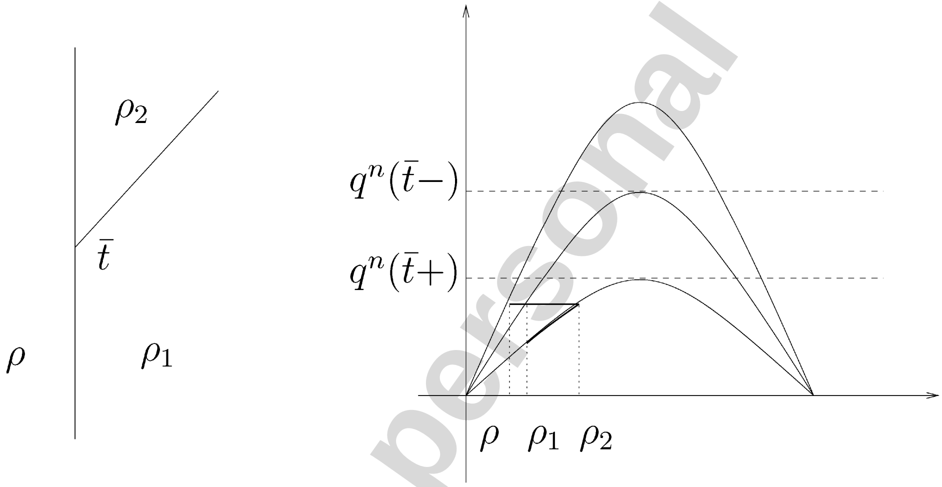

Fig. 11. Notations for case (Iε6).

(Iε6): The constraint q^n jumps downward at $\bar{t}$. If $\rho \leqslant \bar{\rho}$ and $f(\rho) \leqslant q^n(\bar{t}+)$:

$$\begin{aligned}
\Delta\Upsilon_\varepsilon(\bar{t}) &= 4\left|f(\bar{\rho}) - q^n(\bar{t}+)\right| + \left|\Psi_\varepsilon(\rho_2) - \Psi_\varepsilon(\rho_1)\right| - 4\left|f(\bar{\rho}) - q^n(\bar{t}-)\right| \\
&= 4\bigl(q^n(\bar{t}-) - q^n(\bar{t}+)\bigr) + \frac{f(\rho)}{q^n(\bar{t}-)}\bigl(q^n(\bar{t}-) - q^n(\bar{t}+)\bigr) \\
&\leqslant 5\bigl(q^n(\bar{t}-) - q^n(\bar{t}+)\bigr),
\end{aligned}$$

since $k_\varepsilon(\bar{t}+, x)f(\rho_1) = \frac{q^n(\bar{t}+)}{q^n(\bar{t}-)}f(\rho)$ and $k_\varepsilon(\bar{t}+, x)f(\rho_2) = f(\rho)$.

(Iε7): The constraint q^n jumps downward at $\bar{t}$. Assume now that $\rho > \bar{\rho}$, and $q^n(\bar{t}-) \geqslant f(\rho) > q^n(\bar{t}+)$:

$$\begin{aligned}
\Delta\Upsilon_\varepsilon(\bar{t}) &= \left|\Psi(\rho) - \Psi(\rho_2)\right| + 2\left|f(\bar{\rho}) - q^n(\bar{t}+)\right| - 2\left|f(\bar{\rho}) - q^n(\bar{t}-)\right| \\
&= 2\bigl(q^n(\bar{t}-) - q^n(\bar{t}+)\bigr) + \frac{f(\rho)}{q^n(\bar{t}-)}\bigl(q^n(\bar{t}-) - q^n(\bar{t}+)\bigr) \\
&\leqslant 3\bigl(q^n(\bar{t}-) - q^n(\bar{t}+)\bigr),
\end{aligned}$$

since $f(\rho_2) = \frac{q^n(\bar t+)}{q^n(\bar t-)} f(\rho)$. The case $f(\rho) \leqslant q^n(\bar t+)$ is handled similarly.

(Iε8): The constraint q^n jumps upward at $\bar t$. If $\rho \leqslant \bar\rho$:

$$\begin{aligned}\Delta\Upsilon_\varepsilon(\bar t) &= 4\left|f(\bar\rho) - q^n(\bar t+)\right| + \left|\Psi_\varepsilon(\rho_2) - \Psi_\varepsilon(\rho_1)\right| - 4\left|f(\bar\rho) - q^n(\bar t-)\right| \\ &= 4\big(q^n(\bar t-) - q^n(\bar t+)\big) + \frac{f(\rho)}{q^n(\bar t-)}\big(q^n(\bar t+) - q^n(\bar t-)\big) \\ &\leqslant -3\big(q^n(\bar t+) - q^n(\bar t-)\big) < 0.\end{aligned}$$

(Iε9): The constraint q^n jumps upward at $\bar t$. If $\rho > \bar\rho$:

$$\begin{aligned}\Delta\Upsilon_\varepsilon(\bar t) &= \left|\Psi(\rho) - \Psi(\rho_2)\right| + 2\left|f(\bar\rho) - q^n(\bar t+)\right| - 2\left|f(\bar\rho) - q^n(\bar t-)\right| \\ &= 2\big(q^n(\bar t-) - q^n(\bar t+)\big) + \frac{f(\rho)}{q^n(\bar t-)}\big(q^n(\bar t+) - q^n(\bar t-)\big) \\ &\leqslant -\big(q^n(\bar t+) - q^n(\bar t-)\big) < 0.\end{aligned}$$

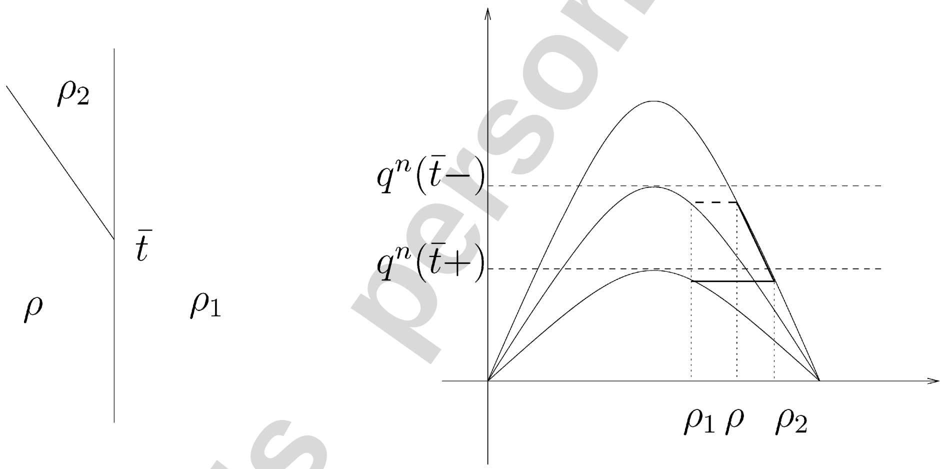

Fig. 12. Notations for case (Iε7).

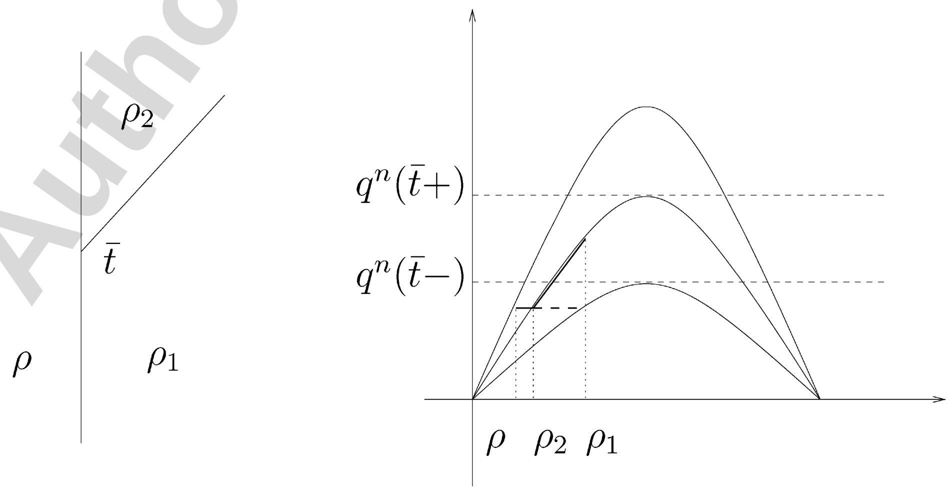

Fig. 13. Notations for case (Iε8).

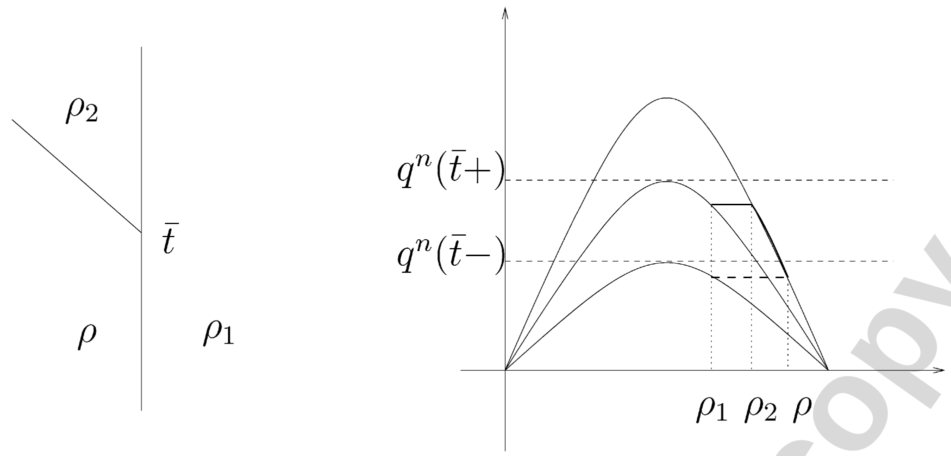

Fig. 14. Notations for case (Iε9).

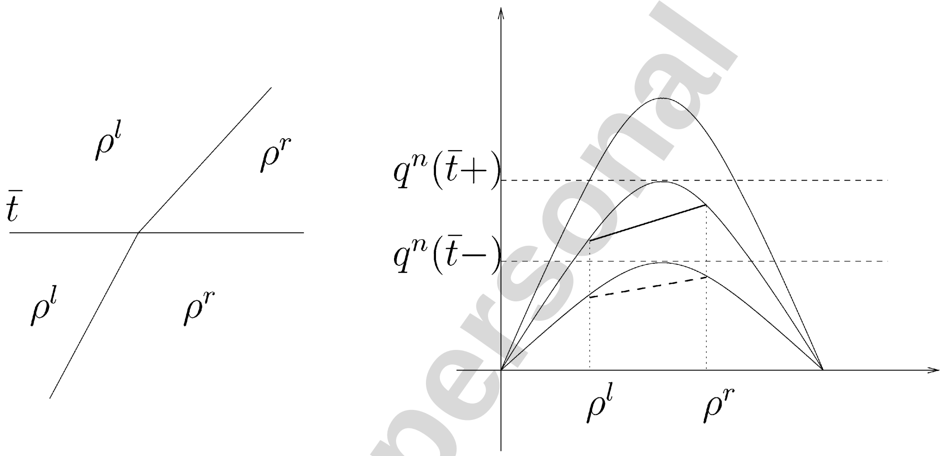

Fig. 15. Notations for case (Iε10).

(Iε10): In the interval $]-\varepsilon, \varepsilon[$, the constraint q^n has a jump at $t = \bar{t}$. Consider a ρ-wave between ρ^l and ρ^r, with $\rho^l \leqslant \bar{\rho} < \rho^r$ (the other cases are similar).

$$\begin{aligned}
\Delta \Upsilon_\varepsilon(\bar{t}) &= \Delta \left|\Psi_\varepsilon\big(\rho^l\big) - \Psi_\varepsilon\big(\rho^r\big)\right| \\
&= 2\big(q^n(\bar{t}+) - q^n(\bar{t}-)\big) + \frac{q^n(\bar{t}-)}{f(\bar{\rho})}\big(f\big(\rho^l\big) + f\big(\rho^r\big)\big)\left(1 - \frac{q^n(\bar{t}+)}{q^n(\bar{t}-)}\right) \\
&\leqslant 4\left|q^n(\bar{t}+) - q^n(\bar{t}-)\right|,
\end{aligned}$$

since $k_\varepsilon(\bar{t}+, x) f(\rho) = \frac{q^n(\bar{t}+)}{q^n(\bar{t}-)} k_\varepsilon(\bar{t}-, x) f(\rho)$.

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